A two-variable Iwasawa main conjecture over the eigencurve

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1 The setup

Fix an odd prime p, an algebraic closure $\overline{\mathbf{Q}_p}$, and an isomorphism $\mathbf{C} \xrightarrow{\sim} \overline{\mathbf{Q}_p}$. Fix an integer $N \geq 1$ prime to p, and let $\mathbf{T}$ be the polynomial algebra over $\mathbf{Z}$ generated by the operators $T_\ell, \ell \nmid Np$, U_p and $\langle d \rangle, d \in (\mathbf{Z}/N\mathbf{Z})^\times$. Set $\mathfrak{W} = \mathrm{Spf}(\mathbf{Z}_p[[\mathbf{Z}_p^\times]])$, and let $\mathscr{W} = \mathfrak{W}^{\mathrm{rig}}$ be the rigid analytic space of characters of $\mathbf{Z}_p^\times$ together with its universal character $\chi_{\mathscr{W}} : \mathbf{Z}_p^\times \to \mathscr{O}(\mathscr{W})^\times$; we embed $\mathbf{Z}$ in $\mathscr{W}(\mathbf{Q}_p)$ by mapping k to the character $t \mapsto t^{k-2}$. For any $\lambda \in \mathscr{W}(\overline{\mathbf{Q}_p})$ we (slightly abusively) write

$$M_\lambda^\dagger(\Gamma_1(N)) \subset \overline{\mathbf{Q}_p}[[q]]$$

for the space of q-expansions of overconvergent modular forms of weight λ and tame level N. Let $\mathscr{C}(N)$ be the tame level N eigencurve, with weight map $w : \mathscr{C}(N) \to \mathscr{W}$ and universal Hecke algebra homomorphism $\phi : \mathbf{T} \to \mathscr{O}(\mathscr{C}(N))$.

Let $\mathscr{C}_0^{M-\mathrm{new}}(N)$, for $M|N$, be the Zariski closure of the points associated with classical cuspidal newforms of level $\Gamma_1(M)$; this sits inside $\mathscr{C}(N)$ as a union of irreducible components. Finally, let $\mathscr{C} = \mathscr{C}_N$ be the normalization of $\mathscr{C}_0^{N-\mathrm{new}}(N)$, with its natural morphism $i : \mathscr{C}_N \to \mathscr{C}(N)$. This is a disjoint union of smooth, reduced rigid analytic curves. We slightly abusively write $w = w \circ i$, $\phi = i^*\phi$, etc.

Let $\mathscr{X} = \mathrm{Spf}(\mathbf{Z}_p[[\mathbf{Z}_p^\times]])^{\mathrm{rig}}$. A point $x \in \mathscr{X}(\overline{\mathbf{Q}_p})$ defines a continuous character $\psi_x : \mathbf{Z}_p^\times \to \overline{\mathbf{Q}_p}^\times$. This space will be our "cyclotomic variable". There is a partition $\mathscr{X} = \mathscr{X}^+ \coprod \mathscr{X}^-$, where $x \in \mathscr{X}^\pm$ according to $\psi_x(-1) = \pm 1$. Set $\mathscr{Y}_N = \mathscr{C}_N \times \mathscr{X}$ and $\mathscr{Y}_N^\pm = \mathscr{C}_N \times \mathscr{X}^\pm \subset \mathscr{Y}$. Let pr and $\mathrm{pr}_\pm$ denote the projections of $\mathscr{Y}_N$ and $\mathscr{Y}_N^\pm$ onto $\mathscr{C}_N$, respectively.

Our goal is to define the following objects:

- Torsion-free coherent sheaves $\mathscr{V}^\pm(N)$ on $\mathscr{C}(N)$. In the notation introduced below, we have

  $$H^0(\mathscr{C}_{\Omega,h}(N), \mathscr{V}^\pm(N)) \cong \mathrm{Symb}_{\Gamma_1(Np)}(\mathscr{D}_\Omega)_h^\pm.$$

  On $\mathscr{C}_0^{N-\mathrm{new}}(N)$ these sheaves have generic rank one.

- Torsion-free coherent sheaves

$$\mathscr{M}^{\pm}(N) = \mathscr{H}\mathrm{om}_{w^{-1}\mathscr{O}_{\mathscr{W}}}\left(\mathscr{V}^{\pm}(N), w^{-1}\mathscr{O}_{\mathscr{W}}\right)$$

on $\mathscr{C}(N)$. We need to be careful about the meaning of the right-hand side, since the morphism w isn't finite. Again, these sheaves have generic rank one on $\mathscr{C}_0^{N-\mathrm{new}}(N)$. Set

$$\begin{aligned} \widetilde{\mathscr{M}}_N^{\pm} &= (i \circ \mathrm{pr}_{\pm})^* \mathscr{M}^{\pm}(N) \\ &= \mathrm{pr}_{\pm}^*(i^* \mathscr{M}^{\pm}(N)). \end{aligned}$$

Since $i^*\mathscr{M}^{\pm}(N)$ is torsion-free of generic rank one on a smooth reduced curve, it is locally free of rank one. Therefore, $\widetilde{\mathscr{M}}_N^{\pm}$ is locally free of rank one (and in fact, $H^0(\mathrm{pr}_{\pm}^{-1}(U), \widetilde{\mathscr{M}}_N^{\pm})$ is free of rank one over $\mathscr{O}(U \times \mathscr{X}^{\pm})$ for suitable affinoids $U \subset \mathscr{C}_N$).

- A canonical global section

$$\mathbf{L} \in H^0\left(\mathscr{Y}_N, \widetilde{\mathscr{M}}_N\right).$$

Here $\widetilde{\mathscr{M}}_N$ denotes the natural line bundle on $\mathscr{Y}_N$ which restricts to $\widetilde{\mathscr{M}}_N^{\pm}$ on $\mathscr{Y}_N^{\pm}$. The element $\mathbf{L}$ is *the canonical two-variable p-adic L-function on* $\mathscr{C}_N \times \mathscr{X}$, in a sense we will make precise.

Since $\mathbf{L}$ is a section of a line bundle on a normal rigid analytic space, it generates a coherent ideal sheaf $\mathscr{I}_{\mathbf{L}} \subset \mathscr{O}_{\mathscr{Y}}$ in the usual way. The general philosophy of Iwasawa theory requires that $\mathscr{I}_{\mathbf{L}}$ coincide with the characteristic ideal of a suitable sheaf of Selmer groups over $\mathscr{Y}$. We have a candidate for this sheaf.

The eigencurve

We very briefly recall the "eigencurve of modular symbols." The main references here are Bellaiche's "Critical p-adic L-functions", Stevens's "Rigid analytic modular symbols", and my eigenvarieties paper.

Let s be a nonnegative integer. Consider the ring of functions

$$\mathbf{A}^s = \{f : \mathbf{Z}_p \to \mathbf{Q}_p \mid f \,\text{analytic on each}\, p^s\mathbf{Z}_p - \text{coset}\}.$$

Recall that by a fundamental result of Amice, the functions $e_j^s(x) = \lfloor p^{-s}j \rfloor! \begin{pmatrix} x \\ j \end{pmatrix}$ define an orthonormal basis of $\mathbf{A}^s$. The ring $\mathbf{A}^s$ is affinoid, and we set $\mathbf{B}_s = \mathrm{Sp}(\mathbf{A}^s)$, so e.g.

$$\mathbf{B}_s(\mathbf{C}_p) = \left\{ x \in \mathbf{C}_p \mid \inf_{a \in \mathbf{Z}_p} |x - a| \leq p^{-s} \right\}.$$

Given an affinoid open $\Omega \subset \mathscr{W}$ with associated character $\chi_\Omega : \mathbf{Z}_p^\times \to \mathscr{O}(\Omega)^\times$, define

$$\begin{aligned}\mathbf{A}^s_\Omega &= \mathscr{O}(\mathbf{B}^s \times \Omega) \\ &= \mathbf{A}^s \widehat{\otimes} \mathscr{O}(\Omega).\end{aligned}$$

For suitably large s, we have the left action $(\gamma \cdot f)(x) = \chi_\Omega(a+cx) f(\frac{b+dx}{a+cx})$ of $\Gamma_0(p)$ on this module. Let $\mathbf{D}^s_\Omega$ be the $\mathscr{O}(\Omega)$-Banach dual of $\mathbf{A}^s_\Omega$ with dual right action, and set

$$\begin{aligned}\mathscr{D}_\Omega &= \lim_{\infty \leftarrow s} \mathbf{D}^s_\Omega \\ &\cong \mathscr{D}(\mathbf{Z}_p) \widehat{\otimes} \mathscr{O}(\Omega).\end{aligned}$$

The assignment $\Omega \rightsquigarrow \mathscr{D}_\Omega$ is a "Frechet sheaf" on $\mathscr{W}$. Define

$$M^\dagger_\Omega(N) = \mathrm{Symb}_{\Gamma_1(N) \cap \Gamma_0(p)}(\mathscr{D}_\Omega).$$

Here, for any neat $\Gamma < \mathrm{SL}_2(\mathbf{Z})$ and any right Γ-module Q,

$$\mathrm{Symb}_\Gamma(Q) := \left\{ f \in \mathrm{Hom}_{\mathbf{Z}}(\mathrm{Div}^0 \mathbf{P}^1(\mathbf{Q}), Q) \mid f(\gamma D) \cdot_Q \gamma = f(D) \, \forall \gamma \in \Gamma, D \in \mathrm{Div}^0 \right\}.$$

The algebra $\mathbf{T}$ acts naturally on $M^\dagger_\Omega(N)$. The matrix $\begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$ conjugates $\Gamma_1(N) \cap \Gamma_0(p)$ to itself, and so induces an order two automorphism ι of $M^\dagger_\Omega(N)$. We write $M^\dagger_\Omega(N)^\pm = \frac{1 \pm \iota}{2} \circ M^\dagger_\Omega(N)$ for the two eigenspaces of ι.

For $h \in \mathbf{Q}$, let $\mathbf{B}[h] = \mathrm{Sp}\mathbf{Q}_p \langle p^h X \rangle$ be the closed rigid ball of radius p^h, and $\mathbf{A}^1 = \cup_h \mathbf{B}[h]$. Let $F(T) \in \mathscr{O}(\mathscr{W})\{\{X\}\}$ be the Fredholm series such that

$$F|_\Omega = \det(1 - U_p X) | M^\dagger_\Omega(N)$$

for all Ω. This cuts out a Fredholm hypersurface $\mathscr{Z} \subset \mathscr{W} \times \mathbf{A}^1$. Let w be the projection $\mathscr{W} \times \mathbf{A}^1 \to \mathscr{W}$. We say (Ω, h) is a *slope datum* if $M^\dagger_\Omega(N)$ admits a slope-$\leq h$ direct summand $M^\dagger_\Omega(N)_h$. This is a finite projective $\mathscr{O}(\Omega)$-module, and is Hecke-stable. Note that (Ω, h) is a slope datum if and only if $\mathscr{Z}_{\Omega,h} := \mathscr{Z} \cap (\Omega \times \mathbf{B}[h])$ is *slope-adapted*, i.e. finite flat over Ω and disconnected from its complement in $\mathscr{Z}_\Omega := \mathscr{Z} \cap w^{-1}(\Omega)$. The set of slope-adapted $\mathscr{Z}_{\Omega,h}$'s give an admissible covering of $\mathscr{Z}$ (this last is a foundational result of Buzzard). The map "$X \to U_p^{-1}$" makes $M^\dagger_\Omega(N)_h$ into a finite $\mathscr{O}(\mathscr{Z}_{\Omega,h})$-module.

Let $\mathbf{T}_{\Omega,h}$ denote the subalgebra of $\mathrm{End}_{\mathscr{O}(\Omega)}\left(M^\dagger_\Omega(N)_h\right)$ generated by the image of $\mathbf{T} \otimes_{\mathbf{Z}} \mathscr{O}(\Omega)$. This is finite over $\mathscr{O}(\Omega)$, so affinoid, and the map $\mathscr{O}(\Omega) \to \mathbf{T}_{\Omega,h}$ factors through $\mathscr{O}(\Omega) \to \mathscr{O}(\mathscr{Z}_{\Omega,h}) \to \mathbf{T}_{\Omega,h}$. The affinoids $\mathscr{C}_{\Omega,h}(N) := \mathrm{Sp}\mathbf{T}_{\Omega,h} \to \mathscr{Z}_{\Omega,h}$ glue over the covering of $\mathscr{Z}$ by the $\mathscr{Z}_{\Omega,h}$'s into $\mathscr{C}(N)$. (And from now on we write $\mathbf{T}_{\Omega,h}$ and $\mathscr{O}(\mathscr{C}_{\Omega,h}(N))$ interchangeably.) Each $M^\dagger_\Omega(N)^\pm_h$ is a finite $\mathbf{T}_{\Omega,h}$-module, and we define $\mathscr{V}^\pm(N)$ to be the coherent sheaf on $\mathscr{C}(N)$ obtained by gluing them up.

The construction

Proposition. The $\mathbf{T}_{\Omega,h}$-modules $\mathscr{M}^{\pm}_{\Omega,h}(N) := \mathrm{Hom}_{\mathscr{O}(\Omega)}\left(M^{\dagger}_{\Omega}(N)^{\pm}_{h}, \mathscr{O}(\Omega)\right)$ glue into coherent sheaves $\mathscr{M}^{\pm}(N)$ over $\mathscr{C}(N)$.

Proof. The key point is the following: suppose $\mathscr{Z}_{\Omega',h'} \subset \mathscr{Z}_{\Omega,h}$ is an inclusion of slope-adapted affinoids. We may assume $\Omega' \subset \Omega$ and $h' \leq h$, so the inclusion $\mathscr{Z}_{\Omega',h'} \subset \mathscr{Z}_{\Omega,h}$ "factors" as $\mathscr{Z}_{\Omega',h'} \subset \mathscr{Z}_{\Omega',h} \subset \mathscr{Z}_{\Omega,h}$ where $\mathscr{Z}_{\Omega',h} = \mathscr{Z}_{\Omega',h'} \coprod \mathscr{Z}^{>h'}_{\Omega',h}$ is slope-adapted with both pieces finite flat over Ω. Then $\mathbf{T}_{\Omega,h} \otimes_{\mathscr{O}(\Omega)} \mathscr{O}(\Omega') \cong \mathbf{T}_{\Omega',h}$, and

$$\begin{aligned}
\mathrm{Hom}_{\mathscr{O}(\Omega)}\left(M^{\dagger}_{\Omega}(N)^{\pm}_{h}, \mathscr{O}(\Omega)\right) \otimes_{\mathbf{T}_{\Omega,h}} \mathbf{T}_{\Omega',h} &\cong \mathrm{Hom}_{\mathscr{O}(\Omega)}\left(M^{\dagger}_{\Omega}(N)^{\pm}_{h}, \mathscr{O}(\Omega)\right) \otimes_{\mathbf{T}_{\Omega,h}} \mathbf{T}_{\Omega,h} \otimes_{\mathscr{O}(\Omega)} \mathscr{O}(\Omega') \\
&\cong \mathrm{Hom}_{\mathscr{O}(\Omega)}\left(M^{\dagger}_{\Omega}(N)^{\pm}_{h}, \mathscr{O}(\Omega)\right) \otimes_{\mathscr{O}(\Omega)} \mathscr{O}(\Omega') \\
&\cong \mathrm{Hom}_{\mathscr{O}(\Omega')}\left(M^{\dagger}_{\Omega}(N)^{\pm}_{h} \otimes_{\mathscr{O}(\Omega)} \mathscr{O}(\Omega'), \mathscr{O}(\Omega')\right) \\
&\cong \mathrm{Hom}_{\mathscr{O}(\Omega')}\left(M^{\dagger}_{\Omega'}(N)^{\pm}_{h}, \mathscr{O}(\Omega')\right)
\end{aligned}$$

as $\mathbf{T}_{\Omega',h}$-modules. The third line here follows from flatness of $\mathscr{O}(\Omega')$ over $\mathscr{O}(\Omega)$, and the fourth line from a basic property of slope decompositions. Now getting from $\mathbf{T}_{\Omega',h}$-modules to $\mathbf{T}_{\Omega',h'}$-modules is easy, because everything in sight has an idempotent decomposition coming from the previously described set-theoretic decomposition of $\mathscr{Z}_{\Omega',h}$. So $\mathscr{M}^{\pm}_{\Omega,h}(N) \otimes_{\mathbf{T}_{\Omega,h}} \mathbf{T}_{\Omega',h'} \cong \mathscr{M}^{\pm}_{\Omega',h'}(N)$ as desired. $\square$

Let $\mathscr{D}(\mathbf{Z}_p^{\times})$ denote the ring of locally analytic $\mathbf{Q}_p$-valued distributions on $\mathbf{Z}_p^{\times}$, and let $\mathscr{D}(\mathbf{Z}_p^{\times})^{\pm}$ denote the subspace of distributions for which

$$\int_{\mathbf{Z}_p^{\times}} f(-x)\mu(x) = \pm \int_{\mathbf{Z}_p^{\times}} f(x)\mu(x).$$

Recall the Amice isomorphism $\mathscr{D}(\mathbf{Z}_p^{\times}) \cong \mathscr{O}(\mathscr{X})$; this is an isomorphism of Frechet $\mathbf{Q}_p$-algebras, and induces isomorphisms $\mathscr{D}(\mathbf{Z}_p^{\times})^{\pm} \cong \mathscr{O}(\mathscr{X}^{\pm})$.

Proposition. There is a natural isomorphism

$$\mathrm{Hom}_{\mathscr{O}(\Omega)}\left(M^{\dagger}_{\Omega}(N)^{\pm}_{h}, \mathscr{O}(\Omega)\widehat{\otimes}\mathscr{D}(\mathbf{Z}_p^{\times})^{\pm}\right) \cong \mathscr{M}^{\pm}_{\Omega,h}(N) \otimes_{\mathscr{O}(\mathscr{C}_{\Omega,h}(N))} \mathscr{O}(\mathscr{C}_{\Omega,h}(N) \times \mathscr{X}^{\pm})$$

compatible with all structures.

Proof. By the Amice isomorphism, so we have

$$\mathrm{Hom}_{\mathscr{O}(\Omega)}\left(M^{\dagger}_{\Omega}(N)^{\pm}_{h}, \mathscr{O}(\Omega)\widehat{\otimes}\mathscr{D}(\mathbf{Z}_p^{\times})^{\pm}\right) \cong \mathrm{Hom}_{\mathscr{O}(\Omega)}\left(M^{\dagger}_{\Omega}(N)^{\pm}_{h}, \mathscr{O}(\Omega \times \mathscr{X}^{\pm})\right).$$

Choose an increasing cover of $\mathscr{X}^{\pm}$ by affinoids $\mathscr{X}_1^{\pm} \subset \cdots \subset \mathscr{X}_n^{\pm} \subset \cdots$. Then

$$\begin{aligned}
\mathrm{Hom}_{\mathscr{O}(\Omega)}\left(M_\Omega^\dagger(N)_h^{\pm}, \mathscr{O}(\Omega \times \mathscr{X}_n^{\pm})\right) &= \mathrm{Hom}_{\mathscr{O}(\Omega)}\left(M_\Omega^\dagger(N)_h^{\pm}, \mathscr{O}(\Omega)\right) \otimes_{\mathscr{O}(\Omega)} \mathscr{O}(\Omega \times \mathscr{X}_n^{\pm}) \\
&= \mathrm{Hom}_{\mathscr{O}(\Omega)}\left(M_\Omega^\dagger(N)_h^{\pm}, \mathscr{O}(\Omega)\right) \otimes_{\mathbf{T}_{\Omega,h}} \mathbf{T}_{\Omega,h} \otimes_{\mathscr{O}(\Omega)} \mathscr{O}(\Omega \times \mathscr{X}_n^{\pm}) \\
&= \mathrm{Hom}_{\mathscr{O}(\Omega)}\left(M_\Omega^\dagger(N)_h^{\pm}, \mathscr{O}(\Omega)\right) \otimes_{\mathbf{T}_{\Omega,h}} \mathbf{T}_{\Omega,h} \otimes_{\mathscr{O}(\Omega)} \mathscr{O}(\Omega)\widehat{\otimes}\mathscr{O}(\mathscr{X}_n^{\pm}) \\
&= \mathrm{Hom}_{\mathscr{O}(\Omega)}\left(M_\Omega^\dagger(N)_h^{\pm}, \mathscr{O}(\Omega)\right) \otimes_{\mathbf{T}_{\Omega,h}} \mathbf{T}_{\Omega,h}\widehat{\otimes}\mathscr{O}(\mathscr{X}_n^{\pm}) \\
&= \mathscr{M}_{\Omega,h}^{\pm}(N) \otimes_{\mathscr{O}(\mathscr{C}_{\Omega,h}(N))} \mathscr{O}(\mathscr{C}_{\Omega,h}(N) \times \mathscr{X}_n^{\pm}).
\end{aligned}$$

The second, third, and fifth equalities here are trivial. The first equality follows from the identification $\mathrm{Hom}_R(M,N) \otimes_R P \cong \mathrm{Hom}_R(M, N \otimes_R P)$ for M, N, P any R-modules with M finitely presented and P flat, together with the flatness of $\mathscr{O}(\Omega \times \mathscr{X}_n^{\pm})$ over $\mathscr{O}(\Omega)$.[1] The fourth equality evidently follows from the identity

$$\mathbf{T}_{\Omega,h} \otimes_{\mathscr{O}(\Omega)} \mathscr{O}(\Omega)\widehat{\otimes}\mathscr{O}(\mathscr{X}_n^{\pm}) \cong \mathbf{T}_{\Omega,h}\widehat{\otimes}\mathscr{O}(\mathscr{X}_n^{\pm}),$$

which is an easy consequence of the ONability of $\mathscr{O}(\mathscr{X}_n^{\pm})$ and the module-finiteness of $\mathbf{T}_{\Omega,h}$ over $\mathscr{O}(\Omega)$ (or, more conceptually, use Propositions 3.7.3/6 and 2.1.7/7 of BGR). Passing to the inverse limit over n, we conclude. □

Definition / Claim. The sheaf

$$\widetilde{\mathscr{M}}^{\pm}(N) := \mathrm{pr}_{\pm}^{*}\mathscr{M}^{\pm}(N)$$

on $\mathscr{C}(N) \times \mathscr{X}^{\pm}$ is characterized by the isomorphism

$$\begin{aligned}
H^0(\mathscr{C}_{\Omega,h}(N) \times \mathscr{X}^{\pm}, \widetilde{\mathscr{M}}^{\pm}(N)) &\cong \mathscr{M}_{\Omega,h}^{\pm}(N) \otimes_{\mathbf{T}_{\Omega,h}} \mathbf{T}_{\Omega,h}\widehat{\otimes}\mathscr{D}(\mathbf{Z}_p^{\times})^{\pm} \\
&= \mathrm{Hom}_{\mathscr{O}(\Omega)}(M_\Omega^\dagger(N)_h^{\pm}, \mathscr{O}(\Omega)\widehat{\otimes}\mathscr{D}(\mathbf{Z}_p^{\times})^{\pm}).
\end{aligned}$$

Proof. This is an easy consequence of the previous proposition. □

Definition. The sheaf $\widetilde{\mathscr{M}}_N^{\pm}$ on $\mathscr{Y}_N^{\pm}$ is the pullback of $\widetilde{\mathscr{M}}^{\pm}(N)$ under $i \times \mathrm{id} : \mathscr{C}_N \times \mathscr{X}^{\pm} \to \mathscr{C}(N) \times \mathscr{X}^{\pm}$.

Given any Ω, let $\mathbf{L}_\Omega(N)$ denote the composite map

$$M_\Omega^\dagger(N) = \mathrm{Symb}_{\Gamma_1(N)\cap\Gamma_0(p)}(\mathscr{D}_\Omega) \overset{\Phi\mapsto\Phi((\infty)-(0))}{\longrightarrow} \mathscr{D}_\Omega \cong \mathscr{D}(\mathbf{Z}_p)\widehat{\otimes}\mathscr{O}(\Omega) \to \mathscr{D}(\mathbf{Z}_p^{\times})\widehat{\otimes}\mathscr{O}(\Omega).$$

[1] This flatness can be deduced by factoring the map as $\mathscr{O}(\Omega) \to \mathscr{O}(\Omega \times \mathbf{X}) \to \mathscr{O}(\Omega \times \mathscr{X}_n)$ where $\mathbf{X}$ is some finite disjoint union of $\mathbf{B}[0]$'s and $\mathscr{X}_n \subset \mathbf{X}$ is an affinoid subdomain. The second arrow is then flat by the flatness of coordinate rings of affinoid subdomains, and the first arrow is flat because $A \to A\langle X\rangle$ is flat for any Tate algebra A.

The final arrow here is induced by the map $\mathscr{D}(\mathbf{Z}_p) \to \mathscr{D}(\mathbf{Z}_p^\times)$ dual to the map $\mathscr{A}(\mathbf{Z}_p^\times) \to \mathscr{A}(\mathbf{Z}_p)$ given by extending functions by zero. Since $\mathbf{L}_\Omega(N)$ is $\mathscr{O}(\Omega)$-linear and $M_\Omega^\dagger(N)_h$ is an $\mathscr{O}(\Omega)$-module direct summand of $M_\Omega^\dagger(N)$ (and likewise the $\pm$-subspaces), we may regard the restriction $\mathbf{L}_{\Omega,h}^\pm(N) := \mathbf{L}_\Omega(N)|_{M_\Omega^\dagger(N)_h^\pm}$ as an element of

$$\mathrm{Hom}_{\mathscr{O}(\Omega)}\left(M_\Omega^\dagger(N)_h^\pm, \mathscr{O}(\Omega)\widehat{\otimes}\mathscr{D}^\pm(\mathbf{Z}_p^\times)\right) = H^0(\mathscr{C}_{\Omega,h}(N) \times \mathscr{X}^\pm, \widetilde{\mathscr{M}}^\pm(N)).$$

These glue together into global sections $\mathbf{L}^\pm(N)$ of the sheaf $\widetilde{\mathscr{M}}^\pm(N)$ on $\mathscr{C}(N) \times \mathscr{X}^\pm$. We finally define $\mathbf{L}^\pm$ as the global sections of $\widetilde{\mathscr{M}}_N^\pm$ given by pullback of $\mathbf{L}^\pm(N)$ under $i \times \mathrm{id}$, and then define

$$\begin{aligned}\mathbf{L} := \mathbf{L}^+ + \mathbf{L}^- &\in H^0(\mathscr{Y}_N, \widetilde{\mathscr{M}}_N)\\ &= H^0(\mathscr{Y}_N^+, \widetilde{\mathscr{M}}_N^+) \oplus H^0(\mathscr{Y}_N^-, \widetilde{\mathscr{M}}_N^-)\end{aligned}$$

(with the obvious meaning).