

# What is a spectral sequence?

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Very short answer: A spectral sequence is a piece of terminology used by algebraists to intimidate other mathematicians.

Short answer / slogan: A spectral sequence is a machine which breaks difficult homological calculations apart into many easier (?) ones, and then glues their results together.

Long answer: A (first quadrant cohomology)  $E_r$ -page spectral sequence consists of the following data:

- A collection of abelian groups  $E_s^{i,j}$  for integers  $i \geq 0, j \geq 0, s \geq r$ . The data of  $E_s^{i,j}$  for  $s$  fixed and varying  $i, j$  is referred to as *the  $E_s$ -page*.
- Homomorphisms of abelian groups  $d_s^{i,j} : E_s^{i,j} \rightarrow E_s^{i+s, j+1-s}$ , the *differentials*, such that  $\text{im} d_s^{i-s, j-1+s} \subset \ker d_s^{i,j}$ . The differentials into  $E_s^{i,j}$  are trivial for  $s > i$ , and the differentials out of  $E_s^{i,j}$  are trivial if  $s > j + 1$ .
- Canonical isomorphisms  $E_{s+1}^{i,j} \simeq \frac{\ker d_s^{i,j}}{\text{im} d_s^{i-s, j-1+s}}$ .

You can and should think of the  $E_s^{i,j}$ 's for fixed  $s$  as indexed by integer points in the first quadrant of the plane; the condition on the differentials says that the groups lying along each line of slope  $-\frac{s-1}{s}$  form a complex, and the  $E_{s+1}$ -page formed by taking the cohomology of the  $E_s$ -page. Note also that each  $E_{s+1}^{i,j}$  is a quotient of a subgroup of  $E_s^{i,j}$ , and that eventually every differential entering or leaving a fixed entry  $E_s^{i,j}$  is identically zero, because either the  $i$ -coordinate of the source or the  $j$ -coordinate of the target is negative. In other words, the entries  $E_s^{i,j}$  evolve as you "turn the pages", but eventually they stop changing: they *stabilize*, and we write  $E_\infty^{i,j}$  for the stable value of  $E_s^{i,j}$ . The following picture must be



*Exercise:* If  $F \rightarrow E \rightarrow B$  is a fiber bundle, with  $B$  path-connected and  $E$  orientable, show the equality

$$\chi(E) = \chi(B)\chi(F)$$

where  $\chi$  denotes Euler characteristic and  $\chi(F)$  is the Euler characteristic of “the” fiber.

*Exercise:* Using the family of fiber bundles  $\mathrm{SO}(n-1) \rightarrow \mathrm{SO}(n) \rightarrow S^{n-1}$ , inductively calculate the cohomology ring of  $\mathrm{SO}(n)$ .

**(Grothendieck)** Let  $X$  be a topological space, with  $\mathcal{F} \rightarrow X$  a sheaf of abelian groups. Let  $\mathcal{U}$  be a covering of  $X$  by open sets. Given any open set  $U \subset X$ , the rule

$$\begin{aligned} \mathcal{H}^j(-, \mathcal{F}) : \text{opens in } X &\rightarrow \text{abelian groups} \\ U &\mapsto H^j(U, \mathcal{F}|_U) \end{aligned}$$

defines a presheaf on  $X$ . Čech cohomology makes sense with coefficients in any presheaf, and we have the *Čech-to-sheaf spectral sequence*

$$E_2^{i,j} = \check{H}^i(\mathcal{U}, \mathcal{H}^j(-, \mathcal{F})) \Rightarrow H^{i+j}(X, \mathcal{F}).$$

*Exercise:* If  $\mathcal{U} = \{U_1, U_2\}$  consists of two open sets, prove that the Čech-to-sheaf spectral sequence yields the Mayer-Vietoris long exact sequence

$$\dots \rightarrow H^i(X, \mathcal{F}) \rightarrow H^i(U_1, \mathcal{F}) \oplus H^i(U_2, \mathcal{F}) \rightarrow H^i(U_1 \cap U_2, \mathcal{F}) \rightarrow H^{i+1}(X, \mathcal{F}) \rightarrow \dots$$

Can you show anything when  $\mathcal{U}$  consists of three open sets?

*Exercise:* The covering  $\mathcal{U}$  is a *Leray covering relative to  $\mathcal{F}$*  if  $H^j(U, \mathcal{F}|_U) = 0$  for all  $j > 0$  and all  $U \in \mathcal{U}$ . Supposing  $\mathcal{U}$  is a Leray covering, show the spectral sequence degenerates to an isomorphism

$$H^i(X, \mathcal{F}) \simeq \check{H}^i(\mathcal{U}, \mathcal{F}).$$

**(Grothendieck)** Let  $X$  be a topological space, with  $\mathcal{F} \rightarrow X$  a sheaf of abelian groups. Recall that  $H^i(X, \mathcal{F})$  is defined as follows: one chooses a long exact sequence  $0 \rightarrow \mathcal{F} \rightarrow \mathcal{I}^0 \rightarrow \mathcal{I}^1 \rightarrow \mathcal{I}^2 \rightarrow \dots$  with each  $\mathcal{I}^i$  an injective sheaf, and  $H^i(X, \mathcal{F})$  is the  $i$ th cohomology group of the complex

$$0 \rightarrow \Gamma(X, \mathcal{I}^0) \rightarrow \Gamma(X, \mathcal{I}^1) \rightarrow \Gamma(X, \mathcal{I}^2) \rightarrow \dots$$

Given a complex  $\mathcal{G}^\bullet = 0 \rightarrow \mathcal{G}^0 \rightarrow \mathcal{G}^1 \rightarrow \dots$  of sheaves, the  $i$ th *cohomology sheaf of  $\mathcal{G}^\bullet$*  is the sheaf  $\mathcal{H}^i(\mathcal{G}^\bullet)$  defined as the sheafification of the presheaf

$$U \mapsto \frac{\ker \Gamma(U, \mathcal{G}^i) \rightarrow \Gamma(U, \mathcal{G}^{i+1})}{\mathrm{im} \Gamma(U, \mathcal{G}^{i-1}) \rightarrow \Gamma(U, \mathcal{G}^i)}.$$

In the setup above, note that the complexes  $\mathcal{F}^\bullet : 0 \rightarrow \mathcal{F} \rightarrow 0$  and  $\mathcal{I}^\bullet : 0 \rightarrow \mathcal{I}^0 \rightarrow \mathcal{I}^1 \rightarrow \mathcal{I}^2 \rightarrow \dots$  have the same cohomology sheaves, namely  $\mathcal{H}^0(\mathcal{F}^\bullet) \simeq \mathcal{H}^0(\mathcal{I}^\bullet) \simeq \mathcal{F}$  and  $\mathcal{H}^i(\mathcal{F}^\bullet) \simeq \mathcal{H}^i(\mathcal{I}^\bullet) \simeq 0 \forall i \geq 1$  - in other words, resolving  $\mathcal{F}$  by injective sheaves is equivalent to finding a complex  $\mathcal{I}^\bullet$  of injective sheaves with  $\mathcal{H}^0(\mathcal{I}^\bullet) \simeq \mathcal{F}$ ,  $\mathcal{H}^i(\mathcal{I}^\bullet) \simeq 0 \forall i \geq 1$ .

**Definition / Theorem:** *Given any complex of sheaves  $\mathcal{F}^\bullet$ , there exists a complex  $\mathcal{I}^\bullet$  of injective sheaves with  $\mathcal{H}^i(\mathcal{I}^\bullet) \simeq \mathcal{H}^i(\mathcal{F}^\bullet) \forall i \geq 0$ ; call any such complex an injective resolution of  $\mathcal{F}^\bullet$ . Choose an injective resolution  $\mathcal{I}^\bullet$  of  $\mathcal{F}^\bullet$  and define  $\mathbf{H}^i(X, \mathcal{F}^\bullet)$  as the  $i$ th cohomology group of the complex*

$$0 \rightarrow \Gamma(X, \mathcal{I}^0) \rightarrow \Gamma(X, \mathcal{I}^1) \rightarrow \Gamma(X, \mathcal{I}^2) \rightarrow \dots;$$

then  $\mathbf{H}^i(X, \mathcal{F}^\bullet)$  a well-defined abelian group (i.e. it doesn't depend on the specific injective resolution  $\mathcal{I}^\bullet$  we chose).

The group  $\mathbf{H}^i(X, \mathcal{F}^\bullet)$  is the  $i$ th hypercohomology group of the complex  $\mathcal{F}^\bullet$ . There are two spectral sequences which abut to  $\mathbf{H}^*(X, \mathcal{F}^\bullet)$ , namely

$$E_2^{i,j} = H^i(X, \mathcal{H}^j(\mathcal{F}^\bullet)) \Rightarrow \mathbf{H}^{i+j}(X, \mathcal{F}^\bullet)$$

and

$$E_1^{i,j} = H^j(X, \mathcal{F}^i) \Rightarrow \mathbf{H}^{i+j}(X, \mathcal{F}^\bullet).$$

*Example:* Let  $X$  be a smooth complex manifold of complex dimension  $n$ . The *holomorphic de Rham complex of  $X$*  is the complex

$$\Omega_X^\bullet : 0 \rightarrow \mathcal{O}_X \xrightarrow{d} \Omega_X^1 \xrightarrow{d} \Omega_X^2 \xrightarrow{d} \dots \rightarrow \Omega_X^n \rightarrow 0,$$

where  $\Omega_X^i$  is the sheaf of holomorphic  $i$ -forms. The holomorphic functions with zero derivative are the locally constant functions, so  $\mathcal{H}^0(\Omega_X^\bullet) \simeq \underline{\mathbf{C}}$ ; furthermore, a holomorphic analogue of the usual Poincare lemma yields  $\mathcal{H}^i(\Omega_X^\bullet) = 0 \forall i \geq 1$ . Hence the *first* hypercohomology spectral sequence degenerates to an isomorphism

$$H_{\text{sing}}^i(X, \mathbf{C}) \simeq H^i(X, \underline{\mathbf{C}}) \simeq \mathbf{H}^i(X, \Omega_X^\bullet),$$

so the *second* hypercohomology spectral sequence now reads

$$E_1^{i,j} = H^j(X, \Omega_X^i) \Rightarrow H_{\text{sing}}^{i+j}(X, \mathbf{C}).$$

In other words, sheaf cohomology of the holomorphic objects  $\Omega_X^i$  suffices to compute the singular cohomology of  $X$ ! This, to me at least, is much more surprising than the de Rham isomorphism - arbitrary smooth differential forms are very mushy objects, but holomorphic differential forms are extremely rigid.