ON p-ADIC L-FUNCTIONS FOR HILBERT MODULAR FORMS

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ABSTRACT. We construct p-adic L-functions associated with p-refined cohomological cuspidal Hilbert modular forms over any totally real field under a mild hypothesis. Our construction is canonical, varies naturally in p-adic families, and does not require any small slope or non-criticality assumptions on the p-refinement. The main new ingredients are an adelic definition of a canonical map from overconvergent cohomology to a space of locally analytic distributions on the relevant Galois group and a smoothness theorem for certain eigenvarieties at critically refined points.

CONTENTS

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1. Introduction

The goal of this article to define canonical p-adic L-functions associated with p-refined cohomological cuspidal automorphic representations of GL_2 over totally real number fields. We make no assumptions on the so-called *slope* (other than finiteness), and our construction varies naturally in p-adic families.

1.1. **The main result.** To state our results we begin by setting notation. Let F be a totally real number field of degree d and write Σ_F for the set of embeddings $F \hookrightarrow \mathbf{R}$. The completion of F at a place v will be written F_v; the ramification index will be written e_v; the residue field will have q_v-many elements. We write π for a cohomological cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_F)$ and λ for its weight. Throughout the introduction we will omit 'cohomological cuspidal' and simply refer to π as an automorphic representation, except when more precision is helpful. In our normalization, the cohomological condition means the weight λ is a pair (κ, w) such that $\kappa = (\kappa_\sigma)_{\sigma\in\Sigma_F}$ is a Σ_F-tuple of non-negative integers, $w \in \mathbf{Z}$, and $\kappa_\sigma \equiv w \bmod 2$. An integer m is called (Deligne-)critical with respect to λ if

$$\frac{w-\kappa_\sigma}{2} \leq m \leq \frac{w+\kappa_\sigma}{2} \qquad (\sigma \in \Sigma_F).$$

For precise explanations of the basic definitions and normalizations, see Sections 2 and 3.

The starting point of our work is a famous algebraicity result of Shimura for special values of the L-functions associated with such π. More precisely, for any finite order Hecke character θ we may consider the completed L-function $\Lambda(\pi \otimes \theta, s)$ associated to the twist of π by θ. It is entire in the variable s, and it satisfies a functional equation. Shimura proved ([72]) that there is a collection of periods $\Omega_\pi^\epsilon \in \mathbf{C}^\times$ indexed by signs $\epsilon = (\epsilon_\sigma) \in \{\pm 1\}^{\Sigma_F}$ with the property that for any integer m critical with respect to λ and any finite order θ, the number

$$\Lambda^{\mathrm{alg}}(\pi \otimes \theta, m+1) := \frac{\left(\prod_{\sigma \in \Sigma_F} \theta_\sigma(-1) i^{1+m+\frac{\kappa_\sigma - w}{2}}\right) \Delta_{F/\mathbf{Q}}^{m+1} \Lambda(\pi \otimes \theta, m+1)}{\Omega_\pi^\epsilon G(\theta)}$$

lies in the field $\mathbf{Q}(\pi,\theta)$ generated by the Hecke eigenvalues of π together with the values of θ. Here $G(\theta)$ is a certain Gauss sum and the sign ϵ is determined by $\epsilon_\sigma = (-1)^m \theta_\sigma(-1)$ for all $\sigma \in \Sigma_F$ (θ_σ being the σ-th component of θ). We will give a complete exposition of this result in Section 4, roughly following Hida ([49]).

Now let p be a prime number. We will fix an isomorphism $\iota : \mathbf{C} \xrightarrow{\sim} \overline{\mathbf{Q}}_p$ where $\overline{\mathbf{Q}}_p$ is a fixed algebraic closure of the field of p-adic numbers $\mathbf{Q}_p$. It then makes sense to try to p-adically interpolate the algebraic special values $\iota\left(\Lambda^{\mathrm{alg}}(\pi \otimes \theta, m+1)\right)$ as m and θ vary.

In order to do this, we introduce a certain p-adic analytic space of characters. Let Γ_F be the Galois group of the maximal abelian extension of F unramified away from p and ∞. This is a compact and abelian topological group. It also contains an open (so finite index) subgroup topologically isomorphic to finitely many copies of the p-adic integers $\mathbf{Z}_p$. Given any such group, there is a canonically associated rigid analytic character variety $\mathscr{X}(\Gamma_F)$ whose $\mathbf{C}_p$-points correspond to continuous characters $\Gamma_F \to \mathbf{C}_p^\times$. In particular, if θ is finite order Hecke character with p-power conductor, then $\theta^\iota := \iota \circ \theta$ defines a point in $\mathscr{X}(\Gamma_F)$. By global class field theory, each character $\chi \in \mathscr{X}(\Gamma_F)$ can be seen as a p-adic Hecke character and so in particular has signs, at infinity, as above. The group Γ_F and its character variety play a key role in this article: our p-adic L-functions will be elements in the ring $\mathscr{O}(\mathscr{X}(\Gamma_F))$ of rigid analytic functions on $\mathscr{X}(\Gamma_F)$.

We also need the notion of a p-refinement. For simplicity, we assume for the remainder of the introduction that π is an unramified principal series at each $v \mid p$. In the body of the text we will also allow π to be an unramified special representation. Let χ_π be the nebentype character of π. If $v \mid p$, then write $a_\pi(v)$ for the v-th eigenvalue in the Hecke eigensystem associated to π and ϖ_v for a uniformizing parameter.

Definition 1.1.1. A p-refinement for π is a tuple $(\alpha_v)_{v|p}$ where α_v is a root of the v-th Hecke polynomial $X^2 - a_\pi(v)X + \chi_\pi(\varpi_v)q_v^{w+1}$.

If α is a p-refinement, we write $(\beta_v)_{v|p}$ for the list of 'other' roots determined by the factorizations

$$X^2 - a_\pi(v)X + \chi_\pi(\varpi_v)q_v^{w+1} = (X - \alpha_v)(X - \beta_v).$$

We often refer to the pair (π,α) as a p-refined automorphic representation (or some minor variant thereof). When $F = \mathbf{Q}$ and π corresponds to a holomorphic eigenform $f(z)$ of level N that is prime to p, a p-refinement α is often instantiated through the eigenform

$$f_\alpha(z) = f(z) - \beta f(pz)$$

which now has level Np. See Section 3.4 for more details.

In Section 1.5, we will define what it means for a p-refined (π,α) to be *non-critical* and, more generally, *decent*. We will call α critical if it is not non-critical.[1] We note immediately that non-critical is implied by a 'small slope' condition on α, but it is certainly not equivalent, and that non-critical implies decent. The condition of being decent is very mild in our estimation. Conjecturally, outside

[1]There are two completely unrelated uses of the word 'critical' in this article, an unfortunate collision. We will stress the context by always referring to an integer as being critical with respect to a weight and a refinement being a (non-)critical refinement.

the non-critical case it should reduce to the condition that α_v and β_v as above are distinct for all $v \mid p$, which is expected to always hold when p is totally split in F. In Section 1.6 we discuss the hypothesis of decency in detail.

Absent the definition of decent we can state our main theorem. We re-iterate that we have assumed π cohomological cuspidal and, for simplicity only, that π is an unramified principal series at each $v \mid p$.

Theorem 1.1.2 (Section 8.2)**.** *Let (π,α) be a decently p-refined automorphic representation of weight λ. Let $E = \mathbf{Q}(\pi,\alpha)$ be the subfield of $\mathbf{C}$ generated by $\mathbf{Q}(\pi)$ and the refinement α, and let $L \subset \overline{\mathbf{Q}}_p$ be the subfield generated by $\iota(E)$.*

Then, for each $\epsilon \in \{\pm 1\}^{\Sigma_F}$ there exists an element $L_p^\epsilon(\pi,\alpha) \in \mathscr{O}(\mathscr{X}(\Gamma_F)) \otimes_{\mathbf{Q}_p} L$ satisfying the following properties.

a. ***Canonicity:*** *The construction of $L_p^\epsilon(\pi,\alpha)$ is canonically specified up to $L^\times$-multiple in general and up to $\iota(E^\times)$-multiple if α is non-critical.*

b. ***Support:*** *$L_p^\epsilon(\pi,\alpha)(\chi) = 0$ if $\operatorname{sgn}(\chi_\sigma) \neq \epsilon_\sigma$ for each $\sigma \in \Sigma_F$.*

c. ***Growth:*** *$L_p^\epsilon(\pi,\alpha)$ has growth bounded by $\sum_{v\mid p} e_v v_p(\iota(\alpha_v)) + \sum_{\sigma\in\Sigma_F} \frac{\kappa_\sigma - w}{2}$.*[2]

d. ***Interpolation:*** *Let m be an integer that is critical with respect to λ, and assume that θ is a finite order Hecke character of p-power conductor with $\epsilon_\sigma = \operatorname{sgn}(\theta_\sigma)(-1)^m$ for each $\sigma \in \Sigma_F$. Then,*

$$L_p^\epsilon(\pi,\alpha)(\theta^\iota \chi_{\mathrm{cycl}}^m) = e_p(\alpha,m)\cdot \iota\left(\Lambda^{\mathrm{alg}}(\pi\otimes\theta, m+1)\right)$$

where the interpolation factor $e_p(\alpha,m) = \prod_{v\mid p} e_v(\alpha,m)$ is defined as follows:

(i) If α is non-critical, then

$$\iota^{-1}(e_v(\alpha,m)) = \begin{cases} \left(1-\theta(\varpi_v)\alpha_v^{-1}q_v^m\right)\left(1-\theta(\varpi_v)\beta_v q_v^{-(m+1)}\right) & \text{(if θ_v is unramified);}\\ \left(\dfrac{q_v^{m+1}}{\alpha_v}\right)^{f_v} & \text{(if θ_v is ramified of conductor $\varpi_v^{f_v}$).}\end{cases}$$

(ii) If α is critical then $e_v(\alpha,m) = 0$ for all $v \mid p$.

e. ***Variation:*** *Suppose the eigenvariety $\mathscr{E}(\mathfrak{n})_{\mathrm{mid}}$ is smooth at the classical point $x_{\pi,\alpha}$ associated with (π,α).*[3] *Then for any sufficiently small open neighborhood U of x in $\mathscr{E}(\mathfrak{n})_{\mathrm{mid}}$ there exists an element $\mathbf{L}_p^\epsilon \in \mathscr{O}(U)\widehat{\otimes}_{\mathbf{Q}_p}\mathscr{O}(\mathscr{X}(\Gamma_F))$ canonically specified up to $\mathscr{O}(U)^\times$-multiple and such that for each decent point $x' \in U$ associated with a p-refined cohomological cuspidal automorphic representation (π',α') we have*

$$\mathbf{L}_p^\epsilon|_{x'} = c_{x'} L_p^\epsilon(\pi',\alpha')$$

for some constant $c_{x'} \in k_{x'}^\times$.

f. ***Uniqueness:*** *If the Leopoldt defect of F at p is zero, then (up to $L^\times$ ambiguity) the assignment $(\pi,\alpha) \rightsquigarrow L_p^\epsilon(\pi,\alpha)$ is uniquely determined by conditions b.-e.*

This article is not the first place a theorem like this has been proven, and we owe a great deal to previous work. We will compare our theorem with the literature in Section 1.8; in order to put these

[2]Growth is defined in Section 7.4.

[3]This is almost always satisfied for decent (π,α). See Theorem 1.7.2 and the discussion following that result.

comparisons in the proper context, however, we should first expand on the definition of decency and the method of our construction. We hope this delay is not taken as a slight.

1.2. **The story when $F = \mathbf{Q}$.** Our strategy is modeled on the case $F = \mathbf{Q}$ which is more or less understood. To motivate our constructions, we outline the necessary ingredients in that case.

1.2.1. *Archimedean considerations.* Let $f = \sum a_n(f)q^n$ be a normalized cuspidal Hecke newform of weight $k \geq 2$ and level $\Gamma_1(N)$ with N prime to p. The construction of Eichler and Shimura associates with f a canonical cohomology class $\omega_f \in H^1_c(Y_1(N), \mathscr{L}_{k-2})$ where $\mathscr{L}_{k-2}$ is a local system on the modular curve $Y_1(N)$ defined by a 'weight $k-2$' action on the space of complex polynomials of degree at most $k-2$ in a single variable. It turns out that when $m = 0, 1, \ldots, k-2$ (i.e., when m is critical with respect to k), the special value $\Lambda(f, m+1)$ can be realized as $\Lambda(f, m+1) = \mathrm{ev}_m(\omega_f)$ where

$$\mathrm{ev}_m : H^1_c(Y_1(N), \mathscr{L}_{k-2}) \to \mathbf{C}$$

is a certain canonical linear functional. The functional ev_m is actually defined over $\mathbf{Q}$, and after renormalizing the Eichler–Shimura construction by a period, everything is defined over a number field. Putting these observations together, one obtains Shimura's result. (One also considers variants of these constructions taking finite-order twists into account, cf. below.) To summarize, this argument for Shimura's result makes use of two essentially distinct ingredients:

(1) Canonical cohomology classes ω_f associated with each f.
(2) Natural functionals ev_m on cohomology which record L-values.

1.2.2. *p-adic considerations.* In the authors' view, the construction of p-adic L-functions should closely mirror the steps (1) and (2) above. The emphasis on a dichotomy like this is largely due to Stevens in the case $F = \mathbf{Q}$. Let us explain the two steps in reverse.

The local systems $\mathscr{L}_{k-2}$ are algebraic, so they can be taken to have p-adic coefficients, and they exist on modular curves of any level. On modular curves of level Np (with $p \nmid N$) there is a second local system $\mathscr{D}_{k-2}$ of locally analytic distributions on $\mathbf{Z}_p$ equipped with a 'weight $k-2$' action of a certain monoid containing $\Gamma := \Gamma_1(N) \cap \Gamma_0(p)$. If $\Phi \in H^1_c(Y(\Gamma), \mathscr{D}_{k-2})$ is any cohomology class, then it makes sense to evaluate Φ on the cycle '$\{\infty\} - \{0\}$' on $Y(\Gamma)$. The output of this evaluation is thus a distribution on $\mathbf{Z}_p$ which can be restricted to $\mathbf{Z}_p^\times$. So, each Φ defines natural elements in the space $\mathscr{D}(\mathbf{Z}_p^\times)$ of locally analytic distributions on $\mathbf{Z}_p^\times$. Now note that $\Gamma_{\mathbf{Q}} \simeq \mathbf{Z}_p^\times$, and so a theorem of Amice from the 1970's ([2]) implies that $\mathscr{D}(\mathbf{Z}_p^\times)$ is canonically isomorphic to $\mathscr{O}(\mathscr{X}(\Gamma_{\mathbf{Q}}))$, which is exactly where our p-adic L-functions are meant to live. This suggests the following (2') as an analog of (2) above:

(2') Consider the linear map

$$\mathscr{P}_{k-2} : H^1_c(Y(\Gamma), \mathscr{D}_{k-2}) \to \mathscr{O}(\mathscr{X}(\Gamma_{\mathbf{Q}}))$$

that associates to each $\Phi \in H^1_c(Y(\Gamma), \mathscr{D}_{k-2})$ the element $\Phi(\{\infty\} - \{0\})|_{\mathbf{Z}_p^\times}$.

To further illuminate the connection with the maps ev_m, note that there is a canonical map $I_{k-2} : \mathscr{D}_{k-2} \to \mathscr{L}_{k-2}$ of local systems over $Y(\Gamma)$ given by recording the first $k-2$ moments of a distribution. It is then not difficult to establish a direct relationship between the map $\mathscr{P}_{k-2}$, the map induced by I_{k-2} on cohomology, the evaluation maps ev_m defined above, and the Hecke operators at p. (More glibly: the cycle '$\{\infty\} - \{0\}$' is 'clearly' related to L-values by the integral representation of L-series as a Mellin transform on the upper-half plane.)

One important point to stress is that the local system $\mathscr{D}_{k-2}$ can only be defined over modular curves with $\Gamma_0(p)$-structure. Thus to an eigenform f of level N with $p \nmid N$, we are naturally led to consider the p-refined eigenform f_α of level Γ, corresponding to some choice of refinement α. An ambitious choice for the p-adic analog to the archimedean step (1) would then be:

(1') 'Canonically' associate with each p-refined eigenform f_α a class $\Phi_{f_\alpha} \in H^1_c(Y(\Gamma), \mathscr{D}_{k-2})$.

If (1') can be carried out, then one may combine (1') and (2') to produce a p-adic L-function as in Theorem 1.1.2.

To what extent is (1') possible? To be sure, the class $\omega_{f_\alpha} \in H^1_c(Y(\Gamma), \mathscr{L}_{k-2})$ is always in the image of the map I_{k-2}, but the kernel of I_{k-2} is infinite-dimensional. One might then try to produce a Hecke eigenclass Φ_{f_α} which maps to ω_{f_α} under I_{k-2}, and one might hope that it is unique; this would certainly pin down a 'canonical' Φ_{f_α}. However, this is only possible some of the time. Specifically, ω_{f_α} can be uniquely lifted to a Hecke eigenclass exactly when the refinement α is non-critical in our sense. In the case $F = \mathbf{Q}$ this combines the two cases commonly referred to as being 'non-critical slope' or 'critical slope but not θ-critical'. These cases were handled by Pollack and Stevens ([62, 63]).

When α is critical, but still decent, Bellaïche ([11]) observed that it is *never* possible to lift ω_{f_α} to a Hecke eigenclass via I_{k-2}. He did this by showing, in an indirect way, that there is still a unique (up to scalar) Hecke eigenclass $\Phi_{f_\alpha} \in H^1_c(Y(\Gamma), \mathscr{D}_{k-2})$ with the same Hecke eigensystem as f_α; it just happens to lie in the kernel of I_{k-2}. This is precisely why one sees 'funny' behavior in the interpolation properties of p-adic L-functions for critical α (Theorem 1.1.2). We will explain Bellaïche's method in more detail below; the argument uses p-adic families in a crucial way.

In any case, we can safely say that when $F = \mathbf{Q}$ the ingredient (1') is available (under the decency hypothesis). The aim of the present paper is to generalize both steps (1') and (2') to any totally real base field F, while maintaining a view towards unrestrictive hypotheses.

1.3. **Basic objects.** Having stated our result and outlined the known methods when $F = \mathbf{Q}$, we now unload the requisite terminology and notations for the general case.

Write $\mathbf{A}_F$ for the adeles of F, $\mathbf{A}_{F,f}$ for the finite adeles. The p-th component of $\mathbf{A}_F$ is $F_p = F \otimes_{\mathbf{Q}} \mathbf{Q}_p \simeq \prod_{v|p} F_v$, and we also write $\mathcal{O}_F \otimes_{\mathbf{Z}} \mathbf{Z}_p = \mathcal{O}_p \subset F_p$ for the corresponding product of rings of integers. The tuple of uniformizers ϖ_v at $v \mid p$ thus defines an element $\varpi_p \in \mathcal{O}_p$. Suppose that $\mathfrak{n} \subset \mathcal{O}_F$ is an integral ideal that is prime to p. We will assume from now on that π has conductor exactly $\mathfrak{n}$. We will write $K = \prod_v K_v$ for the compact open subgroup of $\mathrm{GL}_2(\widehat{\mathcal{O}_F})$ consisting of matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ whose entries satisfy $c \equiv 0 \bmod \varpi_p \mathfrak{n}\widehat{\mathcal{O}_F}$ and $d \equiv 1 \bmod \mathfrak{n}\widehat{\mathcal{O}_F}$. We write Y_K for the open Hilbert modular variety of level K (it is the analog of the modular curve $Y(\Gamma)$ above).

For a fixed cohomological weight $\lambda = (\kappa, w)$, we will consider a finite-dimensional local system $\mathscr{L}_\lambda$ on Y_K of L-vector spaces, where $L \subset \overline{\mathbf{Q}}_p$ is the field generated over $\mathbf{Q}_p$ by all embeddings $\iota(\sigma(F))$. More precisely, $\mathscr{L}_\lambda$ is defined as the finite-dimensional vector space $\mathscr{L}_\lambda \subset L[\{X_\sigma\}_{\sigma \in \Sigma_F}]$ spanned by polynomials whose X_σ-degree is at most κ_σ, and the group $\mathrm{GL}_2(F_p)$ acts by a natural weight λ left action (see Section 2.4 for the precise definition of the action). The cohomology $H^*_c(Y_K, \mathscr{L}_\lambda)$ is naturally acted upon by the Hecke algebra $\mathbf{T}$ generated by the 'standard' Hecke operators T_v ($v \nmid \mathfrak{n}p$), U_v ($v \mid p$), and S_v ($v \nmid \mathfrak{n}$), cf. Definition 3.2.1. If (π, α) is a p-refined automorphic representation, then it has (via ι) an associated $\overline{\mathbf{Q}}_p$-valued $\mathbf{T}$-eigensystem (in particular, the eigenvalue of U_v is $\iota(\alpha_v)$). This defines a maximal ideal $\mathfrak{m}_{\pi,\alpha} \subset \mathbf{T}$ and the Eichler–Shimura construction implies that $H^*_c(Y_K, \mathscr{L}_\lambda)_{\mathfrak{m}_{\pi,\alpha}}$ is non-zero and concentrated in middle degree. More precisely, the cohomology $H^*_c(Y_K, \mathscr{L}_\lambda)$ decomposes into 2^d-many direct summands $H^*_c(Y_K, \mathscr{L}_\lambda)^\epsilon$ indexed by signs $\epsilon \in \{\pm 1\}^{\Sigma_F}$, which correspond to choosing eigenvalues for each of the d 'archimedean Hecke operators' induced by the partial complex conjugations on Y_K (cf. Section 4.1 for a precise discussion). For each ϵ the eigenspace

$$\left(H^*_c(Y_K, \mathscr{L}_\lambda) \otimes_L \overline{\mathbf{Q}}_p\right)^\epsilon [\mathfrak{m}_{\pi,\alpha}]$$

is one-dimensional and concentrated in middle degree.

To introduce p-adic automorphic forms we first consider p-adic weights. For us, this is a pair $\lambda = (\lambda_1, \lambda_2)$ of continuous characters $\lambda_i : \mathcal{O}_p^\times \to \mathbf{C}_p^\times$. If $\lambda = (\kappa, w)$ is cohomological then it defines a p-adic weight (λ_1, λ_2) by the recipe

$$\lambda_1(x) = \prod_{\sigma \in \Sigma_F} (\iota \circ \sigma)(x)^{\frac{w+\kappa_\sigma}{2}}, \qquad \lambda_2(x) = \prod_{\sigma \in \Sigma_F} (\iota \circ \sigma)(x)^{\frac{w-\kappa_\sigma}{2}}.$$

Note that if λ is a cohomological weight, then the values of the characters λ_i generate a field k_λ which is a subfield of L.

For each p-adic weight we then define a k_λ-Frechet space $\mathscr{D}_\lambda$ whose underlying module is the locally analytic distributions $\mathscr{D}(\mathcal{O}_p)$ on $\mathcal{O}_p$ (we also implicitly extend scalars to $\mathbf{C}_p$ for simplicity in the introduction). The subscripted λ indicates that we equip it with a specific left action of the monoid

$$\Delta = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(\mathcal{O}_p) \cap \mathrm{GL}_2(F_p) \mid c \in \varpi_p \mathcal{O}_p \text{ and } d \in \mathcal{O}_p^\times \}.$$

We omit the definition of the action here (see Section 5.3). Now, since $\Delta \supset K_p$, we can also consider the cohomology $H_c^*(Y_K, \mathscr{D}_\lambda)$ for each p-adic weight λ, and the Hecke algebra $\mathbf{T}$ still acts on this cohomology by endomorphisms. Moreover, in the special case that λ is a cohomological weight, there is a natural map

$$I_\lambda : H_c^*(Y_K, \mathscr{D}_\lambda \otimes_{k_\lambda} L) \to H_c^*(Y_K, \mathscr{L}_\lambda)$$

induced by a Δ-equivariant map on the underlying local systems. In particular, I_λ commutes with the $\mathbf{T}$-action, and it commutes with the archimedean Hecke operators.[4]

All of these objects are designed as analogs of the objects we considered when discussing the case $F = \mathbf{Q}$ earlier. Let us now turn towards our ingredients for p-adic L-functions.

1.4. **The period maps.** The portion of this article that requires no hypotheses is the construction of a certain $\mathscr{O}(\mathscr{X}(\Gamma_F))$-valued functional $\mathscr{P}_\lambda$ on the middle-degree distribution-valued cohomology $H_c^d(Y_K, \mathscr{D}_\lambda)$. We call $\mathscr{P}_\lambda$ a period map because of its interaction with the Hecke integrals which compute the completed L-series of automorphic representations in the case where λ is a cohomological weight. We remark ahead of time that is absolutely crucial to the generality of Theorem 1.1.2 that the definition of $\mathscr{P}_\lambda$ works for more general p-adic weights, as well as for affinoid families of weights.

To state a precise result here, we need a little more notation. Let λ be a cohomological weight. Then we can consider the local system $\mathscr{L}_\lambda^\vee$ on Y_K dual to $\mathscr{L}_\lambda$, and then we can take its middle degree Borel–Moore homology $H_d^{\mathrm{BM}}(Y_K, \mathscr{L}_\lambda^\vee)$ (homology defined by locally finite chains). There is a natural pairing

$$\langle -, - \rangle : H_c^d(Y_K, \mathscr{L}_\lambda) \otimes_L H_d^{\mathrm{BM}}(Y_K, \mathscr{L}_\lambda^\vee) \to L \subset \overline{\mathbf{Q}}_p.$$

In Section 7.5 we will define, for each integer m critical with respect to λ, a certain evaluation class $\mathrm{cl}_p(m) \in H_d^{\mathrm{BM}}(Y_K, \mathscr{L}_\lambda^\vee)$. Its purpose is that if $\psi_{\pi,\alpha} \in H_c^d(Y_K, \mathscr{L}_\lambda)$ is the Hecke eigenclass associated with a p-refined cohomological cuspidal automorphic representation (π, α) of weight λ (via Eichler–Shimura), then $\langle \psi_{\pi,\alpha}, \mathrm{cl}_p(m) \rangle$ is a natural scaling (depending on α) of the special value $\Lambda(\pi, m+1)$. In fact, $\psi \mapsto \langle \psi, \mathrm{cl}_p(m) \rangle$ is a p-adic analog of the evaluation maps ev_m.

Theorem 1.4.1. *For each p-adic weight λ, there exists a canonical linear morphism*

$$\mathscr{P}_\lambda : H_c^d(Y_K, \mathscr{D}_\lambda) \to \mathscr{O}(\mathscr{X}(\Gamma_F)) \otimes k_\lambda$$

that, among other things, satisfies the following formal interpolation property:

[4]Strictly speaking, the map I_λ only commutes with the U_v-operators for $v|p$ up to a scaling; we elide this point in the introduction.

If λ is a cohomological weight, m is an integer which is critical with respect to λ, and $\Psi \in H^d_c(Y_K, \mathscr{D}_\lambda)$ is a U_v-eigenvector with eigenvalue $\alpha_v^\sharp$, then

$$\mathscr{P}_\lambda(\Psi)(\chi_{\mathrm{cycl}}^m) = \prod_{v|p}(1 - (\alpha_v^\sharp \varpi_v^{\frac{w-\kappa}{2}})^{-1} q_v^m) \cdot \langle I_\lambda(\Psi), \mathrm{cl}_p(m)\rangle.$$

One should compare the formal interpolation in Theorem 1.4.1 with the interpolation property in Theorem 1.1.2. (The scalar factor $\varpi_v^{\frac{w-\kappa}{2}}$, whose meaning can be found in Section 1.10, appears because of the implicit scaling mentioned in Footnote 4.) The formal interpolation of course generalizes to also allow twists by finite order Hecke characters of p-power conductor; see Theorem 7.6.4 and Corollary 7.6.7 for these more complicated statements. In addition, the period maps enjoy certain growth properties (Section 7.4) and natural interaction with the signs ϵ (Section 7.3). Finally, they also vary naturally in the p-adic weight variable λ (in fact, we define period maps functorially for any affinoid weight). The map described in Theorem 1.4.1 is thus a natural analog of 'evaluating at $\{\infty\} - \{0\}$' in the setting of $F = \mathbf{Q}$. (It is also a short exercise to check that our definition truly generalizes that construction.)

In fact, the definition of $\mathscr{P}_\lambda$ is quite brief once the groundwork is laid. It involves first constructing a natural k_λ-linear map $\mathscr{P}_\lambda : H^d_c(Y_K, \mathscr{D}_\lambda) \to \mathrm{Hom}_{k_\lambda}(\mathscr{A}(\Gamma_F) \otimes k_\lambda, k_\lambda)$ where $\mathscr{A}(\Gamma_F)$ is the ring of locally analytic functions on Γ_F. We then manage to check that the image of $\mathscr{P}_\lambda$ actually lands in the subspace of locally analytic distributions $\mathscr{D}(\Gamma_F)$, which is the continuous (as opposed to abstract) k_λ-linear dual of $\mathscr{A}(\Gamma_F) \otimes k_\lambda$. Once this is proven (Theorem 7.2.3), it is easy to obtain the map described in Theorem 1.4.1 using the theorem of Amice we previously mentioned. The proof of the continuity condition in the definition of $\mathscr{P}_\lambda$ amounts to constructing it canonically enough that it naturally preserves various integral structures on both sides. We refer to Section 7.2 for further details.

1.5. **Control of Hecke eigenclasses.** With Theorem 1.4.1 in hand, we also need a means of canonically associating distribution-valued Hecke eigenclasses with p-refined automorphic representations (π, α). Recall that there is a natural integration map $I_\lambda : H^*_c(Y_K, \mathscr{D}_\lambda \otimes_{k_\lambda} L) \to H^*_c(Y_K, \mathscr{L}_\lambda)$, and that to a pair (π, α) we have a maximal ideal $\mathfrak{m}_{\pi,\alpha} \subset \mathbf{T}$.

Definition 1.5.1 (Non-critical)**.** A p-refined automorphic representation (π, α) is called non-critical if $I_\lambda : H^*_c(Y_K, \mathscr{D}_\lambda \otimes_{k_\lambda} L)_{\mathfrak{m}_{\pi,\alpha}} \to H^*_c(Y_K, \mathscr{L}_\lambda)_{\mathfrak{m}_{\pi,\alpha}}$ is an isomorphism.

A well-known argument shows that non-critical slope implies non-critical, but the two conditions are not equivalent (see Section 6.3). In the case $F = \mathbf{Q}$, non-critical is equivalent to what is sometimes known as being 'not θ-critical' as in [63]. Reasoning with classical facts about automorphic representations, it is easy to prove that if (π, α) is non-critical, then the Hecke eigenspace $\left(H^d_c(Y_K, \mathscr{D}_\lambda) \otimes_{k_\lambda} \overline{\mathbf{Q}}_p\right)^\epsilon [\mathfrak{m}_{\pi,\alpha}]$ is one-dimensional (for any ϵ) and so Theorem 1.4.1 can be used to associate p-adic L-functions $L_p^\epsilon(\pi, \alpha)$ with non-critically refined forms (π, α). More precisely the Eichler–Shimura construction gives us, after scaling by a period, a canonical class in $H^d_c(Y_K, \mathscr{L}_\lambda)^\epsilon[\mathfrak{m}_{\pi,\alpha}]$. We lift this class via the isomorphism I_λ (in the non-critical case) and thus define the p-adic L-function $L_p^\epsilon(\pi, \alpha)$ as the output of $\mathscr{P}_\lambda$ applied to this lift.

In general, and already when $F = \mathbf{Q}$, there definitely exist critically refined (π, α). To handle these cases, our methods demand some input from the theory of Galois representations. Given any π, write ρ_π for the natural two-dimensional irreducible representation of the absolute Galois group $G_F = \mathrm{Gal}(\overline{F}/F)$ associated with π. Recall also that if $\alpha = (\alpha_v)_{v|p}$ is a refinement then there is an evident tuple of 'other roots' $\beta = (\beta_v)_{v|p}$ (Definition 1.1.1).

Definition 1.5.2. A p-refined automorphic representation (π, α) is called decent if at least one of the following two conditions is true.

(1) (π, α) is non-critical.
(2) The following three conditions hold.
   (a) $H^j_c(Y_K, \mathscr{D}_\lambda)_{\mathfrak{m}_{\pi,\alpha}}$ is non-zero if and only if $j = d$ (the middle degree).
   (b) The adjoint Bloch–Kato Selmer group $H^1_f(G_F, \mathrm{ad}\,\rho_\pi)$ is trivial.
   (c) $\alpha_v \neq \beta_v$ for each $v \mid p$.

Before discussing the three conditions in part (2) of this definition, we state our main result on the Hecke eigenspaces in distribution-valued cohomology associated with a decently p-refined (π, α).

Theorem 1.5.3. *If (π, α) is a decently p-refined automorphic representation of weight λ, then*

$$\dim_{\overline{\mathbf{Q}}_p} H^d_c(Y_K, \mathscr{D}_\lambda \otimes_{k_\lambda} \overline{\mathbf{Q}}_p)^\epsilon[\mathfrak{m}_{\pi,\alpha}] = 1$$

for each $\epsilon \in \{\pm 1\}^{\Sigma_F}$.

We already mentioned why Theorem 1.5.3 is true when (π, α) is non-critical, but the fact that it extends to all decently refined (π, α) is rather more difficult. In any case, if we apply the period map of Theorem 1.4.1 to the unique-up-to-scalar Hecke eigenclass provided by Theorem 1.5.3, we get the p-adic L-functions $L^\epsilon_p(\pi, \alpha)$ claimed in Theorem 1.1.2. Note that we make no further claim on how to canonically choose a non-zero vector in the above one-dimensional vector space, so we are ambiguous up to scalars in a p-adic field rather than a number field.

The proof of Theorem 1.5.3 relies in a crucial way on p-adic families of p-refined automorphic representations and their finer geometric properties. Before discussing this further, let us explain what is known about the decency hypothesis.

1.6. **The decency hypothesis.** It is worth detailing what is known about part (2) of the 'decent' hypothesis.[5] In order to orient the discussion from least technical to most technical, let us discuss the conditions in reverse from (c) to (a).

The simplest condition is the condition that $\alpha_v \neq \beta_v$ for each $v \mid p$. Unfortunately, this is also the only condition we do not conjecture always holds. For instance, if $E_{/\mathbf{Q}}$ is an elliptic curve with good supersingular reduction at p, F is a real quadratic field in which p is inert, and π is the parallel weight two automorphic representation associated with the base change $E_{/F}$, then the Hecke polynomial of π at the unique p-adic place is $(X - p)^2$. We do not know if all such examples are non-critical, but we have no strong feeling either way. We do note, however, that when p is totally split in F then it would follow from the Tate conjecture that $\alpha_v \neq \beta_v$ for each $v \mid p$ (cf. [33]). In any case, for a fixed π the condition that $\alpha_v \neq \beta_v$ is surely easy to check depending on how you are handed π, of course.

The next condition we consider is the vanishing of the Selmer group in part (b). This is a well-established consequence of a conjecture of Bloch and Kato ([18]) extending the Birch–Swinnerton-Dyer conjecture. In fact, the condition 2(b) is known to be true in many cases by work of Kisin, when $F = \mathbf{Q}$, and Allen, in general, ([54, 1]). Note as well that hypothesis (b) does not involve the refinement α in any way.

Finally we come to the thorniest of the three hypotheses: the assumption that the distribution-valued eigensystem associated to (π, α) occurs only in the middle degree. This is a classically known fact for the finite-dimensional classical cohomology $H^*_c(Y_K, \mathscr{L}_\lambda)$. So, in particular the non-critical hypothesis overlaps with the middle-degree support hypothesis. Further, when $F = \mathbf{Q}$ the condition 2(a) is also true by a direct analysis: the relevant H^2_c's only contain Eisenstein Hecke eigensystems. One can also check that 2(a) is true when F is a real quadratic field, by using the congruence subgroup property for $\mathrm{SL}_2(\mathcal{O}_F)$ together with Poincaré duality and some other tricks. Based on these evidences,

[5]The terminology is borrowed directly from Bellaïche ([11]).

we conjecture that condition 2(a) *always* holds (remember that π is cuspidal). We admit knowing no affirmative results beyond the cases already discussed.

However, there is hope that condition 2(a) will be verified in the near future, at least under some mild condition on the mod p representation $\overline{\rho}_\pi$. Namely, Caraiani and Scholze ([27]) have proven a 'support in middle degree only' result for the mod p cohomology of certain compact Shimura varieties localized at suitable maximal ideals in the Hecke algebra. This can be bootstrapped to produce vanishing outside middle degree statements for completed cohomology as well. In ongoing work, Caraiani and Scholze are extending their results to the case of non-compact unitary Shimura varieties, and we are told their results and methods in this setting should be directly adaptable to the open Hilbert modular varieties used in this article. Granted such an adaptation, the missing ingredient for verifying 2(a) (again, under some condition on $\overline{\rho}_\pi$) is a comparison between distribution-valued cohomology and completed cohomology. For such a comparison, we refer to forthcoming work of Johansson and the second author.

1.7. **The eigenvariety (proving Theorem 1.5.3).** The method we use to prove Theorem 1.5.3 in the decent, but possibly critical cases, is closely modeled on the method used by Bellaïche in [11]. However, there are a number of new complications that arise in our more general setting. We would like to discuss this in some detail since we expect it will also help explain the role of the hypothesis 2(a) for the reader whose experience with p-adic families is limited to the eigencurve and to other simple situations like groups that are compact-mod-center at infinity.

The first point is the Hecke eigenvarieties parameterizing eigensystems corresponding to (finite slope) automorphic representations for $\mathrm{GL}_{2/F}$ come in different flavors. For instance, there is the parallel weight eigencurve of Kisin and Lai ([55]) and one modeled on overconvergent p-adic Hilbert modular forms by Andreatta, Iovita and Pilloni ([4]). But history (and Theorem 1.4.1) teaches us that the models for eigenvarieties that are closest to seeing p-adic L-functions are those built using distribution-valued cohomology. Beyond the case of $F = \mathbf{Q}$, these appear in the work of Urban ([78]) and the more general construction of the second author ([43]). (They are exposed for $F = \mathbf{Q}$ in [11] following ideas of Stevens).

More precisely, in [43] the second author constructed a rigid analytic space $\mathscr{E}(\mathfrak{n})$ parametrizing the finite slope $\mathbf{T}$-eigensystems appearing in the total cohomology $H^*_c(Y_K, \mathscr{D}_\lambda)$ as λ runs over the space of p-adic weights $\mathscr{W}(1) \subset \mathscr{W}$ which are trivial on the image of the global units (these are the only weights where the cohomology is non-trivial; see Section 6.1). For notation, if ψ is a finite slope $\mathbf{T}$-eigensystem appearing in the total cohomology, then write $x_\psi \in \mathscr{E}(\mathfrak{n})$ for the corresponding point. For instance, if (π, α) is a p-refined automorphic representation as above then its eigensystem appears in the cohomology, in some degree, and thus we get classical points $x_{\pi,\alpha}$ on $\mathscr{E}(\mathfrak{n})$.

The first difficulties are that $\mathscr{E}(\mathfrak{n})$ is certainly not equidimensional if $F \neq \mathbf{Q}$, and it is possibly not reduced. Both the equidimensionality and reducedness of the Coleman-Mazur eigencurve are crucial in the proof of Theorem 1.5.3 given by Bellaïche in [11] for $F = \mathbf{Q}$. One of the theorems we prove is the following.

Theorem 1.7.1 (Section 6.4)**.** *There exists a Zariski-open subspace $\mathscr{E}(\mathfrak{n})_{\mathrm{mid}}$ inside $\mathscr{E}(\mathfrak{n})$ uniquely characterized by the following property: a point x_ψ, of weight λ, is in $\mathscr{E}(\mathfrak{n})_{\mathrm{mid}}$ if and only if the eigensystem ψ appears only in the middle degree $H^d_c(Y_K, \mathscr{D}_\lambda)$.*

Moreover, $\mathscr{E}(\mathfrak{n})_{\mathrm{mid}}$ is reduced, equidimensional of the same dimension as its weight space $\mathscr{W}(1)$, and the classical points (up to twist) are Zariski-dense and accumulating.

The space $\mathscr{E}(\mathfrak{n})_{\mathrm{mid}}$ is defined as the complement of a finite union of closed subspaces in $\mathscr{E}(\mathfrak{n})$, each of which has dimension strictly smaller than the dimension of weight space. The characterization of $\mathscr{E}(\mathfrak{n})_{\mathrm{mid}}$ in Theorem 1.7.1 follows from two spectral sequences developed by the second author in [43].

The density of classical points and the reduced-ness follow standard lines of argument. Finally, the equidimensionality uses a theorem of Newton proved in an appendix to [43].

Now the role of the hypothesis 2(a) comes into view: assuming that (π, α) is decent tells us that the corresponding classical point $x_{\pi,\alpha}$ on $\mathscr{E}(\mathfrak{n})$ in facts lies on the much better behaved sub-eigenvariety $\mathscr{E}(\mathfrak{n})_{\mathrm{mid}}$. We then prove the following statement:

Theorem 1.7.2. *If (π, α) satisfies condition (2) in Definition 1.5.2, then $x_{\pi,\alpha}$ is a smooth point on $\mathscr{E}(\mathfrak{n})_{\mathrm{mid}}$.*

The proof is an argument using deformations of Galois representations; this is where conditions 2(b) and 2(c) come in. The local deformation-theoretic calculations that are needed were carried out by the first author in [14] (see also [23]). We should emphasize that the properties in Theorem 1.7.1, thus condition 2(a), are absolutely crucial to getting the strategy off the ground: they are used not just to guarantee the variation of Galois representations over $\mathscr{E}(\mathfrak{n})_{\mathrm{mid}}$ but also that the key generalizations of Kisin's theorem on crystalline periods ([53, 58]) hold as well.

Theorem 1.7.2 (Theorem 6.6.3 in the text) is also true when (π, α) is non-critical, if it is further assumed that condition 2(c) in Definition 1.5.2 holds. The argument (due to Chenevier) is somewhat different and proves the stronger statement that the weight map is étale. While we expect that étaleness of the weight map definitely fails whenever 2(c) fails, it is open whether or not Theorem 1.7.2 as stated holds without 2(c).

Finally we deduce the one-dimensionality result in Theorem 1.5.3 as a consequence of Theorem 1.7.2 (again, it was already known in the non-critical case). The strategy is to prove that the image $T_{\pi,\alpha}$ of the Hecke algebra $\mathbf{T}$ in the endomorphism ring of $M_{\pi,\alpha} = H^d_c(Y_K, \mathscr{D}_\lambda)_{\mathfrak{m}_{\pi,\alpha}}$ is Gorenstein (of dimension zero), and that each sign eigenspace $M^{\epsilon}_{\pi,\alpha}$ is free of rank one over $T_{\pi,\alpha}$. To carry this out, we first prove analogous structural results for the eigenvariety $\mathscr{E}(\mathfrak{n})_{\mathrm{mid}}$ in a small neighborhood of $x_{\pi,\alpha}$. We then leverage these results to knowledge of the weight fiber over λ; in particular, we manage to prove that the natural map $\mathscr{O}_{\mathscr{E}(\mathfrak{n}),x_{\pi,\alpha}} \otimes_{\mathscr{O}_{\mathscr{W}(1),\lambda}} k_\lambda \to T_{\pi,\alpha}$ is an isomorphism. Note that, quite generally, any point on an eigenvariety gives rise to such a map (suitably defined), but typically one can only prove that these maps are surjective with a nilpotent kernel. The arguments here make use of some classical theorems in commutative algebra (Auslander–Buchsbaum formula and some properties of depth). We refer to the text (Section 8.1) for more details.

1.8. **Comparison to other results.** As we have already indicated, when $F = \mathbf{Q}$ the results we prove can be found in Bellaïche's article. The first paragraph of that article provides more than ample references to the relevant history.

We note, however, that there is something a bit special about $F = \mathbf{Q}$. Precisely, the truth of Leopoldt's conjecture implies that the group Γ_F is a 1-dimensional p-adic Lie group, so a theorem of Amice and Vélu ([3]) implies in turn that the p-adic L-functions described in Theorem 1.1.2 are uniquely determined by their growth and interpolation properties when the growth is sufficiently small. This has the notable advantage that constructions by different methods (for instance, modular symbols vs. Rankin–Selberg methods) necessarily give the same p-adic L-functions in non-critical slope cases, and so only p-adic L-functions beyond non-critical slope have any ambiguity. In the critical slope case, there are constructions by Pollack–Stevens ([63]) and Bellaïche ([11]). These obviously agree on their overlap. There is also a construction, which applies in the critical slope case, using Kato's Euler systems the dual exponential map of Perrin-Riou (cf. the introduction to [57]). This construction agrees with the previous references in the non-theta-critical case by a theorem of the second author [44] (see [79] as well).

Now let us move to a general totally real field F. We would first like to mention the articles of Ash–Ginzburg ([5]), Januszewski ([51, 52, 50]), Manin ([59]), and Haran ([45]), which all give constructions of p-adic L-functions associated with Hilbert modular forms in varying degrees of generality. However, the main goals of these articles are somewhat orthogonal to ours. On the one hand they are more general in some ways. For instance, they actually do not assume the base field is totally real and [5] and [51, 50] construct p-adic L-functions for GL_{2n} and $\mathrm{GL}_{n+1} \times \mathrm{GL}_n$, respectively. On the other hand, of these only the very recent [50] considers variation in families (ordinary, in this case), and none of them go beyond small slope cases. And without input from Leopoldt's conjecture, we can not say for certain that their methods produce the same objects as ours in the overlapping cases.

More closely related to the present article are the recent works of Dimitrov ([36]), Barrera ([8]), Barrera and Williams ([10]), and a very recent article of Dimitrov, Barrera, and Jorza ([9]). Dimitrov's article, in particular, gives a clean and definitive construction of p-adic L-functions associated with ordinary p-refined Hilbert modular forms and with Hida families thereof. In [8], Barrera combined the formalism of overconvergent cohomology with the modular cycles introduced in [36], obtaining a construction of p-adic L-functions in the non-critical case with the correct growth and interpolation properties. This method was generalized in [10] to allow for any number field. (The statements in [8, 10] assume non-critical slope, but it is clear from reading these works that non-criticality is a sufficient hypothesis.) In the course of all these works, and in [9] in particular, one finds a map from eigenclasses in overconvergent cohomology to distributions on a Galois group which bears a resemblance to the period map we have defined and which presumably can be verified to be the *same* map. In particular, even without Leopoldt one might hope that our constructions and those of [8, 9, 10] coincide in the overlapping cases.

The difference between our period map and that of the above works is best illustrated by examining the proofs of the interpolation property. For instance, in [10], the authors check the interpolation property by making use of modular cycles and "hands-on" calculations with group cohomology. These modular cycles do not appear explicitly in our calculations (although they are implicit in some way in what we do). Rather than introduce auxiliary cycles, we instead calculate directly using the adelic chains and cochains introduced by Ash and Stevens (see Section 2). At first glance, this may seem more complicated. However, we believe our approach is quite natural, for at least two reasons.

First, modular cycles were originally introduced in the context of Hida theory, and in particular in a framework where p-adic families can be constructed by considering cohomology with constant coefficients of a $Y_1(\mathfrak{n}p^\infty)$-tower. In this context, it is natural (and in some sense, necessary) to introduce fairly complicated cycles when defining p-adic L-functions and checking their interpolation property. In Stevens's setup, by contrast, there is no tower, but the cohomology has extremely complicated coefficients. Our perspective then is that the difficulty should be shifted from defining the correct modular cycles to defining the correct period map. Second, the details of our construction are consistent with the adelic philosophy which we have adopted. For instance, our definition eliminates the need to choose representatives for various objects, thereby avoiding the ambiguities such choices can engender. This is in contrast to several points in the arguments of the referenced works where one has to check somewhat non-trivial independence of choices. Our approach avoids this kind of issue. (In fact, it is ultimately left as an open question in [10] (see Theorem 9.11 and the final paragraphs of *loc. cit.*) whether the construction of p-adic L-functions given there truly depends on the choice of uniformizers at the p-adic places. The totally real cases seem to have been dealt with in [9].)

1.9. **Organization.** The body of this article is divided into seven main sections. The first three (Sections 2, 3, and 4) are comprised of a verbose discussion of adelic (co)chains on locally symmetric spaces, Hilbert modular forms, and Shimura's algebraicity theorem. Here we have adopted a maximalist approach to the exposition, so that our notations are as precise as possible and to ensure this work is reasonably self-contained.

Starting in Section 5 we turn towards p-adic matters. First we discuss generalities on certain p-adic Lie groups and define various modules of locally analytic functions and distributions.

Section 6 is devoted to an exposition of the middle-degree Hilbert modular eigenvariety mentioned above. We include here (and in the previous section) a lengthy discussion, most of which is moot if we were to assume Leopoldt's conjecture, of twisting classical points by p-adic Hecke characters.

In Section 7 we define and analyze the period maps. The heart of this section is the proof of the abstract interpolation theorem, which is the key ingredient in proving the correct interpolation formula for our p-adic L-functions.

The final section, Section 8, contains the definition of p-adic L-functions and the proofs of Theorem 1.5.3 and Theorem 1.1.2.

1.10. **Notations.** For convenience, we list here notations that will remain in force throughout the paper.

We will always write GL_2 for the general linear group over $\mathbf{Z}$ (and $\mathrm{GL}_{2/R}$ for its base change to a ring R if needed). We write $Z \subset T \subset \mathrm{GL}_2$ for the center, resp. the diagonal torus. If H is a real Lie group we generally write H° for the connected component of H containing the identity.

F is a totally real number field of degree d. Its ring of integers is written $\mathcal{O}_F$. We write $\Sigma_F = \mathrm{Hom}(F, \mathbf{C})$. The adeles of F are written $\mathbf{A}_F$. We write $F_\infty = F \otimes_{\mathbf{Q}} \mathbf{R}$ for the infinite component of $\mathbf{A}_F$. We write $\mathbf{A}_{F,f}$ for the finite component of $\mathbf{A}_F$.

The map $F \to \mathbf{R}^{\Sigma_F}$ given by $\xi \mapsto (\sigma(\xi))$ for $\xi \in F$ extends $\mathbf{R}$-linearly to an isomorphism $F_\infty \simeq \mathbf{R}^{\Sigma_F}$ of $\mathbf{R}$-algebras. If $x \in F_\infty$ we write $x = (x_\sigma)$ for its coordinates in $\mathbf{R}^{\Sigma_F}$. We say $x \in F_\infty$ is totally positive if $x_\sigma > 0$ for all $\sigma \in \Sigma_F$; the set of totally positive elements is written $F_{\infty,+}$. Or, the invertible totally positive elements of F_∞ is equal to $(F_\infty^\times)^\circ$ (our preferred notation in many places).

We fix a prime number p. We write $\overline{\mathbf{Q}}_p$ for an algebraic closure of the p-adic numbers. We also fix an isomorphism $\iota : \mathbf{C} \xrightarrow{\sim} \overline{\mathbf{Q}}_p$. Using ι we have a decomposition

$$\Sigma_F = \bigsqcup_{v \mid p} \Sigma_v \tag{1.10.1}$$

where an element $\sigma \in \Sigma_F$ lies in Σ_v if and only if the composition $\iota \circ \sigma$ induces the p-adic place v on F. Write $F_p = F \otimes_{\mathbf{Q}} \mathbf{Q}_p \simeq \prod_{v \mid p} F_v$ where F_v is the completion of F with respect to $v \mid p$. If $\sigma \in \Sigma_v$ then σ extends to a $\mathbf{Q}_p$-linear embedding $\sigma : F_v \hookrightarrow \overline{\mathbf{Q}}_p$ for which we use the same symbol. In this way Σ_v is identified with $\mathrm{Hom}_{\mathbf{Q}_p}(F_v, \overline{\mathbf{Q}}_p)$.

If $K/\mathbf{Q}_\ell$ is a finite extension, ℓ a prime, we write $\mathrm{Art}_K : K^\times \to G_K^{\mathrm{ab}}$ for the local Artin map, normalized so uniformizers map to geometric Frobenius elements. If π is a smooth, irreducible representation of $\mathrm{GL}_n(K)$ we denote $\mathrm{rec}_K(\pi)$ the Weil–Deligne representation corresponding to π by the local Langlands correspondence as constructed by Harris and Taylor ([47]). We further specify $r(\pi) = \mathrm{rec}_K(\pi \otimes |\det|^{-1/2})$ for the arithmetically normalized local Langlands correspondence. Finally, we write $r^\iota(\pi)$ for the corresponding representation over $\overline{\mathbf{Q}}_p$ obtained via ι.

We also use two shorthand notations for tuple-based operations. First, suppose that S is a set and we are given collections $\{X_s\}_{s \in S}$, $\{Y_s\}_{s\in S}$, and $\{Z_s\}_{s \in S}$ with a binary operations $X_s \times Y_s \xrightarrow{\bullet_s} Z_s$. If $X = \prod_{s \in S} X_s$, $Y = \prod_{s\in S} Y_s$ and $Z = \prod_{s \in S} Z_s$ we then define a binary operation $X \times Y \xrightarrow{\bullet} Z$ by $(x_s) \bullet (y_s) := (x_s \bullet_s y_s)$. A typical situation where we might use this is when, for each $s \in S$, X_s is a group acting on a set Y_s (so $Y_s = Z_s$). The second situation we will find ourselves in is we are given a collection $x = (x_s)_{s \in S}$ of elements of a common ring R, and we are given a collection $n = (n_s)_{s \in S}$ of integers. In that case we define $x^n = \prod_{s \in S} x_s^{n_s}$. This notation satisfies the obvious compatibilities with usual multiplication in a ring.

If v is a place of F then we write $\mathfrak{p}_v$ for the corresponding prime ideal. If p is a prime then we will use the bold letter $\mathbf{p} := \prod_{v|p} \mathfrak{p}_v$ for the product of the primes above p.

1.11. **Acknowledgments.** This project began in May 2012 when D.H. attended William Stein's plenary lecture on elliptic curves over $\mathbf{Q}(\sqrt{5})$ at the Atkin Memorial Conference, and he would like to heartily thank Stein for this crucial initial inspiration. Some of these results (in particular the non-critical case of Theorem 1.1.2) were first announced by D.H. in a conference at UCLA in May 2013. Decisive progress beyond the non-critical case occurred in early 2016, and the authors gave some lectures on these results beginning in Spring 2016. In any case, the authors would like to apologize for the very long delay between the first announcement(s) of these results and the appearance of this manuscript.

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2. Cohomology and local systems

This section concerns the cohomology of local systems on symmetric spaces which arise in the context of Hilbert modular forms. Almost nothing is original in our treatment. However, a number of calculations later in the paper rely on the precise formulas we present and so we found it prudent to expose them in detail. The reader is encouraged to skim the results as needed.

2.1. **Topology.** Throughout this subsection, we write X and Y for topological spaces which are locally compact and Hausdorff. We let R be a fixed principal ideal domain and sheaves are sheaves of R-modules.

If $\mathscr{L}$ is a sheaf on X we consider the cohomology $H^*(X,\mathscr{L})$, homology $H_*(X,\mathscr{L})$, compactly supported cohomology $H^*_c(X,\mathscr{L})$ or Borel–Moore homology $H^{\mathrm{BM}}_*(X,\mathscr{L})$. These are all R-modules. Primary sources for H^*_c and H^{BM}_* are [74, 20]. We refer to [22] for what follows.[6] Along with the usual functorialities in algebraic topology (pushforward in homology, pullback in cohomology, and so forth) we summarize important properties of compactly supported cohomology and Borel–Moore homology.

If $\mathscr{L}$ and $\mathscr{M}$ are two sheaves on X, there is a functorial cup product ([22, Sections II.7])

$$\cup : H^p_?(X,\mathscr{L}) \otimes_R H^q(X,\mathscr{M}) \to H^{p+q}_?(X,\mathscr{L}\otimes_R \mathscr{M}) \tag{2.1.1}$$

for ? either c or the empty symbol. Further, there are two separate cap products ([22, Section V.10])

$$\begin{aligned} H^p_c(X,\mathscr{L}) \otimes_R H^{\mathrm{BM}}_q(X,\mathscr{M}) &\xrightarrow{\cap} H_{q-p}(X,\mathscr{L}\otimes_R \mathscr{M}); \\ H^p(X,\mathscr{L}) \otimes_R H^{\mathrm{BM}}_q(X,\mathscr{M}) &\xrightarrow{\cap} H^{\mathrm{BM}}_{q-p}(X,\mathscr{L}\otimes_R \mathscr{M}). \end{aligned} \tag{2.1.2}$$

The cup and cap products commute in the sense that

$$(\Psi \cup \Psi') \cap \Phi = \Psi \cap (\Psi' \cap \Phi), \tag{2.1.3}$$

[6]We warn the reader that our homology notation is in conflict with [22]. Namely, H^{BM}_* here is written H_* there and H_* here is written H^c_* there (cf. the caution at the start of [22, Section V.3]).

under apparent qualifications on where these elements are defined.

If $\mathscr{L}$ is a sheaf on Y and $f : X \to Y$ is a proper morphism, then there are functorial pushforward and pullback maps

$$H_*^{\mathrm{BM}}(X, f^*\mathscr{L}) \xrightarrow{f_*} H_*^{\mathrm{BM}}(Y, \mathscr{L}); \qquad H_c^*(Y, \mathscr{L}) \xrightarrow{f^*} H_c^*(X, f^*\mathscr{L}). \tag{2.1.4}$$

The cup product commutes with pullbacks. The cap products are compatible with pushfowards and pullbacks along proper morphisms $f : X \to Y$ in that

$$f_*(f^*\Psi \cap \Phi) = \Psi \cap f_*\Phi \tag{2.1.5}$$

for all $\Psi \in H_?^p(Y, \mathscr{L})$ and $\Phi \in H_q^{\mathrm{BM}}(X, f^*\mathscr{M})$.

Now suppose that $p = q$ in (2.1.2) and that we have a pairing $\mathscr{L} \otimes_R \mathscr{M} \to R$. Taking the natural composition

$$H_0^?(X, \mathscr{L} \otimes_R \mathscr{M}) \to H_0^?(X, R) \xrightarrow{\mathrm{tr}} R$$

and combining it with the cap product, $\langle \Psi, \Phi \rangle := \mathrm{tr}(\Psi \cap \Phi)$ defines a functorial R-bilinear pairing

$$\langle -, - \rangle : H_?^p(X, \mathscr{L}) \otimes_R H_p^{\mathrm{BM}}(X, \mathscr{M}) \to R$$

under which f^* and f_* are adjoint (by (2.1.5) and because trace commutes with pushforwards). Thus, our convention is that cap products $\Phi \cap \Psi$ are homology classes and values of pairings $\langle \Phi, \Psi \rangle$ are elements of R.

Suppose now that X is an oriented real manifold of dimension n. Then there is a Borel–Moore fundamental class $[X] \in H_n^{\mathrm{BM}}(X, R)$ with the property that $\mathrm{PD}(\Psi) := \Psi \cap [X]$ defines a functorial morphism

$$\mathrm{PD} : H^q(X, \mathscr{L}) \to H_{n-q}^{\mathrm{BM}}(X, \mathscr{L}) \tag{2.1.6}$$

for each $0 \leq q \leq n$. See [22, Theorem V.10.1 and Corollary V.10.2]. We refer to PD as "Poincaré duality." It satisfies the following properties. First, if $f : X \to X$ is an orientation preserving homeomorphism, then $f_*[X] = [X]$ and so (2.1.5) implies that

$$f_* \,\mathrm{PD}\, f^* = \mathrm{PD}\,. \tag{2.1.7}$$

Second, if $f : X \to Y$ is a proper morphism, $\mathscr{L}$ is a sheaf on X, $\mathscr{M}$ is a sheaf on Y and we have a pairing $\mathscr{L} \otimes_R \mathscr{M} \to R$, then from (2.1.3), (2.1.5), and (2.1.6) we obtain

$$\langle \Phi, f_* \,\mathrm{PD}(\Psi) \rangle = \langle f^*\Phi \cup \Psi, [X] \rangle \tag{2.1.8}$$

for all $\Phi \in H_c^p(Y, \mathscr{L})$ and $\Psi \in H^{n-p}(X, f^*\mathscr{M})$ (the cup product $f^*\Phi \cup \Psi$ is implicitly viewed in $H_c^n(X, R)$ for the purposes of this formula). Finally, when R is a subring of $\mathbf{C}$ there is an integration map $\int_X : H_c^n(X, R) \to R$ which is natural with respect to Poincaré daulity in that $\int_X = \mathrm{tr} \circ \mathrm{PD}$ on $H_c^n(X, R)$.

2.2. **Adelic cochains on symmetric spaces.** In this subsection, we review the adelic (co)chains introduced by Ash and Stevens (see [43, Section 2] and the references there).

Write G for a connected reductive group over $\mathbf{Q}$, $\mathbf{A}$ for the adeles of $\mathbf{Q}$, and $\mathbf{A}_f$ for the finite adeles. Let $G(\mathbf{R})^\circ$ be the connected component of the identity in $G(\mathbf{R})$ and let $K_\infty^\circ \subset G(\mathbf{R})^\circ$ be a subgroup which is either maximal compact or maximal compact mod-center.

Write $D_\infty = G(\mathbf{R})^\circ / K_\infty^\circ$ and $D_{\mathbf{A}} = D_\infty \times G(\mathbf{A}_f)$, which we view as topological spaces where D_∞ gets its structure as a real manifold and $G(\mathbf{A}_f)$ gets the discrete topology. Then, we write $C_\bullet(D_{\mathbf{A}})$ for the chain complex of singular chains in $D_{\mathbf{A}}$. The discrete topology is totally disconnected, so any singular chain in $G(\mathbf{A}_f)$ is a single point, meaning $C_\bullet(D_{\mathbf{A}}) = C_\bullet(D_\infty)\otimes_{\mathbf{Z}}\mathbf{Z}[G(\mathbf{A}_f)]$ with $\partial\otimes 1$ as the boundary map (and we could have also given $G(\mathbf{A}_f)$ its natural topology).

Fix a compact open subgroup $K\subset G(\mathbf{A}_f)$. View $G(\mathbf{Q})^\circ$ diagonally inside $D_{\mathbf{A}}$ and K inside the second coordinate. Then, write Y_K for the double quotient

$$Y_K := G(\mathbf{Q})\backslash G(\mathbf{A})/K_\infty^\circ K = G(\mathbf{Q})^\circ\backslash D_{\mathbf{A}}/K. \tag{2.2.1}$$

Though this may not be a manifold, it is certainly a disconnected orbifold. Specifically, if $\{g_i\}$ is a finite collection of elements $g_i\in G(\mathbf{A}_f)$ such that $G(\mathbf{A}) = \bigsqcup_i G(\mathbf{Q})^\circ G(\mathbf{R})g_iK$, then

$$Y_K = \bigsqcup_i \Gamma(g_i)\backslash D_\infty, \tag{2.2.2}$$

where $\Gamma(g) := gKg^{-1}\cap G(\mathbf{Q})^\circ\subset G(\mathbf{Q})^\circ$ for $g\in G(\mathbf{A}_f)$. When the $\Gamma(g_i)/Z(\Gamma(g_i))$ are without torsion, Y_K is a real manifold of dimension $2d$ (compare with Proposition 2.3.3 below).

Now suppose that N is a $(G(\mathbf{Q})^\circ, K)$-bimodule, meaning:

(1) N is a right K-module whose action we write $n|k$ for $n\in N$ and $k\in K$, and
(2) N is a left $G(\mathbf{Q})^\circ$-module whose actions we write $\gamma\cdot n$ for $n\in N$ or $\gamma\in G(\mathbf{Q})^\circ$.

For instance, the left action of $G(\mathbf{Q})^\circ$, and the right action of K, on $D_{\mathbf{A}}$ equips $C_\bullet(D_{\mathbf{A}})$ with a natural structure of complex of $(G(\mathbf{Q})^\circ, D_{\mathbf{A}})$-bimodules. We consider any N with the discrete topology and write $\underline{N}$ (in the text we will remove underlines for readability) for the local system defined by the sheaf of locally constant sections of the natural projection map

$$G(\mathbf{Q})^\circ\backslash(D_{\mathbf{A}}\times N)/K \to Y_K.$$

We also use the standard abuse of notation to write $\underline{N}$ for the double quotient itself.

The adelic cochain complex associated with N is

$$C^\bullet_{\mathrm{ad}}(K,N) := \mathrm{Hom}_{(G(\mathbf{Q})^\circ,K)}(C_\bullet(D_{\mathbf{A}}),N).$$

Let $g_f\in G(\mathbf{A}_f)$. Then, for each singular chain $\sigma_\infty\in C_\bullet(D_\infty)$ there is a singular chain $\sigma_\infty\otimes[g_f]\in C_\bullet(D_{\mathbf{A}})$. This allows us to define a morphism of abelian groups

$$\begin{aligned}\mathrm{Hom}(C_\bullet(D_{\mathbf{A}}),N) &\to \mathrm{Hom}(C_\bullet(D_\infty),N);\\ \phi &\mapsto \left[\phi_{g_f} : \sigma_\infty \mapsto \phi(\sigma_\infty\otimes[g_f])\right].\end{aligned} \tag{2.2.3}$$

We note that the chain complex $C_\bullet(D_\infty)$ is naturally a chain complex of left $\Gamma(g_f)$-modules, where $\Gamma(g_f)$ acts on D_∞ through the inclusion $\Gamma(g_f)\subset G(\mathbf{Q})^\circ$. On the other hand, we write $N(g_f)$ for the left $\Gamma(g_f)$-module whose underlying abelian group is still N but equipped with a left $\Gamma(g_f)$-action

$$\gamma\cdot_{g_f} n = \gamma\cdot n|(g_f^{-1}\gamma^{-1}g_f).$$

These definitions given, it is straightforward to see that the map (2.2.3) descends to a morphism

$$C^\bullet_{\mathrm{ad}}(K,N)\to \mathrm{Hom}_{\Gamma(g_f)}(C_\bullet(D_\infty),N(g_f)).$$

Finally, let $C^\bullet(D_\infty;N) = \mathrm{Hom}(C_\bullet(D_\infty),N)$ and write $C^\bullet_c(D_\infty;N)\subset C^\bullet(D_\infty;N)$ for the cochains on D_∞ with compact support. We define the compactly supported adelic cochains by

$$C^\bullet_{\mathrm{ad},c}(K,N) := \{\phi\in C^\bullet_{\mathrm{ad}}(K,N) \mid \phi_{g_f}\in C^\bullet_c(D_\infty;N) \text{ for all } g_f\in G(\mathbf{A}_f)\}.$$

Proposition 2.2.1. *There are canonical isomorphisms*

$$\begin{tikzcd} H^*(C^\bullet_{\mathrm{ad},c}(K,N)) \arrow[r, "\simeq"] \arrow[d] & H^*_c(Y_K,\underline{N}) \arrow[d] \\ H^*(C^\bullet_{\mathrm{ad}}(K,N)) \arrow[r, "\simeq"] & H^*(Y_K,\underline{N}) \end{tikzcd}$$

Proof. This follows from the same argument as in [43, Proposition 2.1.1]. □

"Canonical" in Proposition 2.2.1 refers to at least the following functorialities:

(i) If $f : N \to N'$ is a $(G(\mathbf{Q})^\circ, K)$-equivariant morphism, then the natural map $H^*_?(Y_K,\underline{N}) \xrightarrow{f} H^*_?(Y_K,\underline{N}')$ is induced by the morphism of cochain complexes $f\circ - : C^\bullet_{\mathrm{ad},?}(K,N) \to C^\bullet_{\mathrm{ad},?}(K,N')$.
(ii) If $K' \subset K$ is a subgroup then the inclusion $C^\bullet_{\mathrm{ad},?}(K,N) \subset C^\bullet_{\mathrm{ad},?}(K',N)$ induces the pullback $\mathrm{pr}^* : H^*_?(Y_K,\underline{N}) \to H^*_?(Y_K,\underline{N}')$ on cohomology.
(iii) Suppose that $K' \subset K$ is a subgroup of finite index. Then, $\mathrm{pr} : Y_{K'} \to Y_K$ is proper, so it induces a pushfoward map $\mathrm{pr}_* : H^*_?(Y_{K'},\underline{N}) \to H^*_?(Y_K,\underline{N})$. On the other hand, if $K/K' = \{x_iK'\}$ then $\mathrm{tr}(\phi)(\sigma) = \sum \phi(\sigma x_i)|x_i^{-1}$ induces a natural map of cochain complexes $\mathrm{tr} : C^\bullet_{\mathrm{ad},?}(K',N) \to C^\bullet_{\mathrm{ad},?}(K,N)$, whose induced map on cohomology is pr_*.
(iv) Finally, let $g \in G(\mathbf{A}_f)$. Write $N(g^{-1})$ for the $(G(\mathbf{Q})^\circ, g^{-1}Kg)$-module whose right $g^{-1}Kg$-action is given by $n|_{g^{-1}}x = n|gxg^{-1}$. Then, the map $r_g : Y_K \to Y_{g^{-1}Kg}$ given by $x \mapsto xg$ induces a map on cohomology $r_g^* : H^*_?(Y_{g^{-1}Kg},\underline{N(g^{-1})}) \to H^*_?(Y_K,\underline{N})$. On the other hand, if we set $r_g(\phi)(\sigma) = \phi(\sigma g)$ then $r_g : C^\bullet_{\mathrm{ad},?}(g^{-1}Kg,\underline{N(g^{-1})}) \to C^\bullet_{\mathrm{ad},?}(K,N)$ is a map of cochain complexes which induces r_g^* on cohomology.

Suppose now that $\Delta \subset G(\mathbf{A}_f)$ is a monoid and $K \subset \Delta$ so that K and $\delta^{-1}K\delta$ are commensurable for all $\delta \in \Delta$. We assume that N is equipped with a *left* Δ-module structure $\delta \cdot n$ which commutes with the given $G(\mathbf{Q})^\circ$-module structure. We give N the structure of a right K-module by $n|k = k^{-1}\cdot n$ under which we now have a $(G(\mathbf{Q})^\circ, K)$-bimodule again. We equip $\mathrm{Hom}_{G(\mathbf{Q})^\circ}(C_\bullet(D_{\mathbf{A}}), N)$ with the left action of Δ given by $(\delta\cdot\phi)(\sigma) = \delta\cdot\phi(\sigma\delta)$ under which we have $C^\bullet_{\mathrm{ad}}(K,N) = \mathrm{Hom}_{G(\mathbf{Q})^\circ}(C_\bullet(D_{\mathbf{A}}), N)^K$ (and an obvious analog for $C^\bullet_{\mathrm{ad},c}(K,N)$). If $\delta \in \Delta$ and $K\delta K/K = \{\delta_i K\}$ is a decomposition into right cosets and $\phi \in C^\bullet_{\mathrm{ad},?}(K,N)$ then

$$[K\delta K](\phi) = \sum_i \delta_i \cdot \phi \tag{2.2.4}$$

is independent of the choice of δ_i and defines another element of $C^\bullet_{\mathrm{ad},?}(K,N)$. We refer to $[K\delta K]$ as a Hecke operator when we consider its induced map on cohomology. We enumerated the meaning of "canonical" in Proposition 2.2.1 is to justify that this Hecke operator agrees with the usual one defined by the composition

$$\begin{aligned} H^*_?(Y_K,\underline{N}) &\xrightarrow{\mathrm{pr}^*} H^*_?(Y_{K\cap\delta^{-1}K\delta},\underline{N}) \to H^*_?(Y_{K\cap\delta^{-1}K\delta},\underline{N(\delta^{-1})}) \\ &\xrightarrow{r_\delta^*} H^*_?(Y_{K\cap\delta K\delta^{-1}},\underline{N}) \xrightarrow{\mathrm{pr}_*} H^*_?(Y_K,\underline{N}). \end{aligned} \tag{2.2.5}$$

Here, for $\delta \in \Delta$ the morphism $n \mapsto \delta\cdot n$ defines a morphism $N \to N(\delta^{-1})$ which is equivariant for the action of $K \cap \delta^{-1}K\delta$ on either side, giving the unlabeled arrow.

We end our discussion with an algebraic situation. Fix a number field $F/\mathbf{Q}$ and write $\mathscr{N}$ for an F-algebraic representation of G, i.e. an F-vector space $\mathscr{N}$ and a representation $G \to \mathrm{Res}_{F/\mathbf{Q}}\mathrm{GL}(\mathscr{N})$. Recall that we fixed an isomorphism $\iota : \mathbf{C} \simeq \overline{\mathbf{Q}}_p$. Suppose that $E \subset \mathbf{C}$ is a field and $L := \mathbf{Q}_p(\iota(E))$.

Then, we deduce linear representations $G(L) \to \mathrm{GL}_L(N_p)$, and $G(E) \to \mathrm{GL}_E(N_\infty)$ where $N_p := \mathscr{N} \otimes_{\mathbf{Q}} L$ and $N_\infty := \mathscr{N} \otimes_{\mathbf{Q}} E$. By construction, ι induces a morphism of $\mathbf{Q}$-vector spaces $\iota : N_\infty \to N_p$, which becomes an isomorphism $\iota : N_\infty \otimes_{E,\iota} L \simeq N_p$. Let K be a compact open subgroup of $G(\mathbf{A}_f)$, and write $K_p \subset G(\mathbf{Q}_p)$ for its p-th component. Using the inclusion $G(\mathbf{Q})^\circ \subset G(E)$ we thus get a local system $\underline{N}_\infty$ on Y_K; or we can use the inclusion $K_p \subset G(\mathbf{Q}_p) \subset G(L)$ to get a local system $\underline{N}_p$. Note that $k_p \in K_p$ acts on the right of N_p via $n|k_p = k_p^{-1} \cdot n$.

Proposition 2.2.2.

(1) If $\gamma \in G(\mathbf{Q})$ then $\gamma_p \iota(n) = \iota(\gamma_\infty n)$ for all $n \in N_\infty$.
(2) The map $\iota((g,n)) = (g, g_p^{-1}\iota(n))$ defines a morphism $\iota : \underline{N}_\infty \longrightarrow \underline{N}_p$ of local systems on Y_K.
(3) The map $\iota(\phi)(\sigma_\infty \otimes [g_f]) = g_p^{-1}\iota(\phi(\sigma_\infty \otimes [g_f]))$ defines a morphism $\iota : C^\bullet_{\mathrm{ad},?}(K, N_\infty) \to C^\bullet_{\mathrm{ad},?}(K, N_p)$ of cochain complexes.
(4) The maps in parts (2) and (3) induce a canonical commuting diagram

$$\begin{array}{ccc} H^*_?(C^\bullet_{\mathrm{ad}}(K,N_\infty)) & \xrightarrow{\simeq} & H^*_?(Y_K, \underline{N}_\infty) \\ \downarrow{\scriptstyle\iota} & & \downarrow{\scriptstyle\iota} \\ H^*_?(C^\bullet_{\mathrm{ad}}(K,N_p)) & \xrightarrow{\simeq} & H^*_?(Y_K, \underline{N}_p). \end{array}$$

Proof. Everything is straightforward to check. □

2.3. **Symmetric spaces for F.** Here we specialize the above discussion to the setting of this article.

First, let $G = \mathrm{Res}_{F/\mathbf{Q}} \mathrm{GL}_1$. Write $\widehat{\mathcal{O}}_F$ for the profinite completion of $\mathcal{O}_F$ and $K_\infty^\circ = \{1\} \subset (F_\infty^\times)^\circ$ (maximal compact) and $K = \widehat{\mathcal{O}}_F^\times \subset \mathrm{GL}_1(\mathbf{A}_{F,f})$. The corresponding symmetric space is written $\mathrm{C}_\infty := F^\times \backslash \mathbf{A}_F^\times / \widehat{\mathcal{O}}_F^\times$.

Write $\mathbf{A}_{F,+}^\times := (F_\infty^\times)^\circ \times \mathbf{A}_{F,f}^\times$ and $F_+^\times = F^\times \cap (F_\infty^\times)^\circ$. By weak approximation, $F^\times\backslash \mathbf{A}_F^\times \simeq F_+^\times \backslash \mathbf{A}_{F,+}^\times$ and so we may also write

$$\mathrm{C}_\infty = F^\times \backslash \mathbf{A}_F^\times / \widehat{\mathcal{O}}_F^\times \simeq F_+^\times \backslash \mathbf{A}_{F,+}^\times / \widehat{\mathcal{O}}_F^\times. \tag{2.3.1}$$

This is a real Lie group that sits inside an exact sequence

$$1 \to (F_\infty^\times)^\circ / \mathcal{O}_{F,+}^\times \to \mathrm{C}_\infty \to \mathrm{Cl}_F^+ \to 1 \tag{2.3.2}$$

where Cl_F^+ is the narrow class group and $\mathcal{O}_{F,+}^\times$ are the totally positive units in F.

We will write $\frac{dx_\infty}{x_\infty}$ for the choice of a volume form on $(F_\infty^\times)^\circ$, which then induces a translation-invariant orientation on C_∞. This fixes a Borel–Moore fundamental class $[\mathrm{C}_\infty] \in H_d^{\mathrm{BM}}(\mathrm{C}_\infty, \mathbf{Z})$. We record this discussion as a proposition.

Proposition 2.3.1. *If $x \in \mathbf{A}_F^\times$ then right multiplication $r_x : \mathrm{C}_\infty \to \mathrm{C}_\infty$ is orientation preserving. In particular, $(r_x)_*[\mathrm{C}_\infty] = [\mathrm{C}_\infty]$.*

Now let $G = \mathrm{Res}_{F/\mathbf{Q}} \mathrm{GL}_2$. Here, we take $K_\infty^\circ = \mathrm{SO}_2(F_\infty) Z(F_\infty) \subset \mathrm{GL}_2(F_\infty)^\circ$ (maximal compact mod-center). For $K \subset \mathrm{GL}_2(\mathbf{A}_{F,f})$ we write Y_K for the symmetric space as in (2.2.1). We will be a bit more concrete regarding Y_K. Let $\mathfrak{h}$ denote the complex upper half plane. Then, $\mathrm{GL}_2(F_\infty)^\circ$ acts on $\mathfrak{h}^{\Sigma_F}$ via fractional linear transformations

$$g \cdot z := \frac{az+b}{cz+d} \tag{2.3.3}$$

for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(F_\infty)^\circ$ and $z \in \mathfrak{h}^{\Sigma_F}$. If $i \in \mathfrak{h}^{\Sigma_F}$ means the complex number i diagonally embedded then K_∞° is the stabilizer of i so that $D_\infty = \mathrm{GL}_2(F_\infty)^\circ / K_\infty^\circ \simeq \mathfrak{h}^{\Sigma_F}$. Thus

$$Y_K = \mathrm{GL}_2(F) \backslash \mathrm{GL}_2(\mathbf{A}_F)/K_\infty^\circ K \simeq \mathrm{GL}_2^+(F) \backslash D_\infty \times \mathrm{GL}_2(\mathbf{A}_{F,f})/K, \tag{2.3.4}$$

and Y_K is a $2d$-dimensional real orbifold, decomposing into a finite disjoint union of quotients $\Gamma(g)\backslash D_\infty$ where $\Gamma(g) = gKg^{-1} \cap \mathrm{GL}_2^+(F)$ (see (2.2.2)). We make the following definition.

Definition 2.3.2. Let $K \subset \mathrm{GL}_2(\mathbf{A}_{F,f})$ be a compact open subgroup.

1. K is neat if $\Gamma(g)/Z(\Gamma(g))$ is torsion-free for all $g \in \mathrm{GL}_2(\mathbf{A}_{F,f})$.
2. K is t-good if $\begin{pmatrix} \widehat{\mathcal{O}}_F^\times & \\ & 1 \end{pmatrix} \subset K$.

As mentioned above, if K is a neat level then Y_K is a manifold. The purpose of the t-good definition is that for t-good levels K, the map $\mathbf{A}_F^\times \to \mathrm{GL}_2(\mathbf{A}_F)$ given by $x \mapsto \left(\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\right)$ descends to a closed (thus, proper) embedding

$$\mathrm{t} : \mathrm{C}_\infty \hookrightarrow Y_K. \tag{2.3.5}$$

In particular, for such K one gets pullbacks (resp. pushforwards) along t on compactly supported cohomology (resp. Borel–Moore homology).

Beginning in Section 3.2 we will mostly be concerned with level subgroups of the form

$$K_1(\mathfrak{n}) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\widehat{\mathcal{O}}_F) \mid c \equiv 0 \bmod \mathfrak{n}\widehat{\mathcal{O}}_F, d \equiv 1 \bmod \mathfrak{n}\widehat{\mathcal{O}}_F \right\} \tag{2.3.6}$$

with $\mathfrak{n}$ an integral ideal.

Proposition 2.3.3. *Let $\mathfrak{n} \subset \mathcal{O}_F$ be an integral ideal.*

1. *There exists $\mathfrak{n}' \subset \mathfrak{n}$ such that $K_1(\mathfrak{n}')$ is neat.*
2. *$K_1(\mathfrak{n})$ is* t*-good.*

Proof. (1) follows from [35, Lemma 2.1]. (2) is clear. □

2.4. **Weights and algebraic local systems.** Here we specify a collection algebraic local systems.

Definition 2.4.1. A cohomological weight $\lambda = (\lambda_1, \lambda_2)$ is a pair of characters $\lambda_i : F^\times \to \mathbf{C}^\times$ of the form

$$\lambda_i(\xi) = \prod_{\sigma \in \Sigma_F} \sigma(\xi)^{e_i(\sigma)}$$

for $e_i(\sigma) \in \mathbf{Z}$ such that:

1. If $\omega_\lambda = \lambda_1\lambda_2 : F^\times \to \mathbf{C}^\times$ then ω_λ is trivial on a finite index subgroup of $\mathcal{O}_F^\times$, and
2. $e_1(\sigma) \geq e_2(\sigma)$ for all $\sigma \in \Sigma_F$.

Let λ be a cohomological weight. An argument of Weil implies that $w(\sigma) = e_1(\sigma) + e_2(\sigma)$ is independent of $\sigma \in \Sigma_F$. Set $\kappa_\sigma = e_1(\sigma) - e_2(\sigma)$; this is a non-negative integer. Thus a cohomological weight λ is the same data as a pair $(\kappa, w) \in \mathbf{Z}_{\geq 0}^{\Sigma_F} \times \mathbf{Z}$ with $\kappa_\sigma \equiv w \bmod 2$ for each $\sigma \in \Sigma_F$. We will almost always write $\lambda = (\kappa, w)$ to indicate a cohomological weight in this way.

If n is a non-negative integer, write $\mathscr{L}_n$ for the space of polynomials over $\mathbf{Z}$ with degree at most n. If R is a ring, write $\mathscr{L}_n(R) = \mathscr{L}_n \otimes_{\mathbf{Z}} R$. We equip $\mathscr{L}_n$ with an algebraic left-action of GL_2 via

$$(g \cdot P)(X) = (a + cX)^n P\left(\frac{b + dX}{a + cX}\right) \tag{2.4.1}$$

for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(R)$ and $P \in \mathscr{L}_n(R)$. Given a cohomological weight $\lambda = (\kappa, w)$ we write

$$\mathscr{L}_\lambda := \bigotimes_{\sigma \in \Sigma_F} \left(\mathscr{L}_{\kappa_\sigma}(F) \otimes \det^{\frac{w-\kappa_\sigma}{2}} \right) \tag{2.4.2}$$

(where $\det : \mathrm{GL}_2 \to \mathbf{G}_m$ is the determinant character). Thus $\mathscr{L}_\lambda$ is an F-vector space equipped with an algebraic representation of the F-algebraic group $(\mathrm{Res}_{F/\mathbf{Q}} \mathrm{GL}_2) \times_{\mathbf{Q}} F$, and so we can apply the discussion at the end of Section 2.2 to $G = \mathrm{Res}_{F/\mathbf{Q}} \mathrm{GL}_2$ and $\mathscr{N} = \mathscr{L}_\lambda$.

Specifically, suppose that $E \subset \mathbf{C}$ contains $\sigma(F)$ for all $\sigma \in \Sigma_F$, and let $L = \mathbf{Q}_p(\iota(E))$. Then, $G(E) = \mathrm{GL}_2(F \otimes_{\mathbf{Q}} E) \simeq \mathrm{GL}_2(E)^{\Sigma_F}$ and and the action of $\mathrm{GL}_2(E)^{\Sigma_F}$ on

$$\mathscr{L}_\lambda(E) := \bigotimes_{\sigma \in \Sigma_F} \mathscr{L}_{\kappa_\sigma}(E) \otimes \det^{\frac{w-\kappa_\sigma}{2}}$$

is the one where the σ-th factor acts on the σ-th tensorand as in (2.4.1). On the other hand,

$$G(L) = \mathrm{GL}_2(F \otimes_{\mathbf{Q}} L) \simeq \mathrm{GL}_2(F_p \otimes_{\mathbf{Q}_p} L) \simeq \prod_{v|p} \mathrm{GL}_2(F_v \otimes_{\mathbf{Q}_p} L) \simeq \prod_{v|p} \mathrm{GL}_2(L)^{\Sigma_v}$$

and $G(L)$ acts on the L-vector space

$$\mathscr{L}_\lambda(L) := \bigotimes_{v|p} \bigotimes_{\sigma \in \Sigma_v} \mathscr{L}_{\kappa_\sigma}(L) \otimes \det^{\frac{w-\kappa_\sigma}{2}} \tag{2.4.3}$$

in the analogous way, tensorand-by-tensorand.

Remark 2.4.2. For any compact open subgroup $K \subset \mathrm{GL}_2(\mathbf{A}_{F,f})$, the above representations define local systems $\mathscr{L}_\lambda(E)$ and $\mathscr{L}_\lambda(L)$ on Y_K, and ι induces a $\mathbf{Q}$-linear morphism of local systems $\iota : \mathscr{L}_\lambda(E) \to \mathscr{L}_\lambda(L)$ by Proposition 2.2.2. However, we note that the ι-transfer from $\mathscr{L}_\lambda(E)$ to $\mathscr{L}_\lambda(L)$ has a non-trivial effect on certain formulas (cf. Section 5.5).

For instance, suppose that $g \in \mathrm{GL}_2(\mathbf{A}_{F,f})$, $K \subset \mathrm{GL}_2(\mathbf{A}_{F,f})$ is a compact open subgroup and $K' \subset K$ is another compact open subgroup so that $g^{-1}K'g \subset K$. Write $\mathscr{L}_\lambda(L)(g)$ for the left $G(L)$-representation whose action is given by $h \cdot_g P := g_p^{-1} h g_p \cdot P$ for $P \in \mathscr{L}_\lambda(L)$ and $h \in G(L)$. Then $P \mapsto g_p^{-1} \cdot P$ defines a $G(L)$-equivariant isomorphism $\mathscr{L}_\lambda(L) \simeq \mathscr{L}_\lambda(L)(g)$ (compare with (2.2.5)) that fits into a diagram of local systems whose bases are as indicated:

$$\begin{tikzcd}
\mathscr{L}_\lambda(E)_{/Y_{K'}} \arrow[r, "\iota"] \arrow[d, "r_g"'] & \mathscr{L}_\lambda(L)_{/Y_{K'}} \arrow[dr, "P \mapsto g_p^{-1} \cdot P"] & \\
\mathscr{L}_\lambda(E)_{/Y_{g^{-1}K'g}} \arrow[d, "\mathrm{pr}"'] & \mathscr{L}_\lambda(L)_{/Y_{g^{-1}K'g}} \arrow[d, "\mathrm{pr}"] & \mathscr{L}_\lambda(L)(g)_{/Y_{K'}} \arrow[l, "r_g"'] \\
\mathscr{L}_\lambda(E)_{/Y_K} \arrow[r, "\iota"'] & \mathscr{L}_\lambda(L)_{/Y_K}. &
\end{tikzcd} \tag{2.4.4}$$

3. Hilbert modular forms

3.1. **Recollection of definitions.** The goal of this subsection is to describe the three points of view that we need to adopt regarding Hilbert modular forms. General references for automorphic representation theory are [19, 25]. Specific to Hilbert modular forms, we refer to [48, Section 2] or [49, Section 3]. More precise references will be given if confusion could arise.

Let t be a real number. We write ω_t for the character of $F_\infty^\times$ given by $\omega_t(x_\infty) = \prod_\sigma x_\sigma^t$ for $x_\infty = (x_\sigma) \in F_\infty^\times$. When $t = w$ is an integer, the restriction to $F^\times \subset F_\infty^\times$ is what we called ω_λ in

Definition 2.4.1. Suppose that $\omega : F_\infty^\times \to \mathbf{C}^\times$ is a continuous character such that $\omega|_{(F_\infty^\times)^\circ} = \omega_t|_{(F_\infty^\times)^\circ}$. We write $L^2(\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbf{A}_F), \omega)$ for the space of functions $f : \mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbf{A}_F) \to \mathbf{C}$ that satisfy the following two properties:

(1) $f(x_\infty g) = \omega^{-1}(x_\infty) f(g)$ for all $g \in \mathrm{GL}_2(\mathbf{A}_F)$ and $x_\infty \in F_\infty^\times$.
(2) $|\det g|^{t/2}|f(g)|$ is square-integrable on $(F_\infty^\times)^\circ \mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbf{A}_F)$.

The condition in (2) is well-defined by the condition (1) and the assumption on ω. We further write $L_0^2(\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbf{A}_F), \omega)$ for those $f \in L^2(\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbf{A}_F), \omega)$ which are cuspidal, meaning that

$$\int_{F\backslash \mathbf{A}_F} f\left(\left(\begin{smallmatrix} 1 & u \\ & 1 \end{smallmatrix}\right) g\right) du = 0 \qquad (\text{for all } g \in \mathrm{GL}_2(\mathbf{A}_F)). \tag{3.1.1}$$

Note that the group $\mathrm{GL}_2(\mathbf{A}_F)$ acts on these L^2-spaces by right translation in the domain.

Definition 3.1.1. A cuspidal automorphic representation π for $\mathrm{GL}_2(\mathbf{A}_F)$ is an irreducible (admissible) $\mathrm{GL}_2(\mathbf{A}_F)$-subrepresentation of $L_0^2(\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbf{A}_F), \omega)$ for some ω.

By admissible here, we mean the induced $(\mathfrak{gl}_2(F_\infty), K_\infty^\circ) \times \mathrm{GL}_2(\mathbf{A}_{F,f})$-module on the K_∞°-finite vectors of π are admissible in the usual sense ([25, Section 3.3]). For a cuspidal automorphic representation π, we write $\pi = \bigotimes_v' \pi_v$ for its factorization as a restricted tensor product ([38]). We further specify the notation $\pi_\infty := \bigotimes_{\sigma\in\Sigma_F} \pi_\sigma$, and $\pi_f := \bigotimes_v' \pi_v$ where v runs over finite places of F, so $\pi = \pi_\infty \otimes \pi_f$.

For the rest of this subsection, fix a cohomological weight $\lambda = (\kappa, w)$. We need two representations associated to λ. First, $\mathbf{C}_\lambda$ is the 1-dimensional $\mathbf{C}$-vector space $\mathbf{C}_\lambda = \mathbf{C}\cdot v$ on which we let K_∞° act by

$$v|_{k_\infty} := \omega_w^{-1}(x_\infty) e^{i\theta_\infty(\kappa+2)} \cdot v. \tag{3.1.2}$$

Here, $k_\infty \in K_\infty^\circ$ is written $k_\infty = x_\infty r_\infty$ with $x_\infty \in F_\infty^\times$ and $r_\infty = \left(\begin{smallmatrix} \cos\theta_\infty & \sin\theta_\infty \\ -\sin\theta_\infty & \cos\theta_\infty \end{smallmatrix}\right) \in \mathrm{SO}_2(F_\infty)$. Second, for $\sigma \in \Sigma_F$ we write $D_{\kappa_\sigma+2,w}$ for the weight $\kappa_\sigma + 2$ discrete series representation of $\mathrm{GL}_2(\mathbf{R})$ with central character $x \mapsto x^{-w}$ (see [56, Section 11] for example). Then, we define $D_\lambda := \bigotimes_{\sigma\in\Sigma_F} D_{\kappa_\sigma+2,w}$ (a representation of $\mathrm{GL}_2(F_\infty)$).

Definition 3.1.2. A cuspidal automorphic representation π is cohomological of weight λ if $\pi_\infty \simeq D_\lambda$.

We recall that there is a unique K_∞°-equivariant embedding $\mathbf{C}_\lambda \subset D_\lambda$, the image of which generates D_λ as a $\mathrm{GL}_2(F_\infty)$-representation. Given π, cohomological of weight λ, we write $\pi_\infty^+ \subset \pi_\infty$ for the corresponding line. We also note that the irreducibility and admissibility of such a π implies that $\mathbf{A}_F^\times$ acts on π through a Hecke character ω_π (the central character). Of course, $\omega_{\pi,\infty} := \omega_\pi|_{F_\infty^\times} = \omega_w^{-1}$ and thus $\pi \subset L_0^2(\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbf{A}_F), \omega_w)$.

We now turn towards automorphic forms.

Definition 3.1.3. Let $K \subset \mathrm{GL}_2(\mathbf{A}_{F,f})$ be a compact open subgroup. The space of cohomological cuspidal automorphic forms of weight λ and level K is the set $S_\lambda(K)$ of all functions $\phi : \mathrm{GL}_2(\mathbf{A}_F) \to \mathbf{C}_\lambda$ satisfying the following conditions.

(1) If $g_f \in \mathrm{GL}_2(\mathbf{A}_{F,f})$, then the function $g_\infty \mapsto \phi(g_\infty g_f)$ is a smooth function on $\mathrm{GL}_2(F_\infty)$.
(2) If $\sigma \in \Sigma_F$, then $C_\sigma(\phi) = \left(\kappa_\sigma + \frac{1}{2}\kappa_\sigma^2\right)\phi$, where C_σ denotes the Casimir operator.[7]
(3) If $\gamma \in \mathrm{GL}_2(F)$, $g \in \mathrm{GL}_2(\mathbf{A}_F)$, $k_\infty \in K_\infty^\circ$ and $k \in K$, then $\phi(\gamma g k_\infty k) = \phi(g)|_{k_\infty}$.
(4) ϕ is cuspidal in the sense that (3.1.1) holds for $f = \phi$ and all $g \in \mathrm{GL}_2(\mathbf{A}_F)$.

[7]The Casimir operator is the element $XY + YX + \frac{1}{2}H^2$ in the center of $\mathrm{U}(\mathfrak{sl}_2(\mathbf{R}) \otimes_{\mathbf{R}} \mathbf{C})$ where $X = \frac{1}{2}\left(\begin{smallmatrix} 1 & i \\ i & -1 \end{smallmatrix}\right)$, $Y = \frac{1}{2}\left(\begin{smallmatrix} 1 & -i \\ -i & -1 \end{smallmatrix}\right)$ and $H = \left(\begin{smallmatrix} 0 & i \\ -i & 0 \end{smallmatrix}\right)$. It acts as a differential operator on smooth functions $\mathrm{GL}_2(\mathbf{R}) \to \mathbf{C}$. What we mean by C_σ is the Casimir operator acting on the σ-th component of functions $\mathrm{GL}_2(F_\infty) \to \mathbf{C}$.

The $\mathbf{C}$-vector space $S_\lambda(K)$ is finite-dimensional, but it is not a representation of $\mathrm{GL}_2(\mathbf{A}_F)$. Instead, if $g \in \mathrm{GL}_2(\mathbf{A}_F)$ and $\phi \in S_\lambda(K)$ then $(g \cdot \phi)(g') := \phi(g'g)$ defines a natural $\mathbf{C}$-linear map $S_\lambda(K) \to S_\lambda(gKg^{-1})$. Note as well that $S_\lambda(K) \subset L^2_0(\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbf{A}_F), \omega_w)$. Indeed, this is true by [19, Section 4.4] when $\phi \in S_\lambda(K)$ has a central character (i.e. there exists a Hecke character ω_ϕ such that $\phi(zg) = \omega(z)\phi(g)$ for all $z \in \mathbf{A}_F^\times$) and it is not difficult to see that any ϕ is a finite sum of ϕ's with central character (because $S_\lambda(K)$ is finite-dimensional). Moreover, the discussion in [19] implies:

Proposition 3.1.4. *Let $\mathcal{A}^0_\lambda$ be the set of all cohomological cuspidal automorphic representations of weight λ. Then, for each compact open subgroup $K \subset \mathrm{GL}_2(\mathbf{A}_{F,f})$ there is a canonical isomorphism*

$$S_\lambda(K) \simeq \bigoplus_{\pi \in \mathcal{A}^0_\lambda} \pi_\infty^+ \otimes_{\mathbf{C}} \pi_f^K \tag{3.1.3}$$

as subspaces of $L^2_0(\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbf{A}_F), \omega_w)$.

In order to describe the Eichler–Shimura construction (Section 4.2), we also need a holomorphic version of the previous notion. Recall from Section 2.3 that we write $D_\infty := \mathfrak{h}^{\Sigma_F}$. If $g = \left(\begin{smallmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{smallmatrix}\right) \in \mathrm{GL}_2(F_\infty)$ and $z = (z_\sigma) \in D_\infty$, then we define an automorphy factor

$$j(g,z) = (c_\sigma z_\sigma + d_\sigma)_{\sigma \in \Sigma_F} \in \mathbf{C}^{\Sigma_F}.$$

In particular, one can take $g = \gamma \in \mathrm{GL}_2(F)$ embedded diagonally into $\mathrm{GL}_2(F_\infty)$. Recall also that $\gamma \in \mathrm{GL}_2^+(F)$ acts on $z \in D_\infty$ by fractional linear transformation $z \mapsto \gamma \cdot z$.

Definition 3.1.5. Let $K \subset \mathrm{GL}_2(\mathbf{A}_{F,f})$ be a compact open subgroup. A holomorphic Hilbert cuspform $\mathbf{f}$ of weight $(\kappa+2, w)$ and level K is a function

$$\mathbf{f} : D_\infty \times \mathrm{GL}_2(\mathbf{A}_{F,f}) \to \mathbf{C}$$

satisfying the following conditions.

(1) If $g_f \in \mathrm{GL}_2(\mathbf{A}_{F,f})$, then the function $z \mapsto \mathbf{f}(z, g_f)$ is holomorphic in z.
(2) If $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(F)$, $k \in K$ and $g_f \in \mathrm{GL}_2(\mathbf{A}_{F,f})$ then

$$\mathbf{f}(\gamma \cdot z, \gamma g_f k) = \det(\gamma)^{\frac{w-\kappa}{2}-1} j(\gamma, z)^{\kappa+2} \mathbf{f}(z, g_f). \tag{3.1.4}$$

(3) $\mathbf{f}$ is cuspidal in that $\phi_{\mathbf{f}}$ defined below satisfies (3.1.1).

We write $S_\lambda^{\mathrm{hol}}(K)$ for the holomorphic Hilbert cuspforms $\mathbf{f}$ of weight $(\kappa+2, w)$. As indicated by part (3) of Definition 3.1.5, one can easily compare $S_\lambda^{\mathrm{hol}}(K)$ and $S_\lambda(K)$. Namely, given $\phi \in S_\lambda(K)$ we define

$$\mathbf{f}_\phi(g_\infty, g_f) := \det(g_\infty)^{\frac{w-\kappa}{2}-1} \cdot j(g_\infty, i)^{\kappa+2} \phi(g_\infty g_f).$$

Here $g_\infty \in \mathrm{GL}_2(F_\infty)^\circ$ and $g_f \in \mathrm{GL}_2(\mathbf{A}_{F,f})$. It is straightforward to see that $g_\infty \mapsto \mathbf{f}_\phi(g_\infty, g_f)$ is invariant under right-multiplication by K_∞° and thus descends to a function on $D_\infty \times \mathrm{GL}_2(\mathbf{A}_{F,f})$.[8] It is also readily verified that $\mathbf{f}_\phi \in S^{\mathrm{hol}}(K)$. To go backwards, given $\mathbf{f} \in S_\lambda^{\mathrm{hol}}(K)$, view it as a function on $\mathrm{GL}_2(F_\infty)^\circ \times \mathrm{GL}_2(\mathbf{A}_{F,f})$. Then define $\phi_{\mathbf{f}}$ on the same domain by

$$\phi_{\mathbf{f}}(g) := \det(g_\infty)^{1-\frac{w-\kappa}{2}} j(g_\infty, i)^{-\kappa-2} \mathbf{f}(g_\infty, g_f)$$

for $g = g_\infty g_f \in \mathrm{GL}_2(F_\infty)^\circ \times \mathrm{GL}_2(\mathbf{A}_{F,f})$. Finally, extend $\phi_{\mathbf{f}}$ to all of $\mathrm{GL}_2(\mathbf{A}_F)$ by (2.3.4). We finally remark that $\phi \leftrightarrow \mathbf{f}_\phi$ and $\mathbf{f} \leftrightarrow \phi_{\mathbf{f}}$ are clearly compatible with right translation by $g_f \in \mathrm{GL}_2(\mathbf{A}_{F,f})$.

[8] To be clear: to compute $\mathbf{f}_\phi(z, g_f)$ one finds a $g_\infty \in \mathrm{GL}_2(F_\infty)^\circ$ such that $g_\infty \cdot i = z$ and then computes $\mathbf{f}_\phi(g_\infty, g_f)$ by the formula we just gave.

3.2. **Hecke operators, Fourier expansions and newforms.** The main goal of this subsection to make precise the notion of the newform associated to a cohomological cuspidal automorphic representation π. We will also record information about Hecke operators and Fourier expansions. We leave transcription of the discussion to $S_\lambda^{\mathrm{hol}}(K)$ to the reader.

Let K be a compact open subgroup in $\mathrm{GL}_2(\mathbf{A}_{F,f})$ and $g \in \mathrm{GL}_2(\mathbf{A}_{F,f})$. The double coset KgK can be decomposed $KgK = \bigcup_i x_i K$ into a finite disjoint union of right K-cosets. Then, for any cohomological weight λ we get a Hecke operator $[KgK]$ acting on the space $S_\lambda(K)$ by

$$([KgK]\phi)(g) = \sum \phi(gx_i) \qquad (\phi \in S_\lambda(K)). \tag{3.2.1}$$

The operator $[KgK]$ is independent of the choice of the x_i's.

For the rest of the subsection we are interested in K of the form $K_1(\mathfrak{n})$ (see (2.3.6)) for $\mathfrak{n} \subset \mathcal{O}_F$ an integral ideal.

Definition 3.2.1. Let $\mathfrak{m} \subset \mathcal{O}_F$ be an integral ideal, written $\mathfrak{m} = \prod_v \mathfrak{p}_v^{m_v}$, and $\varpi_{\mathfrak{m}} = \prod_v \varpi_v^{m_v} \in \mathbf{A}_{F,f}^\times$.

1. $T_{\mathfrak{m}} := [K_1(\mathfrak{n}) \left(\begin{smallmatrix} \varpi_{\mathfrak{m}} & \\ & 1 \end{smallmatrix}\right) K_1(\mathfrak{n})]$.
2. If $(\mathfrak{m}, \mathfrak{n}) = 1$, $S_{\mathfrak{m}} := [K_1(\mathfrak{n}) \left(\begin{smallmatrix} \varpi_{\mathfrak{m}} & \\ & \varpi_{\mathfrak{m}} \end{smallmatrix}\right) K_1(\mathfrak{n})]$.
3. When $\mathfrak{m} = \mathfrak{p}_v$ is a prime ideal we write $T_v := T_{\mathfrak{p}_v}$ and $S_v = S_{\mathfrak{p}_v}$ (when $(\mathfrak{p}_v, \mathfrak{n}) = 1$).

We denote $\mathbf{T}_{\mathbf{Z}}(K_1(\mathfrak{n}))$ the $\mathbf{Z}$-algebra abstractly generated by the Hecke operators. So, for each cohomological weight λ we have a natural morphism of $\mathbf{C}$-algebras

$$\mathbf{T}_{\mathbf{C}}(K_1(\mathfrak{n})) := \mathbf{T}_{\mathbf{Z}}(K_1(\mathfrak{n})) \otimes_{\mathbf{Z}} \mathbf{C} \to \mathrm{End}_{\mathbf{C}}(S_\lambda(K_1(\mathfrak{n}))).$$

Remark 3.2.2. We will assume the reader is familiar with basic properties of the $T_{\mathfrak{m}}$ (see [40, Section 5.6] for example). For instance, $T_{\mathfrak{m}}$ and $S_{\mathfrak{m}}$, when defined, are independent of the choice of uniformizers and they are multiplicative over co-prime ideals $\mathfrak{m}$ because the double coset representatives x_i as in (3.2.1) are calculated "locally at $\mathfrak{m}$" in that they can be chosen to be $\left(\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}\right)$ at each place v where $\mathfrak{p}_v \nmid \mathfrak{n}$.

Remark 3.2.3. If $\mathfrak{m} \mid \mathfrak{n}$, then we will sometimes use the notations $U_{\mathfrak{m}} := T_{\mathfrak{m}}$, $U_v := T_v$, etc. Let us recall an explicit formula in that case. When $\mathfrak{m} \mid \mathfrak{n}$, one may check that the representatives $K_1(\mathfrak{n}) \left(\begin{smallmatrix} \varpi_{\mathfrak{m}} & \\ & 1 \end{smallmatrix}\right) K_1(\mathfrak{n})/K_1(\mathfrak{n})$ can be chosen to be of the form $\left(\begin{smallmatrix} \varpi_{\mathfrak{m}} & a \\ & 1 \end{smallmatrix}\right)$ where a runs over a choice of representatives in $\prod_{v \mid \mathfrak{m}} \mathcal{O}_v$ for $\prod_{\mathfrak{p}_v \mid \mathfrak{m}} \mathcal{O}_v/\mathfrak{m}\mathcal{O}_v$. So, we will often write expressions like

$$(U_{\mathfrak{m}}\phi)(g) = \sum_{a \in \mathcal{O}_v/\mathfrak{m}\mathcal{O}_v} \phi\left(g \left(\begin{smallmatrix} \varpi_v & a \\ & 1 \end{smallmatrix}\right)\right), \tag{3.2.2}$$

omitting the choices of lifts. This makes clear, for instance, that $U_{\mathfrak{p}_v^j} = U_v^j$ for all integers $j \geq 0$.

Remark 3.2.4. If $\mathfrak{p}_v \nmid \mathfrak{n}$ then there is a formula similar to (3.2.2) for T_v. Specifically,

$$(T_v\phi)(g) = \phi(g \left(\begin{smallmatrix} 1 & \\ & \varpi_v \end{smallmatrix}\right)) + \sum_{a \in \mathcal{O}_v/\varpi_v\mathcal{O}_v} \phi\left(g \left(\begin{smallmatrix} \varpi_v & a \\ & 1 \end{smallmatrix}\right)\right).$$

Thus T_v "is equal to" $U_v + V_v^-$ where V_v^- means translation by $\left(\begin{smallmatrix} 1 & \\ & \varpi_v \end{smallmatrix}\right)$ (see Section 3.4 below). The quotes refer to T_v being the bona fide endomorphism of $S_\lambda(K_1(\mathfrak{n}))$ given by Definition 3.2.1 whereas U_v (resp. V_v^-) means the formal operator on functions $\mathrm{GL}_2(\mathbf{A}_F) \to \mathbf{C}$ given by (3.2.2) (resp. right translation by $\left(\begin{smallmatrix} 1 & \\ & \varpi_v \end{smallmatrix}\right)$). Their sum $U_v + V_v^-$ happens to be well-defined on $S_\lambda(K_1(\mathfrak{n}))$. See the calculation in Proposition 3.4.4 below.

In this article, an eigenform means an element $\phi \in S_\lambda(K_1(\mathfrak{n}))$ such that there exists a $\mathbf{C}$-algebra morphism $\psi : \mathbf{T}_{\mathbf{C}}(K_1(\mathfrak{n})) \to \mathbf{C}$ such that $T\phi = \psi(T)\phi$ for all $T \in \mathbf{T}(K_1(\mathfrak{n}))$. If ϕ is an eigenform then we refer to $\psi = \psi_\phi$ as its Hecke eigensystem.

An eigenform is only possibly unique up to scalar, but we can normalize it in a natural way using Fourier expansions. Start by writing $e_{\mathbf{Q}} : \mathbf{A}_{\mathbf{Q}} \to \mathbf{C}^\times$ for the natural non-degenerate character

$$e_{\mathbf{Q}}(x) = e^{2\pi i x_\infty} e^{-2\pi i \{x_f\}},$$

where $\{-\}$ is the morphism on the finite adeles given by the composition

$$\{-\} : \mathbf{A}_{\mathbf{Q},f} \to \mathbf{A}_{\mathbf{Q},f}/\widehat{\mathbf{Z}} \simeq \mathbf{Q}/\mathbf{Z} \hookrightarrow \mathbf{R}/\mathbf{Z}.$$

Then, define $e_F : \mathbf{A}_F \to \mathbf{C}^\times$ to be the composition $e_F := e_{\mathbf{Q}} \circ \mathrm{tr}_{F/\mathbf{Q}}$. Next, if $\lambda = (\kappa, w)$ is a cohomological weight, then we define $W_\lambda : F_\infty^\times \to \mathbf{C}$ (an Archimedean Whittaker function) to be

$$W_\lambda(x_\infty) := \prod_{\sigma \in \Sigma_F} |x_\sigma|^{\frac{\kappa_\sigma - w}{2}} e^{-2\pi |x_\sigma|}.$$

Finally, we set two more notations. If $x_f \in \mathbf{A}_{F,f}$, then we define $[x_f]$ to be the fractional ideal $F \cap x_f \widehat{\mathcal{O}}_F$ and we also write $\mathcal{D}_{F/\mathbf{Q}}$ for the different ideal associated to the extension $F/\mathbf{Q}$.

Proposition 3.2.5. *For each $\phi \in S_\lambda(K_1(\mathfrak{n}))$ there exists a uniquely determined function $\widetilde{a}_\phi : \mathbf{A}_{F,f}^\times \to \mathbf{C}$ such that $\widetilde{a}_\phi(x_f)$ depends only on $[x_f]$ and*

$$\phi\left(\left(\begin{smallmatrix} x & y \\ & 1 \end{smallmatrix}\right)\right) = |x|_{\mathbf{A}_F} \sum_{\xi \in F_+^\times} \widetilde{a}_\phi(\xi x_f) W_\lambda(\xi x_\infty) e_F(\xi y). \tag{3.2.3}$$

Moreover, $\widetilde{a}_\phi(x_f) = 0$ if $[x_f]\mathcal{D}_{F/\mathbf{Q}}$ is not integral.

Proof. See [40, Theorem 5.8] (also, [49, Theorem 6.1]). ☐

Definition 3.2.6. Let $\phi \in S_\lambda(K_1(\mathfrak{n}))$.

(1) If $\mathfrak{m} \subset \mathcal{O}_F$ is an integral ideal, then $a_\phi(\mathfrak{m}) := \widetilde{a}_\phi(\xi x_f)$ for any choice of $\xi \in F_+^\times$ and $x_f \in \mathbf{A}_{F,f}^\times$ such that $\mathfrak{m} = [\xi x_f]\mathcal{D}_{F/\mathbf{Q}}$.
(2) We say that ϕ is a normalized if $a_\phi(\mathcal{O}_F) = 1$.

Remark 3.2.7. For each $\mathfrak{m}$, the function $\phi \mapsto a_\phi(\mathfrak{m})$ is linear. It is also helpful to note that $a_\phi(\mathfrak{m}) = a_{T_{\mathfrak{m}}\phi}(\mathcal{O}_F)$ (see [49, Corollary 6.2] where the central character is not fixed and [80, Chapter VI]). Combining these points, if ϕ is an eigenvector for $T_{\mathfrak{m}}$ and $a_\phi(\mathcal{O}_F) = 0$, then $a_\phi(\mathfrak{m}) = 0$ as well.

Proposition 3.2.8. *Let $\phi \in S_\lambda(K_1(\mathfrak{n}))$ be a normalized eigenform.*

(1) If $\mathfrak{m}$ is an integral ideal, then $a_\phi(\mathfrak{m}) = \psi_\phi(T_{\mathfrak{m}})$.
(2) ϕ has a central character ω_ϕ of conductor dividing $\mathfrak{n}$, and $\omega_\phi(\varpi_v) = \psi_\phi(S_v)$ for $\mathfrak{p}_v \nmid \mathfrak{n}$.

Proof. For (1), see the end of [40, Section 5.9] (and [49, Corollary 6.2]). For part (2), we give a standard argument. If $x \in \mathbf{A}_{F,f}^\times$ then the translate $x \cdot \phi$ is a $T_{\mathfrak{m}}$-eigenvector with the same eigenvalue as ϕ, so Remark 3.2.7 above implies $a_{x\cdot\phi}(\mathcal{O}) \neq 0$. So, by multiplicity one, $x \cdot \phi = \omega_\phi(x)\phi$ for some non-zero constant $\omega_\phi(x)$. The assertions about ω_ϕ follow immediately from Definitions 3.1.3 and 3.2.1. ☐

If $\delta \in \widehat{\mathcal{O}}_F$ and $\mathfrak{n}'$ is an integral ideal with $\mathfrak{n}\widehat{\mathcal{O}}_F \subset \delta\mathfrak{n}'\widehat{\mathcal{O}}_F$, then $\phi \mapsto \phi_\delta(g) := \phi\left(g \left(\begin{smallmatrix} 1/\delta & \\ & 1 \end{smallmatrix}\right)\right)$ gives a well-defined morphism $j_{\mathfrak{n}',\delta} : S_\lambda(K_1(\mathfrak{n}')) \to S_\lambda(K_1(\mathfrak{n}))$. The Hecke-stable subspace $S_\lambda^{\mathrm{new}}(K_1(\mathfrak{n})) \subset S_\lambda(K_1(\mathfrak{n}))$ is the orthogonal complement of $\sum_{\mathfrak{n} \subsetneq \mathfrak{n}'} \mathrm{im}(j_{\mathfrak{n}',\delta})$ under the Petersson product (see [49, Section 3] or [40, Sections 5.7-8]). We highlight our convention for the word "newform":[9]

Definition 3.2.9. A newform ϕ of level $\mathfrak{n}$ is a normalized eigenform $\phi \in S_\lambda^{\mathrm{new}}(K_1(\mathfrak{n}))$.

[9]Note that by [40, Theorem 5.7], an equivalent definition would be to require that $\phi \in S_\lambda^{\mathrm{new}}(K_1(\mathfrak{n}))$ which is normalized and an eigenform just for almost all the Hecke operators T_v.

If π is a cohomological cuspidal automorphic representation then there exists an ideal $\mathfrak{n}$, called the conductor of π, which is maximal among all ideals with $\pi_f^{K_1(\mathfrak{n})} \neq (0)$. A famous result of Casselman ([29]) implies in fact that $\dim_{\mathbf{C}} \pi_f^{K_1(\mathfrak{n})} = 1$.

Definition/Proposition 3.2.10. *If π is a cohomological cuspidal automorphic representation of conductor $\mathfrak{n}$, then there exists a unique newform ϕ_π of level $\mathfrak{n}$ such that ϕ_π generates the representation π under the isomorphism (3.1.3). We call ϕ_π the newform associated to π.*

Proof. From Casselman's theorem, we immediately get a unique normalized cuspform $\phi_\pi \in S_\lambda(K_1(\mathfrak{n}))$ which generates π under (3.1.3). Its unicity implies it is a normalized eigenform, and checking it is a newform is straightforward (see [40, Theorem E.1] for instance). □

Now let π be a cohomological cuspidal automorphic representation. We define its Hecke eigensystem ψ_π to be $\psi_\pi = \psi_{\phi_\pi}$ where ϕ_π is the associated newform, $a_\pi(\mathfrak{m}) = \psi_\pi(T_\mathfrak{m})$ for each integral ideal $\mathfrak{m}$, and the Hecke field of π is $\mathbf{Q}(\pi) := \mathbf{Q}(\psi_\pi(T) \mid T \in \mathbf{T}_{\mathbf{Z}}(K_1(\mathfrak{n})))$.

Proposition 3.2.11. *If π is a cohomological cuspidal automorphic representation then $\mathbf{Q}(\pi)$ is a finite extension of $\mathbf{Q}$.*

Proof. See [72, Proposition 2.8] (and replace ϕ_π by $\mathbf{f}_{\phi_\pi}$). □

3.3. ***L*-functions.** Suppose that $\phi \in S_\lambda(K_1(\mathfrak{n}))$. Its L-series is defined to be

$$L(\phi, s) := \sum_{\mathfrak{m} \subset \mathcal{O}_F} a_\phi(\mathfrak{m}) N_{F/\mathbf{Q}}(\mathfrak{m})^{-s}, \tag{3.3.1}$$

where the sum $\mathfrak{m}$ runs over integral ideals of F and $N_{F/\mathbf{Q}}(-)$ means the absolute norm. The series (3.3.1) converges absolutely for the real part of s sufficiently large. Further, it admits analytic continuation to all $s \in \mathbf{C}$ as we now recall.

Define $\Gamma_{\mathbf{C}}(s) = (2\pi)^{-s}\Gamma(s)$ and then complete $L(\phi, s)$ by defining

$$\Lambda(\phi, s) := \Gamma_{\mathbf{C}}\left(s + \frac{\kappa - w}{2}\right) L(\phi, s) = \left(\prod_{\sigma \in \Sigma_F} \Gamma_{\mathbf{C}}\left(s + \frac{\kappa_\sigma - w}{2}\right)\right) L(\phi, s).$$

We can also define the Mellin transform of ϕ

$$\mathbf{M}(\phi, s) := \int_{F^\times \backslash \mathbf{A}_F^\times} \phi\left(\left(\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\right)\right) |x|^s d^\times x. \tag{3.3.2}$$

The integral (3.3.2) is absolutely convergent for all $s \in \mathbf{C}$ ([25, Section 3.5]). Here, $d^\times x$ is the natural Haar measure on $\mathbf{A}_F^\times$: $d^\times x_\infty$ is the canonical measure $\prod_\sigma \frac{dx_\sigma}{|x_\sigma|}$ on $F_\infty^\times$ and $d^\times x_v$ is the unique multiple of $\frac{dx_v}{|x_v|_v}$ on $F_v^\times$ such that $\mathcal{O}_v^\times$ has measure one.

Now write $\Delta_{F/\mathbf{Q}}$ for the absolute discriminant $\Delta_{F/\mathbf{Q}} = N_{F/\mathbf{Q}}(\mathcal{D}_{F/\mathbf{Q}})$. The analytic continuation of $\Lambda(\phi, s)$ follows from the proposition.

Proposition 3.3.1. *If $\phi \in S_\lambda(K_1(\mathfrak{n}))$, then $\mathbf{M}(\phi, s) = \Delta_{F/\mathbf{Q}}^{s+1}\Lambda(\phi, s+1)$.*

We include a proof of this proposition for completeness, especially as this integral expression of the (completed) L-function is crucial for the algebraicity of the special values (see Section 4.5).

Proof of Proposition 3.3.1. By weak approximation, the integral (3.3.2) is unchanged by replacing $F^\times \backslash \mathbf{A}_F^\times$ by $F_+^\times \backslash \mathbf{A}_{F,+}^\times$. Further, $x \mapsto \phi\left(\left(\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\right)\right) |x|_{\mathbf{A}_F}^s$ is invariant under $x \mapsto \xi x$ for $\xi \in F^\times$. Thus, using the Fourier expansion (Proposition 3.2.5) and unfolding the integral (3.3.2), we get

$$\begin{aligned}\int_{F_+^\times\backslash \mathbf{A}_{F,+}^\times} \phi\left(\begin{pmatrix} x & \\ & 1\end{pmatrix}\right)|x|_{\mathbf{A}_F}^s d^\times x &= \int_{F_+^\times\backslash \mathbf{A}_{F,+}^\times}\left(\sum_{\xi\in F_+^\times}\tilde{a}_\phi(\xi x_f)|x|_{\mathbf{A}_F}^{s+1}W_\lambda(\xi x_\infty)\right)d^\times x\\ &= \int_{\mathbf{A}_{F,+}^\times}\tilde{a}_\phi(x_f)|x|_{\mathbf{A}_F}^{s+1}W_\lambda(x_\infty)d^\times x\\ &= \left(\int_{(F_\infty^\times)^\circ} x_\infty^{\frac{1+s+\kappa-w}{2}}e^{-2\pi x_\infty}\frac{dx_\infty}{x_\infty}\right)\cdot\left(\int_{\mathbf{A}_{F,f}^\times}\tilde{a}_\phi(x_f)|x_f|_{\mathbf{A}_F}^{s+1}d^\times x_f\right).\end{aligned} \tag{3.3.3}$$

The first integral in the product (3.3.3) is clearly

$$\begin{aligned}\int_{(F_\infty^\times)^\circ} x_\infty^{\frac{1+s+\kappa-w}{2}}e^{-2\pi x_\infty}\frac{dx_\infty}{x_\infty} &= \prod_{\sigma\in\Sigma_F}\int_0^\infty\left(\frac{x_\sigma}{2\pi}\right)^{1+s+\frac{\kappa_\sigma-w}{2}}e^{-x_\sigma}\frac{dx_\sigma}{x_\sigma}\\ &= \Gamma_{\mathbf{C}}\left(1+s+\frac{\kappa-w}{2}\right).\end{aligned} \tag{3.3.4}$$

For the second integral in (3.3.3), recall that $\tilde{a}_\phi(x_f)$ depends only on $[x_f]$ and is trivial unless $[x_f]\mathcal{D}_{F/\mathbf{Q}}$ is an integral ideal. Thus we may compute the integral

$$\begin{aligned}\int_{\mathbf{A}_{F,f}^\times}\tilde{a}_\phi(x_f)|x_f|^{s+1}d^\times x_f &= \sum_{\mathfrak{m}\subset\mathcal{O}_F}a_\phi(\mathfrak{m})\int_{\mathfrak{m}\mathcal{D}_{F/\mathbf{Q}}^{-1}\widehat{\mathcal{O}}_F^\times}|x_f|_{\mathbf{A}_F}^{s+1}d^\times x_f\\ &= \Delta_{F/\mathbf{Q}}^{s+1}\sum_{\mathfrak{m}\subset\mathcal{O}_F}a_\phi(\mathfrak{m})N_{F/\mathbf{Q}}(\mathfrak{m})^{-(1+s)}.\end{aligned} \tag{3.3.5}$$

For the final equality we used that $x_f\in\mathfrak{m}\mathcal{D}_{F/\mathbf{Q}}^{-1}\widehat{\mathcal{O}}_F^\times$ if and only if $|x_f|_{\mathbf{A}_F}=\Delta_{F/\mathbf{Q}}N_{F/\mathbf{Q}}(\mathfrak{m})^{-1}$. Putting (3.3.4) and (3.3.5) back into (3.3.3), the proof is complete. □

If ϕ is a normalized eigenform with central character ω_ϕ (Proposition 3.2.8), the Dirichlet series $L(\phi,s)$ admits an Euler product expansion $L(\phi,s)=\prod_v L_v(\phi,s)$, where

$$L_v(\phi,s)^{-1}=\begin{cases}1-a_\phi(\mathfrak{p}_v)q_v^{-s}+\omega_\phi(\varpi_v)q_v^{1-2s} & (\text{if } \mathfrak{p}_v\nmid\mathfrak{n});\\ 1-a_\phi(\mathfrak{p}_v)q_v^{-s} & (\text{if } \mathfrak{p}_v\mid\mathfrak{n}).\end{cases} \tag{3.3.6}$$

See [40, Section 5.12.1]. If, furthermore, $\phi=\phi_\pi$ is the newform associated to a cohomological cuspidal automorphic representation π (Proposition 3.2.10) then this is the same as the Euler product expresssion

$$L(\phi,s)=L(\pi,s):=\prod_v L_v(\pi_v,s) \tag{3.3.7}$$

where the product runs over finite places v of F and the local L-factor $L_v(\pi_v,s)$ is defined to be

$$L_v(\pi_v,s):=\det\left(1-q_v^{-s}\operatorname{Frob}_v|_{r(\pi_v)^{I_v,N=0}}\right)^{-1}.$$

Here, $r(\pi_v)$ is Weil–Deligne representation associated to π_v via the normalized local Langlands correspondence (see Section 1.10).

3.4. **Refinements.** In this subsection we discuss the notion of (p-)refinements of cohomological cuspidal automorphic representations. Fix a cohomological weight λ.

If v is a finite place of F and ϖ_v is a choice of uniformizer then write $V_v^- = \left(\begin{smallmatrix} 1 & \\ & \varpi_v \end{smallmatrix}\right)$. If $\phi \in S_\lambda(K)$, then the translate $V_v^-\phi$ belongs to $S_\lambda(V_v^- K (V_v^-)^{-1})$ and explicitly depends on the choice of ϖ_v. Its independence of ϖ_v can be shown if the level is prime to v.

Lemma 3.4.1. *Let $\mathfrak{n}$ be an integral ideal, $\phi \in S_\lambda(K_1(\mathfrak{n}))$, and assume that $\mathfrak{p}_v \nmid \mathfrak{n}$.*

(1) $V_v^-\phi$ belongs to $S_\lambda(K_1(\mathfrak{n}\mathfrak{p}_v))$ and it is independent of the choice of ϖ_v.
(2) If $c \in \mathbf{C}$, then $a_\phi(\mathcal{O}) = a_{(1-cV_v^-)\phi}(\mathcal{O})$. In particular, if ϕ is normalized then so is $(1-cV_v^-)\phi$.
(3) $U_v V_v^- \phi = q_v S_v \phi$.
(4) If $\mathfrak{m}$ is an integral ideal and $\mathfrak{p}_v \nmid \mathfrak{m}$, then $V_v^- T_\mathfrak{m}\phi = T_\mathfrak{m} V_v^-\phi$.

Proof. Since $\mathfrak{p}_v \nmid \mathfrak{n}$, $\begin{pmatrix} 1 & \\ & \mathcal{O}_v^\times \end{pmatrix} \subset K_1(\mathfrak{n})$ and thus $V_v^-\phi$ is independent of the choice of ϖ_v. That it is an automorphic form of level $K_1(\mathfrak{n}\mathfrak{p}_v)$ follows from the straightforward inclusion $K_1(\mathfrak{n}\mathfrak{p}_v) \subset V_v^- K_1(\mathfrak{n})(V_v^-)^{-1}$. This completes the proof of (1).

We will check (2) using Fourier expansions. As mentioned in Remark 3.2.7, $\phi \mapsto a_\phi(\mathfrak{m})$ is linear. So, it suffices to show that $a_{V_v^-\phi}(\mathcal{O}_F) = 0$. To this end, we note the relation

$$\left(\begin{smallmatrix} x & y \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi_v \end{smallmatrix}\right) = \begin{pmatrix} x\varpi_v^{-1} & y \\ 0 & 1 \end{pmatrix}\left(\begin{smallmatrix} \varpi_v & 0 \\ 0 & \varpi_v \end{smallmatrix}\right). \tag{3.4.1}$$

By (3.4.1) and Proposition 3.2.5 we deduce that

$$\widetilde{a}_{V_v^-\phi}(\xi x_f) = |\varpi_v^{-1}|_{\mathbf{A}_F} \widetilde{a}_{S_v\phi}(\xi x_f \varpi_v^{-1}). \tag{3.4.2}$$

In particular, if ξ and x_f are chosen so that $[\xi x_f]\mathcal{D}_{F/\mathbf{Q}} = \mathcal{O}_F$ then certainly $[\xi x_f \varpi_v^{-1}]\mathcal{D}_{F/\mathbf{Q}}$ is not an integral ideal. But then the quantity (3.4.2) vanishes by Proposition 3.2.5, completing the proof of (2).

For part (3), we have already checked in part (1) that $V_v^-\phi \in S_\lambda(K_1(\mathfrak{n}\mathfrak{p}_v))$. Thus by Remark 3.2.3 and (3.4.1) we get

$$(U_v V_v^-\phi)(g) = \sum_{a \in \mathcal{O}_v/\varpi_v\mathcal{O}_v} \phi\left(g\left(\begin{smallmatrix} \varpi_v & a \\ & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & \\ & \varpi_v \end{smallmatrix}\right)\right) = \sum_{a \in \mathcal{O}_v/\varpi_v\mathcal{O}_v} \phi(g\left(\begin{smallmatrix} 1 & a \\ & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} \varpi_v & \\ & \varpi_v \end{smallmatrix}\right)). \tag{3.4.3}$$

The a-th term in the sum (3.4.3) is equal to $(S_v\phi)(g\left(\begin{smallmatrix} 1 & a \\ & 1 \end{smallmatrix}\right))$ which equals $(S_v\phi)(g)$ because $\left(\begin{smallmatrix} 1 & a \\ & 1 \end{smallmatrix}\right) \in K_1(\mathfrak{n})$ and $S_v\phi \in S_\lambda(K_1(\mathfrak{n}))$. Thus from (3.4.3) we get

$$(U_v V_v^-\phi)(g) = \sum_{a \in \mathcal{O}_v/\varpi_v\mathcal{O}_v} (S_v\phi)(g) = (q_v S_v\phi)(g),$$

as promised.

Part (4) is clear. Indeed, the matrices involved in the definition of $T_\mathfrak{m}$ are the identity at v because $\mathfrak{p}_v \nmid \mathfrak{m}$ (Remark 3.2.2), so they obviously commute with the action of V_v^-. □

For the rest of this subsection, we fix a cohomological cuspidal automorphic representation π and a prime p. We write $\mathfrak{n}$ for the conductor of π (not necessarily prime to p) and assume that π has weight λ. We also denote $\phi_\pi \in S_\lambda(K_1(\mathfrak{n}))$ for the associated newform (Proposition 3.2.10).

Definition 3.4.2.

(1) π is called p-refinable if for each place $v \mid p$, π_v is either an unramified principal series representation or an unramified twist of the Steinberg.[10]

[10]There is a more general notion of π being "finite slope" at p (we will not use it). Specifically one could say that π is finite slope at p provided the smooth $\mathrm{GL}_2(F_v)$-representation π_v has non-zero Jacquet module $(\pi_v)_{N_v}$ for all $v \mid p$

(2) If π is p-refinable, then a p-refinement α for π is the choice of $\alpha = (\alpha_v)_{v|p}$ of one of the following equivalent data.
   (a) For each v where π_v is an unramified principal series, α_v is a root of $X^2 - a_\pi(\mathfrak{p}_v)X + \omega_\pi(\varpi_v)q_v$, and for each v where π_v is Steinberg, $\alpha_v = a_\pi(\mathfrak{p}_v)$.
   (b) $\alpha_v = \chi_v(\varpi_v)$ where χ_v is the choice of smooth character $\chi_v : F_v^\times \to \mathbf{C}^\times$ such that $\chi_v \circ \mathrm{Art}_{F_v}^{-1}$ is a subrepresentation of $r(\pi_v)$.

(3) If α is a p-refinement of π, then the associated refined eigenform is

$$\phi_{\pi,\alpha} := \prod_{\substack{v|p \\ \mathfrak{p}_v \nmid \mathfrak{n}}} (1 - \alpha_v^{-1} V_v^-) \cdot \phi_\pi.$$

The equivalence in parts (a) and (b) of Definition 3.4.2(2) is the same unwinding of definitions that goes into (3.3.7). As a matter of course, we will often abuse language and simply say things like "Let α be a p-refinement for π..." by which we mean "Assume that π is p-refineable and that α is a p-refinement for π..." (we already did this in part (3) of Definition 3.4.2 for instance).

Remark 3.4.3. We stress that if $v \mid p$ and $\mathfrak{p}_v \mid \mathfrak{n}$ then π_v is necessarily a Steinberg representation, so $\alpha_v = a_\pi(\mathfrak{p}_v)$ already, and $\mathfrak{p}_v^2 \nmid \mathfrak{n}$.

Recall that we write $\mathbf{p} = \prod_{v|p} \mathfrak{p}_v$ for the product of the primes above p in F.

Proposition 3.4.4. *Let α be a p-refinement for π.*

(1) $\phi_{\pi,\alpha} \in S_\lambda(K_1(\mathfrak{n} \cap \mathbf{p}))$

(2) $\phi_{\pi,\alpha}$ is a normalized eigenform which generates the representation π under (3.1.3) and the Fourier coefficients/Hecke eigenvalues of $\phi_{\pi,\alpha}$ are given by

$$a_{\phi_{\pi,\alpha}}(\mathfrak{p}_v^j) = \begin{cases} a_\pi(\mathfrak{p}_v^j) & \text{if } v \nmid p; \\ \alpha_v^j & \text{if } v \mid p. \end{cases}$$

In particular, $U_v(\phi_{\pi,\alpha}) = \alpha_v \phi_{\pi,\alpha}$ for each $v \mid p$.

Proof. The fact that $\phi_{\pi,\alpha}$ lies in $S_\lambda(K_1(\mathfrak{n}\cap\mathbf{p}))$ and is normalized (thus non-zero!) follows from repeated uses of parts (1) and (2) in Lemma 3.4.1. Since $\phi_{\pi,\alpha}$ is a $\mathrm{GL}_2(\mathbf{A}_{F,f})$-translate of ϕ_π, it lies in π under (3.1.3) and thus generates π since π is irreducible and $\phi_{\pi,\alpha}$ is non-zero. This proves parts (1) and the normalized portion of part (2).

It remains to check that $\phi_{\pi,\alpha}$ is an eigenform with the prescribed Hecke eigensystem. For that, it is enough to show that $\phi_{\pi,\alpha}$ is a U_v-eigenvector with eigenvalue α_v when $v \mid p$ and $\mathfrak{p}_v \nmid \mathfrak{n}$ (by Lemma 3.4.1(4) and the end of Remark 3.2.3). So, fix $v \mid p$ and $\mathfrak{p}_v \nmid \mathfrak{n}$. Then, α_v is a root of $X^2 - a_\pi(\mathfrak{p}_v)X + \omega_\pi(\varpi_v)q_v$. Write $\beta_v = a_\pi(\mathfrak{p}_v) - \alpha_v = \alpha_v^{-1}\omega_\pi(\varpi_v)q_v$ for the other root. Then,

$$U_v(1 - \alpha_v^{-1}V_v^-)\phi_\pi = U_v\phi_\pi - \beta_v\phi_\pi \tag{3.4.4}$$

by Lemma 3.4.1(3). Since the operator T_v on $S_\lambda(K_1(\mathfrak{n}))$ decomposes into a sum $T_v = U_v + V_v^-$ (Remark 3.2.4) we can continue (3.4.4) and get

$$\begin{aligned} U_v(1 - \alpha_v^{-1}V_v^-)\phi_\pi = U_v\phi_\pi - \beta_v\phi_\pi = (T_v - V_v^-)\phi_\pi - \beta_v\phi_\pi = a_\pi(\mathfrak{p}_v)\phi_\pi - V_v^-\phi_\pi - \beta_v\phi_\pi \\ = (\alpha_v - V_v^-)\phi_\pi. \end{aligned}$$

Thus, $(1 - \alpha_v^{-1}V_v)\phi_\pi$ is a U_v-eigenvector with eigenvalue α_v, completing the proof. ∎

([30, Section 3.2]). It follows from Frobenius reciprocity that a p-refinement as in Definition 3.4.2(2) is the equivalent to an eigenspace for the torus action on $(\pi_v)_{N_v}$.

4. Algebraicity of special values

4.1. **Archimedean Hecke operators.** We denote by K any compact open subgroup of $\mathrm{GL}_2(\mathbf{A}_{F,f})$ and N any $(\mathrm{GL}_2^+(F), K)$-bimodule with a left action of a monoid $\Delta \subset \mathrm{GL}_2(\mathbf{A}_{F,f})$ as in Section 2.2. Write $\pi_0(F_\infty^\times) = F_\infty^\times / F_{\infty,+}^\times$ for the component group of $F_\infty^\times$. There is a natural isomorphism $\widehat{\pi_0(F_\infty^\times)} \simeq \{\pm 1\}^{\Sigma_F}$ where $\widehat{\pi_0(F_\infty^\times)}$ is the character group of $\pi_0(F_\infty^\times)$. So, we will often confuse signs $\epsilon \in \{\pm 1\}^{\Sigma_F}$ with the corresponding character of $\pi_0(F_\infty^\times)$. On the other hand, the function $\mathrm{sgn} : F_\infty^\times \to \{\pm 1\}^{\Sigma_F}$ defines a section $\pi_0(F_\infty^\times) \hookrightarrow F_\infty^\times$ of the natural quotient map. We fix this identification. By doing so, we may consider the double coset operator $T_\zeta = [K_\infty^\circ \left(\begin{smallmatrix} \zeta & \\ & 1 \end{smallmatrix}\right) K_\infty^\circ]$ acting on the cohomology $H_c^*(Y_K, N)$ (trivially on N). Since $\left(\begin{smallmatrix} \zeta & \\ & 1 \end{smallmatrix}\right)$ normalizes K_∞°, this operator is just pullback under right multiplication by $\left(\begin{smallmatrix} \zeta & \\ & 1 \end{smallmatrix}\right)$. Since $\left(\begin{smallmatrix} \zeta & \\ & 1 \end{smallmatrix}\right) \subset \mathrm{GL}_2(F_\infty)$, T_ζ obviously commutes with any Hecke action arising from elements of $\Delta \subset \mathrm{GL}_2(\mathbf{A}_{F,f})$. Further, if $\zeta, \zeta' \in \pi_0(F_\infty^\times)$, then $T_\zeta T_{\zeta'} = T_{\zeta\zeta'}$. In particular T_ζ commutes with $T_{\zeta'}$ and $T_\zeta^2 = 1$. Thus T_ζ has only eigenvalues ± 1. If 2 acts invertibly on N, then for each $\epsilon \in \{\pm 1\}^{\Sigma_F}$ we define

$$\mathrm{pr}^\epsilon = \frac{1}{2^d} \sum_{\zeta \in \pi_0(F_\infty^\times)} \epsilon(\zeta) T_\zeta$$

as an endomorphism of $H_c^*(Y_K, N)$. It is an idempotent projector mapping onto

$$H_c^*(Y_K, N)^\epsilon = \{v \in H_c^*(Y_K, N) \mid T_\zeta(v) = \epsilon(\zeta) v \text{ for all } \zeta \in \pi_0(F_\infty^\times)\}.$$

4.2. **The Eichler–Shimura construction.** We now recall a transcendental construction associating a certain differential form to a holomorphic Hilbert modular form. Throughout this subsection we fix a cohomological weight λ.

Recall that $D_\infty = \mathfrak{h}^{\Sigma_F}$. Denote by $\Omega^d(D_\infty)$ the space of $\mathbf{C}$-valued smooth differential forms on D_∞ (as a real manifold). For $z = (z_\sigma)$ the canonical coordinate on D_∞, we define $dz := \wedge_\sigma dz_\sigma \in \Omega^d(D_\infty)$. Here we have to choose an ordering of Σ_F, technically, and so we do that by insisting that dz restricts to dx_∞ / x_∞ along $(F_\infty^\times)^\circ \hookrightarrow D_\infty$ (see Section 2.3). Before the next lemma, we remind ourselves that $\mathrm{GL}_2^+(F)$ acts on both D_∞ and the algebraic local system $\mathscr{L}_\lambda(\mathbf{C})$ defined in Section 2.4.

Lemma 4.2.1. *If $z \in D_\infty$ and $P_z \in \mathscr{L}_\lambda(\mathbf{C})$ is defined by $P_z = (z + X)^\kappa$, then*

$$P_{\gamma(z)} = (\det \gamma)^{\frac{\kappa - w}{2}} j(\gamma, z)^\kappa (\gamma \cdot P_z)$$

for all $\gamma \in \mathrm{GL}_2^+(F)$.

Proof. Clear. ☐

Now denote by $\Omega^d(D_\infty, \mathscr{L}_\lambda(\mathbf{C})) = \Omega^d(D_\infty) \otimes_{\mathbf{C}} \mathscr{L}_\lambda(\mathbf{C})$ the smooth $\mathscr{L}_\lambda(\mathbf{C})$-valued differential forms on D_∞. If K is a neat level, so that Y_K is a smooth real manifold, then we denote by $\Omega^d(Y_K, \mathscr{L}_\lambda(\mathbf{C}))$ the smooth $\mathscr{L}_\lambda(\mathbf{C})$-valued d-forms on Y_K.

Proposition 4.2.2. *Let $K \subset \mathrm{GL}_2(\mathbf{A}_{F,f})$ be a neat compact open subgroup and $\mathbf{f} \in S_\lambda^{\mathrm{hol}}(K)$.*

1. *$\omega_{\mathbf{f}}(z, g_f) := \mathbf{f}(z, g_f)(z + X)^\kappa dz \in \Omega^d(D_\infty, \mathscr{L}_\lambda(\mathbf{C})) \otimes_{\mathbf{C}} \mathcal{C}^\infty(\mathrm{GL}_2(\mathbf{A}_{F,f}), \mathbf{C})$ descends to a closed and rapidly decreasing d-form in $\Omega^d(Y_K, \mathscr{L}_\lambda(\mathbf{C}))$, thus defining a canonical element $\omega_{\mathbf{f}} \in H_c^d(Y_K, \mathscr{L}_\lambda(\mathbf{C}))$.*
2. *If $g \in \mathrm{GL}_2(\mathbf{A}_{F,f})$, then gKg^{-1} is also neat and if $r_g : Y_{gKg^{-1}} \to Y_K$ is right multiplication by g, then $r_g^* \omega_{\mathbf{f}} = \omega_{g \cdot \mathbf{f}}$.*
3. *If $K' \subset K$ is an open subgroup and $\mathrm{pr} : Y_{K'} \to Y_K$ is the projection map, then $\mathrm{pr}^* \omega_{\mathbf{f}} = \omega_{\mathbf{f}}$.*

Proof. Parts (2) and (3) of the proposition are formal. For (1), the descent of $\omega_{\mathbf{f}}$ to Y_K follows from (3.1.4), Lemma 4.2.1 and the chain rule. See [40, Proposition 6.6] for the rest of the claims. ☐

Now let K be any compact open subgroup. We may choose a finite index normal subgroup $K' \subset K$ so that K' is neat. Then we have a natural map $S_\lambda^{\mathrm{hol}}(K') \to H^d_c(Y_{K'}, \mathscr{L}_\lambda(\mathbf{C}))$ given by $\mathbf{f} \mapsto \omega_{\mathbf{f}}$. By part (2) of Proposition 4.2.2, it is equivariant with respect to the action of K/K' on either side, so descends to well-defined map $S_\lambda^{\mathrm{hol}}(K) \to H^d_c(Y_K, \mathscr{L}_\lambda(\mathbf{C}))$. By part (3) of Proposition 4.2.2, construction is independent of the choice of K'.

Definition 4.2.3. If $K \subset \mathrm{GL}_2(\mathbf{A}_{F,f})$ is a compact open subgroup, then the Eichler–Shimura map is the map $\mathrm{ES} : S_\lambda^{\mathrm{hol}}(K) \to H^d_c(Y_K, \mathscr{L}_\lambda(\mathbf{C}))$ defined above.

We will sometimes also write ES for the map $\mathrm{ES} : S_\lambda(K) \to H^d_c(Y_K, \mathscr{L}_\lambda(\mathbf{C}))$ obtained by pre-composing with $\phi \mapsto \mathbf{f}_\phi$. This should cause no confusion. Note as well that parts (2) and (3) of Proposition 4.2.2 imply that ES is Hecke-equivariant. We now state a theorem proven by Hida and its apparent applications.

Theorem 4.2.4. *For any choice of sign $\epsilon \in \{\pm 1\}^{\Sigma_F}$, the composition*

$$\mathrm{pr}^\epsilon \circ \mathrm{ES} : S_\lambda^{\mathrm{hol}}(K) \to H^d_c(Y_K, \mathscr{L}_\lambda(\mathbf{C}))^\epsilon$$

is a Hecke-equivariant injection onto the ϵ-component of the cuspidal cohomology.

Proof. See (4.2) in [49, Section 4] (and also [48, Section 6]). □

Corollary 4.2.5. *Suppose that π is a cohomological cuspidal automorphic representation of weight λ and conductor $\mathfrak{n}$. Assume that $E \subset \mathbf{C}$ is any subfield containing $\mathbf{Q}(\pi)$ and the Galois closure of F inside $\mathbf{C}$. Then, for each choice of sign $\epsilon \in \{\pm 1\}^{\Sigma_F}$,*

$$\dim_E H^d_c(Y_1(\mathfrak{n}), \mathscr{L}_\lambda(E))^\epsilon[\psi_\pi] = 1,$$

where $(-)[\psi_\pi]$ denotes subspace on which the Hecke operators acts through the character ψ_π.

Proof. Since ψ_π has image in E it suffices to check the claim when $E = \mathbf{C}$. By Theorem 4.2.4, we are reduced to showing that $\dim_{\mathbf{C}} S_\lambda(K_1(\mathfrak{n}))[\psi_\pi] = 1$, which in turn reduces to the existence and uniqueness of the newform (Proposition 3.2.10). □

Corollary 4.2.6. *Suppose that π is a cohomological cuspidal automorphic representation of weight λ and conductor $\mathfrak{n}$. Write E of the subfield of $\mathbf{C}$ generated by $\mathbf{Q}(\pi)$ and the Galois closure of F inside $\mathbf{C}$. Then, for each sign character $\epsilon \in \{\pm 1\}^{\Sigma_F}$, there exists $\Omega_\pi^\epsilon \in \mathbf{C}^\times$ such that*

$$\frac{\mathrm{pr}^\epsilon\, \mathrm{ES}(\mathbf{f}_\pi)}{\Omega_\pi^\epsilon} \in H^d_c(Y_1(\mathfrak{n}), \mathscr{L}_\lambda(E))^\epsilon[\psi_\pi].$$

Proof. This follows immediately Corollary 4.2.5. □

Remark 4.2.7. By Corollary 4.2.5, the choice of Ω_π^ϵ in Corollary 4.2.6 is unique up to an element of $E^\times$ (for E as in Corollary 4.2.6). We do not discuss further how to possibly specify these periods.

4.3. **Twisting.** In this subsection we discuss twisting by finite order Hecke characters. We will do this carefully since we will need a less familiar p-adic version of these ideas in Section 5.5. Our treatment here is inspired by [40, Sections 5.10 and 9.4]. Throughout, we fix a cohomological weight λ and an integral ideal $\mathfrak{n}$. We will also let E denote a variable subfield of $\mathbf{C}$ containing the Galois closure of F.

To start, if $t \in \mathbf{A}_{F,f}$ then write $u_t := \left(\begin{smallmatrix} 1 & t \\ & 1 \end{smallmatrix}\right)$. For an integral ideal $\mathfrak{m}$, we write

$$K_{11}(\mathfrak{m}) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\widehat{\mathcal{O}}_F) \mid a, d \equiv 1 \bmod \mathfrak{m}\widehat{\mathcal{O}}_F \text{ and } c \equiv 0 \bmod \mathfrak{m}\widehat{\mathcal{O}}_F\}.$$

Now let $\mathfrak{f}$ be an integral ideal and $t \in \mathfrak{f}^{-1}\widehat{\mathcal{O}}_F$. Then, $K_{11}(\mathfrak{n}\mathfrak{f}^2)_t := u_t^{-1}K_{11}(\mathfrak{n}\mathfrak{f}^2)u_t \subset K_1(\mathfrak{n})$. In particular if $\phi \in S_\lambda(K_1(\mathfrak{n}))$, then $\phi_t(g) := \phi(gu_t)$ is in $S_\lambda(K_{11}(\mathfrak{n}\mathfrak{f}^2))$. We also have a diagram of Hilbert modular varieties

$$\begin{tikzcd} & Y_{11}(\mathfrak{n}\mathfrak{f}^2) \arrow[r, "r_{u_t}"] \arrow[dl, "\mathrm{pr}"'] \arrow[drr, dashed, "v_t"] & Y_{K_{11}(\mathfrak{n}\mathfrak{f}^2)_t} \arrow[dr, "\mathrm{pr}"] & \\ Y_1(\mathfrak{n}\mathfrak{f}^2) & & & Y_1(\mathfrak{n}) \end{tikzcd}$$

where v_t is defined to be the composition as indicated. Since $u_t \in \mathrm{GL}_2(\mathbf{A}_{F,f})$, the identity map defines an isomorphism $v_t^*\mathscr{L}_\lambda(E) \simeq \mathscr{L}_\lambda(E)$ of local systems on $Y_{11}(\mathfrak{n}\mathfrak{f}^2)$.

Lemma 4.3.1. *For each $t \in \mathfrak{f}^{-1}\widehat{\mathcal{O}}_F$, the diagram*

$$\begin{tikzcd} S_\lambda(K_1(\mathfrak{n})) \arrow[r, "\mathrm{ES}"] \arrow[d, "\phi\mapsto\phi_t"'] & H_c^d(Y_1(\mathfrak{n}), \mathscr{L}_\lambda(\mathbf{C})) \arrow[d, "v_t^*"] \\ S_\lambda(K_{11}(\mathfrak{n}\mathfrak{f}^2)) \arrow[r, "\mathrm{ES}"'] & H_c^d(Y_{11}(\mathfrak{n}\mathfrak{f}^2), \mathscr{L}_\lambda(\mathbf{C})). \end{tikzcd}$$

is commutative.

Proof. See parts (2) and (3) of Proposition 4.2.2. □

Now consider a finite order Hecke character θ and let $\mathfrak{f}$ be an ideal dividing the conductor of θ.[11] Write $\Upsilon_\mathfrak{f} = \mathfrak{f}^{-1}\widehat{\mathcal{O}}_F/\widehat{\mathcal{O}}_F$ and $\Upsilon_\mathfrak{f}^\times$ for cosets represented by x/f with $f \in \mathfrak{f}$ and $x \in \widehat{\mathcal{O}}_F^\times$. We naturally view θ as a character on $\Upsilon_\mathfrak{f}^\times$. If $t \in \Upsilon_\mathfrak{f}^\times$ write $t_0 \in \widehat{\mathcal{O}}_F$ for a lift of t which is zero outside of $v \mid \mathfrak{f}$. Then, for $\phi \in S_\lambda(K_1(\mathfrak{n}))$ then we define $\mathrm{tw}_\theta(\phi)$ by

$$\mathrm{tw}_\theta(\phi)(g) = \theta(\det g) \sum_{t \in \Upsilon_\mathfrak{f}^\times} \theta(t)\phi\left(gu_{t_0}\right).$$

By [40, Proposition 5.11], this defines a linear map $\mathrm{tw}_\theta : S_\lambda(K_1(\mathfrak{n})) \to S_\lambda(K_1(\mathfrak{n}\mathfrak{f}^2))$.

On the other hand, suppose E contains the values of θ. Then, $\theta_{\det}(g) := \theta(\det g)$ defines an element of $H^0(Y_{11}(\mathfrak{n}\mathfrak{f}^2), E)$ (compare with Remark 4.3.3 below). So, cup product with $\theta_{\det}$ defines an endomorphism of $H_c^*(Y_{11}(\mathfrak{n}\mathfrak{f}^2), \mathscr{L}_\lambda(E))$ and we get a natural map

$$\mathrm{tw}_\theta : H_c^*(Y_1(\mathfrak{n}), \mathscr{L}_\lambda(E)) \to H_c^*(Y_{11}(\mathfrak{n}\mathfrak{f}^2), \mathscr{L}_\lambda(E))$$

given by

$$\mathrm{tw}_\theta = \theta_{\det} \cup \sum_{t \in \Upsilon_\mathfrak{f}^\times} \theta(t) v_{t_0}^*. \tag{4.3.1}$$

We claim that (4.3.1) descends to the cohomology at level $K_1(\mathfrak{n}\mathfrak{f}^2)$. To see that, note that $Y_{11}(\mathfrak{n}\mathfrak{f}^2) \to Y_1(\mathfrak{n}\mathfrak{f}^2)$ is a Galois cover with Galois group $(\widehat{\mathcal{O}}_F/\mathfrak{n}\mathfrak{f}^2\widehat{\mathcal{O}}_F)^\times$. Specifically, if $a \in \widehat{\mathcal{O}}_F^\times$ then $\eta_a := \left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right)$ normalizes $K_{11}(\mathfrak{n}\mathfrak{f}^2)$ and so right multiplication by η_a defines an automorphism of $Y_{11}(\mathfrak{n}\mathfrak{f}^2)$ over $Y_1(\mathfrak{n}\mathfrak{f}^2)$ which depends only on the image of a inside $(\widehat{\mathcal{O}}_F/\mathfrak{n}\mathfrak{f}^2\widehat{\mathcal{O}}_F)^\times$. Since $(\widehat{\mathcal{O}}_F/\mathfrak{n}\mathfrak{f}^2\widehat{\mathcal{O}}_F)^\times$ is a finite group, and E has characteristic zero, we may identify $H_c^*(Y_1(\mathfrak{n}\mathfrak{f}^2), \mathscr{L}_\lambda(E))$ as the $(\widehat{\mathcal{O}}_F/\mathfrak{n}\mathfrak{f}^2\widehat{\mathcal{O}}_F)^\times$-invariants in $H_c^*(Y_{11}(\mathfrak{n}\mathfrak{f}^2), \mathscr{L}_\lambda(E))$ (with a acting via pullback η_a^*).

[11] Recall this means that $\theta(1 + \mathfrak{f}\widehat{\mathcal{O}}_F) = \{1\}$.

Lemma 4.3.2. $\mathrm{tw}_\theta\left(H_c^*(Y_1(\mathfrak{n})), \mathscr{L}_\lambda(E)\right) \subset H_c^*(Y_1(\mathfrak{n}\mathfrak{f}^2), \mathscr{L}_\lambda(E))$ *and the diagram*

$$\begin{tikzcd} S_\lambda(K_1(\mathfrak{n})) \arrow[r, "\mathrm{ES}"] \arrow[d, "\mathrm{tw}_\theta"'] & H_c^d(Y_1(\mathfrak{n}), \mathscr{L}_\lambda(\mathbf{C})) \arrow[d, "\mathrm{tw}_\theta"] \\ S_\lambda(K_1(\mathfrak{n}\mathfrak{f}^2)) \arrow[r, "\mathrm{ES}"'] & H_c^d(Y_1(\mathfrak{n}\mathfrak{f}^2), \mathscr{L}_\lambda(\mathbf{C})) \end{tikzcd}$$

is commutative.

Proof. We need to show that $\eta_a^* \mathrm{tw}_\theta = \mathrm{tw}_\theta$ for each $a \in \widehat{\mathcal{O}}_F^\times$. If $t \in \mathbf{A}_{F,f}$ then

$$\eta_a u_t = \left(\begin{smallmatrix} a & at \\ & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & at \\ & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right) \in u_{at}K_1(\mathfrak{n}),$$

so $\eta_a^* v_t^* = v_{at}^*$. Moreover, $\eta_a^*\theta_{\det} = \theta(a)\theta_{\det}$. So, since $at_0 = (at)_0$ for $t \in \Upsilon_{\mathfrak{f}}^\times$, we can finally compute:

$$\eta_a^* \mathrm{tw}_\theta = \eta_a^*\theta_{\det} \cup \sum_{t \in \Upsilon_{\mathfrak{f}}^\times} \theta(t)\eta_a^* v_{t_0}^* = \theta(a)\theta_{\det} \cup \sum_{t \in \Upsilon_{\mathfrak{f}}^\times} \theta(t) v_{(at)_0}^* = \theta_{\det} \cup \sum_{t \in \Upsilon_{\mathfrak{f}}^\times} \theta(at) v_{(at)_0}^* = \mathrm{tw}_\theta\,.$$

The commutativity of tw_θ with ES follows from Lemma 4.3.1. □

Remark 4.3.3. One may also consider twisting by characters of the form $|\cdot|_{\mathbf{A}_F}^n \theta$ where θ is finite order and n is an integer. Namely, there is a suitable modification of $\theta_{\det}$ (compare with Definition 4.4.5) so that the cup product (4.3.1) induces a linear map

$$\mathrm{tw}_{|\cdot|_{\mathbf{A}_F}^n \theta} : H_c^*(Y_1(\mathfrak{n}), \mathscr{L}_{\kappa,w}(E)) \to H_c^*(Y_1(\mathfrak{n}\mathfrak{f}^2), \mathscr{L}_{\kappa,w-2n}(E)). \tag{4.3.2}$$

We omit an explicit description, but in Section 5.5 we will explain the same idea.

We note for later (Lemma 4.5.5) the interaction between twisting and Archimedean Hecke operators.

Proposition 4.3.4. *If $\zeta \in \pi_0(F_\infty^\times)$, then $T_\zeta \circ \mathrm{tw}_\theta = \theta(\zeta)\,\mathrm{tw}_\theta \circ T_\zeta$.*[12]

Proof. Recall that T_ζ is pullback along right-multiplication by $\left(\begin{smallmatrix} \zeta & \\ & 1 \end{smallmatrix}\right)$ on Y_K (for any K). In the definition (4.3.1) of tw_θ, the pullbacks $v_{t_0}^*$ are pullbacks along multiplication by elements of $\mathrm{GL}_2(\mathbf{A}_{F,f})$, so they obviously commute with T_ζ. Since pullbacks commute over cup products, the result is a straightforward check after noticing that $T_\zeta \circ \theta_{\det} = \theta(\zeta)\theta_{\det}$. □

We continue to assume that θ is a finite order Hecke character as above. We define a Gauss sum

$$G(\theta^{-1}) = \sum_{t \in \Upsilon_{\mathfrak{f}}^\times} \theta(\delta^{-1})\theta(t) e_F(\delta^{-1}t)$$

where $\delta_{F/\mathbf{Q}} \in \mathbf{A}_{F,f}^\times$ is any choice of finite idele with $[\delta_{F/\mathbf{Q}}] = \mathcal{D}_{F/\mathbf{Q}}$ (notations as in Section 3.2). We note now that if θ has conductor exactly $\mathfrak{f}$, then

$$G(\theta^{-1}) = \frac{\mathrm{sgn}(\theta_\infty) N_{F/\mathbf{Q}}(\mathfrak{f})}{G(\theta)}, \tag{4.3.3}$$

where $N_{F/\mathbf{Q}}(-)$ is the absolute norm. (This is a classical calculation.)

By [40, Proposition 5.11], if $\phi \in S_\lambda(K_1(\mathfrak{n}))$ then $G(\theta^{-1})^{-1}\mathrm{tw}_\theta(\phi) =: \phi \otimes \theta$ is what one usually thinks of as the "twist": the Fourier coefficients of $\phi \otimes \theta$ are given by

$$a_{\phi\otimes\theta}(\mathfrak{m}) = \begin{cases} \theta(\mathfrak{m}) a_\phi(\mathfrak{m}) & \text{if } (\mathfrak{m}, \mathfrak{f}) = 1; \\ 0 & \text{otherwise.} \end{cases} \tag{4.3.4}$$

[12] The tw_θ here means the one on cohomology. It must be, since the T_ζ are not defined on automorphic forms.

Here we have descended θ to a character of the prime-to-$\mathfrak{f}$ part of the ideal class group. It follows from (4.3.4) that if ϕ is a normalized eigenform of level $\mathfrak{n}$ then $\phi\otimes\theta$ is a normalized eigenform of level $\mathfrak{n}\mathfrak{f}^2$.

We end with the following synopsis of the relationship between twisting and p-refinements.

Proposition 4.3.5. *Let p be a prime. Suppose that π is a cohomological cuspidal automorphic representation of conductor $\mathfrak{n}$, α is a p-refinement of π, and θ is a finite order Hecke character with conductor of the form $\mathfrak{f}=\prod_{v|p}\mathfrak{p}_v^{f_v}$ with $f_v\geq 0$. If $v\mid p$ and π_v is a principal series representation, then write $\beta_v=a_\pi(\mathfrak{p}_v)-\alpha_v$.*

1. *$\phi_\pi\otimes\theta$ and $\phi_{\pi,\alpha}\otimes\theta$ are normalized eigenforms of levels $\mathfrak{n}\mathfrak{f}^2$ and $(\mathfrak{n}\cap\mathbf{p})\mathfrak{f}^2$, respectively.*
2. *If $v\nmid p$ or $f_v>0$ or $\mathfrak{p}_v\mid\mathfrak{n}$ then $L_v(\phi_{\pi,\alpha}\otimes\theta,s)=L_v(\phi_\pi\otimes\theta,s)$.*
3. *If $v\mid p$ and $f_v=0$ and $\mathfrak{p}_v\nmid\mathfrak{n}$ then*
$$L_v(\phi_{\pi,\alpha}\otimes\theta,s)=(1-\theta_v(\varpi_v)\beta_vq_v^{-s})L_v(\phi_\pi\otimes\theta,s).$$
4. *$L_v(\phi_\pi\otimes\theta,s)=L_v(\pi\otimes\theta,s)$ for all v.*[13]
5. $$\mathbf{M}(\phi_{\pi,\alpha}\otimes\theta,s)=\Big[\prod_{\substack{v|p\\ \mathfrak{p}_v\nmid\mathfrak{n}\mathfrak{f}}}(1-\beta_v\theta_v(\varpi_v)q_v^{-(s+1)})\Big]\Delta_F^{s+1}\Lambda(\pi\otimes\theta,s+1).$$

Proof. As mentioned above, twisting by θ preserves the property of being a normalized eigenform. Since ϕ_π is a normalized eigenform, and $\phi_{\pi,\alpha}$ is one by Proposition 3.4.4, part (1) is proven.

We will prove (2) and (3) at the same time. First note that since $\mathfrak{f}$ is divisible only by primes above p, the level of $\phi_\pi\otimes\theta$ and the level of $\phi_{\pi,\alpha}\otimes\theta$ are the same away from p. Note as well that the central characters are the same: they are both $\omega_\pi\theta^2$. Thus we see that (2) is true in the case $v\nmid p$ by Proposition 3.4.4 and (4.3.4).

Now we consider $v\mid p$. If $f_v>0$ or $\mathfrak{p}_v\mid\mathfrak{n}$ then $\mathfrak{p}_v$ divides the level of both $\phi_{\pi,\alpha}\otimes\theta$ and $\phi_\pi\otimes\theta$, and the v-th Fourier coefficient of either eigenform is the same: if $f_v>0$ then the coefficients are both zero, and if $f_v=0$ but $\mathfrak{p}_v\mid\mathfrak{n}$ then both coefficients are $\theta(\varpi_v)\alpha_v=\theta(\varpi_v)a_\pi(\mathfrak{p}_v)$ (compare with Remark 3.4.3). This completes the proof of (2).

Finally suppose that $v\mid p$ and $f_v=0$ and $\mathfrak{p}_v\nmid\mathfrak{n}$. Since $\mathfrak{p}_v$ is then co-prime to the level of $\phi_\pi\otimes\theta$, we have from (4.3.4) that

$$L_v(\phi_\pi\otimes\theta,s)=\frac{1}{1-\theta(\varpi_v)a_\pi(\mathfrak{p}_v)q_v^{-s}+\omega_\pi\theta^2(\varpi_v)q_v^{1-2s}}=\frac{1}{(1-\theta(\varpi_v)\alpha_vq_v^{-s})(1-\theta(\varpi_v)\beta_vq_v^{-s})}.$$

On the other hand, by Proposition 3.4.4 and (4.3.4) we have $a_{\phi_{\pi,\alpha}\otimes\theta}(\mathfrak{p}_v)=\theta(\varpi_v)\alpha_v$. Since $\phi_{\pi,\alpha}\otimes\theta$ has level divisible by $\mathfrak{p}_v$, its local L-factor is

$$L_v(\phi_{\pi,\alpha}\otimes\theta,s)=\frac{1}{1-a_{\phi_{\pi,\alpha}\otimes\theta}(\mathfrak{p}_v)q_v^{-s}}=\frac{1}{1-\theta(\varpi_v)\alpha_vq_v^{-s}}.$$

Comparing the previous two displayed equations completes the proof of (3).

Point (4) is obvious if $f_v=0$. Otherwise θ is ramified at v and in particular $v\mid p$. We claim that $L_v(\pi\otimes\theta,s)=1=L_v(\phi_\pi\otimes\theta,s)$. Since π is p-refineable and $v\mid p$, the first equality follows because twisting an unramified principal series or an unramified twist of the Steinberg by a ramified character trivializes the local L-factor. For the second equality, note that if θ_v is ramified then $\mathfrak{p}_v$ divides the level of $\phi_\pi\otimes\theta$ and $a_{\phi_\pi\otimes\theta}(\mathfrak{p}_v)=0$ by (4.3.4). The second inequality now follows from (3.3.6).

Finally, (5) follows from the previous parts and Proposition 3.3.1 ∎

[13]Here, $\pi\otimes\theta$ is the automorphic representation on which the action of $\mathrm{GL}_2(\mathbf{A}_F)$ on π is twisted by $\theta(\det g)$.

4.4. **Evaluation classes.** In this subsection, E denotes a subfield of $\mathbf{C}$ that contains the Galois closure of F. We will also fix a cohomological weight $\lambda = (\kappa, w)$. Our goal is to define an evaluation class in homology which is used to detect L-values.

Recall $\mathscr{L}_\lambda(E)$ is equipped with a left action of $\mathrm{GL}_2(F)$. We write $\mathscr{L}_\lambda(E)^\vee$ for E-linear dual space of $\mathscr{L}_\lambda(E)$ with its canonical right action of $\mathrm{GL}_2(F)$

$$\mu\big|_g(P) = \mu(g \cdot P)$$

if $\mu \in \mathscr{L}_\lambda(E)^\vee$, $g \in \mathrm{GL}_2(F)$ and $P \in \mathscr{L}_\lambda(E)$.

Lemma 4.4.1. *If $x \in F^\times$ and $P \in \mathscr{L}_\lambda(E)$, then $\left(\left(\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\right) \cdot P\right)(X) = x^{\frac{w+\kappa}{2}} P\left(\frac{X}{x}\right)$.*

Proof. See definition (2.4.2). □

We now make two definitions.

Definition 4.4.2. An integer m is critical with respect to λ if

$$\frac{w - \kappa_\sigma}{2} \leq m \leq \frac{w + \kappa_\sigma}{2}$$

for all $\sigma \in \Sigma_F$.

Definition 4.4.3. Let m be critical with respect to λ. Then, $\delta_m^\star \in \mathscr{L}_\lambda(E)^\vee$ is defined by

$$\delta_m^\star(X^j) = \begin{cases} \binom{\kappa}{j}^{-1} & \text{if } j = \frac{\kappa+w}{2} - m, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 4.4.4. *If $x \in F^\times$, then $\delta_m^\star\big|_{\left(\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\right)} = x^m \delta_m^\star$.*

Proof. By Lemma 4.4.1, if $0 \leq j \leq \kappa$ then

$$\delta_m^\star\big|_{\left(\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\right)}(X^j) = x^{\frac{\kappa+w}{2} - j}\delta_m^\star(X^j). \tag{4.4.1}$$

If $j \neq \frac{\kappa+w}{2} - m$, then both $x^m\delta_m^\star(X^j)$ and the right-hand side of (4.4.1) vanish. And if $j = \frac{\kappa+w}{2} - m$ then clearly $x^m\delta_m^\star(X^j)$ is equal to the right-hand side of (4.4.1). The result follows. □

Recall the definition (2.3.1) of the Shintani cone $\mathrm{C}_\infty = F^\times\backslash\mathbf{A}_F^\times/\widehat{\mathcal{O}}_F^\times$. Above we took a right action of $\mathrm{GL}_2(F)$ on $\mathscr{L}_\lambda(E)^\vee$ but now we restrict this to the *left* action of $F^\times$ where $x \in F^\times$ acts by $x \cdot \mu = \mu|_{\left(\begin{smallmatrix} x^{-1} & \\ & 1 \end{smallmatrix}\right)}$. Using this action, we define a local system

$$\mathrm{t}^*\mathscr{L}_\lambda(E)^\vee = F^\times\backslash\mathbf{A}_F^\times \times \mathscr{L}_\lambda(E)^\vee/\widehat{\mathcal{O}}_F^\times \twoheadrightarrow \mathrm{C}_\infty.$$

Definition/Proposition 4.4.5. *If m is critical with respect to λ, then*

$$\delta_m(x) := \left(\mathrm{sgn}(x_\infty)|x_f|_{\mathbf{A}_F}\right)^m \delta_m^\star$$

defines an element of $H^0(\mathrm{C}_\infty, \mathrm{t}^*\mathscr{L}_\lambda(E)^\vee)$.

Proof. Since $\delta_m(x)$ is clearly constant on the connected component $(F_\infty^\times)^\circ$, what we need to show is that if $\xi \in F^\times$, $x \in \mathbf{A}_F^\times$ and $u \in \widehat{\mathcal{O}}_F^\times$ then

$$\delta_m(\xi x u) = \delta_m(x)\big|_{\left(\begin{smallmatrix} \xi^{-1} & \\ & 1 \end{smallmatrix}\right)}. \tag{4.4.2}$$

Since elements of $\widehat{\mathcal{O}}_F^\times$ have trivial adelic norm and no infinite component, we see that δ_m is right $\widehat{\mathcal{O}}_F^\times$-invariant. On the other hand, the product formula implies that

$$\delta_m(\xi x) = (\mathrm{sgn}(\xi_\infty)|\xi_f|_{\mathbf{A}_F})^m\delta_m(x) = \xi_\infty^{-m}\delta_m(x),$$

if $\xi \in F^\times$. But this is exactly the right-hand side of (4.4.2) by Lemma 4.4.4. $\square$

Now suppose that $K \subset \mathrm{GL}_2(\mathbf{A}_{F,f})$ is a t-good subgroup (Definition 2.3.2). As in (2.3.5) we consider the proper embedding $\mathrm{t} : \mathrm{C}_\infty \to Y_K$ given by $\mathrm{t}(x) = \left(\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\right)$. The local system $\mathscr{L}_\lambda(E)^\vee$ on Y_K defined by the left-action of $\mathrm{GL}_2(F)$ on $\mathscr{L}_\lambda(E)^\vee$ pulls back exactly to the local system $\mathrm{t}^*\mathscr{L}_\lambda(E)^\vee$ on C_∞ which we just considered.[14] Since t is proper, we get a pushforward map

$$\mathrm{t}_* : H_*^{\mathrm{BM}}(\mathrm{C}_\infty, \mathrm{t}^*\mathscr{L}_\lambda(E)^\vee) \to H_*^{\mathrm{BM}}(Y_K, \mathscr{L}_\lambda(E)^\vee)$$

on the level of Borel–Moore homology. Furthermore, we also have a Poincaré duality map (see (2.1.6))

$$\mathrm{PD} : H^0(\mathrm{C}_\infty, \mathrm{t}^*\mathscr{L}_\lambda(E)^\vee) \to H_d^{\mathrm{BM}}(\mathrm{C}_\infty, \mathrm{t}^*\mathscr{L}_\lambda(E)^\vee)$$

given by cap product with a Borel–Moore fundamental class $[\mathrm{C}_\infty]$.

Definition 4.4.6. If m is critical with respect to λ, and K is a t-good subgroup, then we define

$$\mathrm{cl}_\infty(m) = \mathrm{t}_*(\mathrm{PD}(\delta_m)) \in H_d^{\mathrm{BM}}(Y_K, \mathscr{L}_\lambda(E)^\vee).$$

We call $\mathrm{cl}_\infty(m)$ an Archimedean evaluation class.

Note that strictly speaking we should write something like $\mathrm{cl}_\infty^K(m)$ to indicate the dependence on K. But, the local systems $\mathscr{L}_\lambda(E)^\vee$ live at all levels simultaneously and the next lemma shows we do not need this extra notation.

Lemma 4.4.7. *If $K' \subset K$ are two compact open subgroups of $\mathrm{GL}_2(\mathbf{A}_{F,f})$ and K' is* t*-good, then* $\mathrm{pr}^*(\mathrm{cl}_\infty^K(m)) = \mathrm{cl}_\infty^{K'}(m)$.

Proof. The two possible embeddings t commute with the projection $Y_{K'} \to Y_K$. $\square$

We end by recording how Archimedean Hecke operators act on the Archimedean evaluation classes.

Proposition 4.4.8. *If $\zeta \in \pi_0(F_\infty^\times)$ then $T_\zeta\,\mathrm{cl}_\infty(m) = \zeta^{-m}\,\mathrm{cl}_\infty(m)$.*

Proof. Write $\rho_\zeta : Y_K \to Y_K$ for right-multiplication by $\left(\begin{smallmatrix} \zeta & \\ & 1 \end{smallmatrix}\right)$, so T_ζ acting on homology is the pushfoward $(\rho_\zeta)_*$. Also write $r_\zeta : \mathrm{C}_\infty \to \mathrm{C}_\infty$ for right multiplication by ζ so that $\rho_\zeta \circ \mathrm{t} = \mathrm{t} \circ r_\zeta$. Since $\zeta = \mathrm{sgn}(\zeta_\infty)$, it follows from the definition of δ_m that $r_\zeta^*\delta_m = \zeta^m\delta_m$. Using this, we get

$$T_\zeta\,\mathrm{cl}_\infty(m) = (\rho_\zeta)_*\mathrm{t}_*\,\mathrm{PD}(\delta_m) = \mathrm{t}_*(r_\zeta)_*\,\mathrm{PD}(\zeta^{-m}r_\zeta^*\delta_m) = \zeta^{-m}\mathrm{t}_*(r_\zeta)_*\,\mathrm{PD}(r_\zeta^*\delta_m).$$

The proposition now follows from Proposition 2.3.1 and (2.1.7). $\square$

4.5. **Special values of L-functions.** Throughout this subsection we will use λ to denote a cohomological weight, m an integer that is critical with respect to λ, and $\mathfrak{n}$ an integral ideal. Further, we will use $\langle -, - \rangle$ to denote the natural pairing (see Section 2.1)

$$\langle -, - \rangle : H_c^d(Y_K, \mathscr{L}_\lambda(E)) \otimes_E H_d^{\mathrm{BM}}(Y_K, \mathscr{L}_\lambda(E)^\vee) \to E.$$

We combine our previous results to compute pairing between the image of the Eichler–Shimura map and Archimedean evaluation classes.

Theorem 4.5.1. *If $\phi \in S_\lambda(K_1(\mathfrak{n}))$, then $\langle \mathrm{ES}(\phi), \mathrm{cl}_\infty(m) \rangle = i^{1+m+\frac{\kappa-w}{2}}\mathbf{M}(\phi, m)$.*

Remark 4.5.2. Note that since $\kappa = (\kappa_\sigma)$ is a Σ_F-tuple, $i^{1+m+\frac{\kappa-w}{2}}$ means the product $\prod_\sigma i^{1+m+\frac{\kappa_\sigma-w}{2}}$.

[14]We consider left actions in order to define the local systems on Y_K because the quotient by $\mathrm{GL}_2^+(F)$ is on the left.

Proof of Theorem 4.5.1. By Proposition 4.2.2(3), Lemma 4.4.7, Proposition 2.3.3, and because $\mathbf{M}(\phi, s)$ only depends on the underlying automorphic form ϕ, we can and will assume that $K_1(\mathfrak{n})$ is a neat level subgroup. Then, we will write $\mathbf{f} = \mathbf{f}_\phi \in S_\lambda^{\mathrm{hol}}(K_1(\mathfrak{n}))$ for the holomorphic Hilbert modular form corresponding to ϕ, and $\omega_{\mathbf{f}} = \mathrm{ES}(\mathbf{f})$ for the bona fide differential form on $Y_1(\mathfrak{n})$ constructed in Proposition 4.2.2. Now we turn towards computation. By the push-pull formula (2.1.8) we have

$$\langle \omega_{\mathbf{f}}, \mathrm{cl}_\infty(m)\rangle = \langle \mathrm{t}^*\omega_{\mathbf{f}} \cup \delta_m, [\mathrm{C}_\infty]\rangle \tag{4.5.1}$$

where $\cup$ is the cup product

$$\cup : H_c^d(\mathrm{C}_\infty, \mathrm{t}^*\mathscr{L}_\lambda(E)) \otimes_E H^0(\mathrm{C}_\infty, \mathrm{t}^*\mathscr{L}_\lambda(E)^\vee) \to H_c^d(\mathrm{C}_\infty, E).$$

Let us first compute the E-valued differential form $\mathrm{t}^*\omega_{\mathbf{f}} \cup \delta_m$ on C_∞. We recall that we have fixed our coordinate z at the start of Section 4.2 to be compatible with the canonical coordinate x_∞ on $(F_\infty^\times)^\circ$. Thus, $\mathrm{t}^*\omega_{\mathbf{f}}$ is the d-form on C_∞ given in coordinates on $\mathbf{A}_{F,+}^\times = F_{\infty,+}^\times \times \mathbf{A}_{F,f}^\times$ by

$$\mathrm{t}^*\omega_{\mathbf{f}}(x_\infty, x_f) = \mathbf{f}\left(ix_\infty, \left(\begin{smallmatrix} x_f & \\ & 1\end{smallmatrix}\right)\right)(ix_\infty + X)^\kappa d(ix_\infty)$$

for $x = x_\infty x_f \in \mathbf{A}_{F,+}^\times$. Further, by definition, $\delta_m^\star\left((ix_\infty + X)^\kappa\right) = (ix_\infty)^{\frac{\kappa - w}{2} + m}$. So, in coordinates we have

$$\begin{aligned}(\mathrm{t}^*\omega_{\mathbf{f}} \cup \delta_m)(x_\infty, x_f) &= \delta_m(x)\left(\mathbf{f}\left(ix_\infty, \left(\begin{smallmatrix} x_f & \\ & 1\end{smallmatrix}\right)\right)(ix_\infty + X)^\kappa\right) d(ix_\infty) \\ &= i^d \mathbf{f}\left(ix_\infty, \left(\begin{smallmatrix} x_f & \\ & 1\end{smallmatrix}\right)\right)|x_f|_{\mathbf{A}_F}^m (ix_\infty)^{\frac{\kappa-w}{2}+m} dx_\infty \\ &= i^{1+m+\frac{\kappa-w}{2}} |x|_{\mathbf{A}_F}^m \phi\left(\left(\begin{smallmatrix} x & \\ & 1\end{smallmatrix}\right)\right)\frac{dx_\infty}{x_\infty}.\end{aligned} \tag{4.5.2}$$

Now we note that the pairing (4.5.1) is computed by integrating $\mathrm{t}^*\omega_{\mathbf{f}} \cup \delta_m$ over C_∞. Since $x \mapsto |x|_{\mathbf{A}_F}^m \phi\left(\left(\begin{smallmatrix} x & \\ & 1\end{smallmatrix}\right)\right)$ is invariant under right multiplication by $\widehat{\mathcal{O}}_F^\times$, we get from (4.5.2) that

$$\begin{aligned}\langle \mathrm{t}^*\omega_{\mathbf{f}} \cup \delta_m, [\mathrm{C}_\infty]\rangle &= \int_{\mathrm{C}_\infty} \mathrm{t}^*\omega_{\mathbf{f}} \cup \delta_m \\ &= i^{1+m+\frac{\kappa-w}{2}} \int_{F_+^\times \backslash \mathbf{A}_{F,+}^\times} \phi\left(\left(\begin{smallmatrix} x & \\ & 1\end{smallmatrix}\right)\right)|x|_{\mathbf{A}_F}^m d^\times x \\ &= i^{1+m+\frac{\kappa-w}{2}} \mathbf{M}(\phi, m).\end{aligned}$$

This completes the proof. $\square$

Corollary 4.5.3. *If $\phi \in S_\lambda(K_1(\mathfrak{n}))$, then*

$$\langle \mathrm{ES}(\phi), \mathrm{cl}_\infty(m)\rangle = i^{1+m+\frac{\kappa-w}{2}} \Delta_{F/\mathbf{Q}}^{m+1} \Lambda(\phi, m+1).$$

Proof. This is immediate from Proposition 3.3.1 and Theorem 4.5.1. $\square$

In the special case of a p-refined newform, we have the following.

Corollary 4.5.4. *Let p be a prime. Suppose that π is a cohomological cuspidal automorphic representation of conductor $\mathfrak{n}$, α is a p-refinement of π, and θ is a finite order Hecke character with conductor of the form $\mathfrak{f} = \prod_{v|p} \mathfrak{p}_v^{f_v}$ with $f_v \geq 0$. If $v \mid p$ and π_v is a principal series representation, then write $\beta_v = a_\pi(\mathfrak{p}_v) - \alpha_v$. Then,*

$$\langle \mathrm{ES}(\phi_{\pi,\alpha} \otimes \theta), \mathrm{cl}_\infty(m)\rangle = \Big(\prod_{\substack{v|p \\ \mathfrak{p}_v \nmid \mathfrak{n}\mathfrak{f}}} (1 - \beta_v\theta_v(\varpi_v) q_v^{-(m+1)})\Big) i^{1+m+\frac{\kappa-w}{2}} \Delta_{F/\mathbf{Q}}^{m+1} \Lambda(\pi \otimes \theta, m+1).$$

Proof. Apply Theorem 4.5.1 to $\phi = \phi_{\pi,\alpha} \otimes \theta$, and then apply Proposition 4.3.5. $\square$

Prior to the final result of this section, we need one more calculation.

Lemma 4.5.5. *Let θ be a finite order Hecke character and $E \subset \mathbf{C}$ a field containing the Galois closure of F and the values of θ. For each $\omega \in H_c^d(Y_1(\mathfrak{n}), \mathscr{L}_\lambda(E))$ and $\zeta \in \pi_0(F_\infty)$ we have*

$$\langle \mathrm{tw}_\theta(T_\zeta\omega), \mathrm{cl}_\infty(m)\rangle = \theta(\zeta)\zeta^{-m}\langle \mathrm{tw}_\theta(\omega), \mathrm{cl}_\infty(m)\rangle. \tag{4.5.3}$$

In particular, if $\epsilon \in \{\pm 1\}^{\Sigma_F}$ is uniquely defined by $\epsilon(\zeta) = \theta^{-1}(\zeta)\zeta^m$ for all $\zeta \in \pi_0(F_\infty^\times)$, then

$$\langle \mathrm{tw}_\theta(\omega), \mathrm{cl}_\infty(m)\rangle = \langle \mathrm{tw}_\theta(\mathrm{pr}^\epsilon \omega), \mathrm{cl}_\infty(m)\rangle.$$

Proof. Proposition 4.3.4 and the adjointness of pushfowards/pullbacks under $\langle -, -\rangle$ implies that

$$\langle \mathrm{tw}_\theta(T_\zeta\omega), \mathrm{cl}_\infty(m)\rangle = \theta(\zeta)\langle T_\zeta \,\mathrm{tw}_\theta(\omega), \mathrm{cl}_\infty(m)\rangle = \theta(\zeta)\langle \mathrm{tw}_\theta(\omega), T_\zeta \,\mathrm{cl}_\infty(m)\rangle.$$

So, (4.5.3) follows from Proposition 4.4.8. □

Remark 4.5.6. The next result is originally due to Shimura [72]. The method we have explained is due to Hida. See [49].

Theorem 4.5.7. *Let π be a cohomological cuspidal automorphic representation of weight λ. Write E for the smallest subfield of $\mathbf{C}$ containing $\mathbf{Q}(\pi)$ and the Galois closure of F. Then, for each $\epsilon \in \{\pm 1\}^{\Sigma_F}$ there exists $\Omega_\pi^\epsilon \in \mathbf{C}^\times$ such that, if θ is a finite order Hecke character of conductor $\mathfrak{f}$, then*

$$\frac{\mathrm{sgn}(\theta_\infty)N_{F/\mathbf{Q}}(\mathfrak{f})i^{1+m+\frac{\kappa-w}{2}}\Delta_{F/\mathbf{Q}}^{m+1}\Lambda(\pi\otimes\theta, m+1)}{G(\theta)\Omega_\pi^\epsilon} \in E(\theta), \tag{4.5.4}$$

where

(1) *$E(\theta)$ is the field generated by E and the values of θ, and*
(2) *ϵ is chosen so that $\epsilon(\zeta) = \theta^{-1}(\zeta)\zeta^m$ for all $\zeta \in \pi_0(F_\infty^\times)$.*

Proof. Write ϕ_π for the newform associated to π. For each $\epsilon \in \{\pm 1\}^{\Sigma_F}$ choose the period Ω_π^ϵ as in Corollary 4.2.6. We claim that, given θ, (4.5.4) now holds for the specific ϵ as in (3).

To see the claim, let $\omega = \mathrm{ES}(\phi_\pi)/\Omega_\pi^\epsilon \in H_c^d(Y_1(\mathfrak{n}), \mathscr{L}_\lambda(\mathbf{C}))$. The choice of period Ω_π^ϵ means that $\mathrm{pr}^\epsilon \omega$ is actually defined over E and so Lemma 4.5.5 implies that

$$\langle \mathrm{tw}_\theta(\omega), \mathrm{cl}_\infty(m)\rangle \in E(\theta). \tag{4.5.5}$$

On the other hand,

$$\mathrm{tw}_\theta(\omega) = \frac{1}{\Omega_\pi^\epsilon}\mathrm{tw}_\theta(\mathrm{ES}(\phi_\pi)) = \frac{1}{\Omega_\pi^\epsilon}\mathrm{ES}(\mathrm{tw}_\theta\,\phi_\pi) = \frac{G(\theta^{-1})}{\Omega_\pi^\varepsilon}\mathrm{ES}(\phi_\pi\otimes\theta).$$

Here we used Lemma 4.3.2 for the second equality. Combining Corollary 4.5.3 and (4.5.5), we conclude

$$\frac{G(\theta^{-1})i^{1+m+\frac{\kappa-w}{2}}\Delta_{F/\mathbf{Q}}^{m+1}\Lambda(\phi_\pi\otimes\theta, m+1)}{\Omega_\pi^\epsilon} \in E(\theta).$$

The translation between this and (4.5.4) follows from (4.3.3). Finally, ϕ_π and π have the same L-function up to elements of E so we can replace $\Lambda(\phi_\pi\otimes\theta, m+1)$ with $\Lambda(\pi\otimes\theta, m+1)$ as well. □

5. Locally analytic distributions and p-adic weights

5.1. Compact abelian p-adic Lie groups.

Definition 5.1.1.

(1) A compact abelian p-adic Lie group G (CPA group for short) is an abelian topological group G which is compact and which contains an open subgroup topologically isomorphic to $\mathbf{Z}_p^n$ for some $0 \leq n < \infty$.
(2) The dimension of a CPA G is the integer $\dim G := n$.
(3) A chart of a CPA group G is an injective and open group morphism $\mathbf{Z}_p^{\dim G} \hookrightarrow G$.

We note that CPA groups are exactly the p-adic Lie groups which are compact and abelian ([71]) and the dimension is the dimension of the underlying p-adic manifold.[15] The salient facts are contained in the next lemma. The proofs are left to the reader.

Lemma 5.1.2.

(1) If G and H are CPA groups then $G \times H$ is a CPA group.
(2) If G is a CPA group and H is a closed subgroup then H and G/H are CPA groups.
(3) If $f : G \to H$ is a group morphism between CPA groups then f is continuous, $\ker(f) \subset G$ and $\mathrm{im}(f) \subset H$ are closed subgroups and the group isomorphism $G/\ker(f) \simeq \mathrm{im}(f)$ is a homeomorphism.
(4) Let $0 \to G \to H \to J \to 0$ be any short exact sequence of abelian groups. If any two of the groups are CPA, then all three are CPA and the morphisms in the sequence are continuous. In particular, any abelian group which is an extension of one CPA group by another is automatically CPA.

For the rest of this subsection we fix a CPA group G and write $n = \dim G$. We also fix a $\mathbf{Q}_p$-Banach algebra R.

For each integer $s \geq 0$ and each chart $\nu : \mathbf{Z}_p^n \hookrightarrow G$, we write $\mathbf{A}^s(G, \nu, R)$ for the functions $f : G \to R$ with the following property: for each $g \in G$, the function $z \mapsto f(g\nu(p^s z))$ is an R-valued rigid analytic function in the variable $z = (z_1, \dots, z_n) \in \mathbf{Z}_p^n$. If $f \in \mathbf{A}^s(G, \nu, R)$ then $f(g\nu(p^s z))$ is defined by an element in the Tate-algebra $R\langle z_1, \dots, z_n\rangle$ (for each g) and so $\mathbf{A}^s(G, \nu, R)$ is naturally an R-Banach algebra by considering the largest of the the pullback norms from $R\langle z_1, \dots, z_n\rangle$ for any finite choice of coset representatives of $G/\nu(p^s\mathbf{Z}_p^n)$. Further, for $s' \geq s$ the canonical map $\mathbf{A}^s(G, \nu, R) \to \mathbf{A}^{s'}(G, \nu, R)$ is injective with dense image and compact if $s' > s$. We define the R-valued locally analytic functions on G as the compact type space (see [37, Section 1.1])

$$\mathscr{A}(G, R) := \varinjlim_{s\to\infty} \mathbf{A}^s(G, \nu, R).$$

This is independent of the chart ν.

Next, we define $\mathbf{D}^s(G, \nu, R) := \mathbf{A}^s(G, \nu, R)'$ as the R-Banach module dual (equipped with the operator topology). This is also an R-Banach algebra under the convolution product $(\mu_1, \mu_2) \mapsto \mu_1 * \mu_2$. If $s' \geq s$ then the canonical map $\mathbf{D}^{s'}(G, \nu, R) \to \mathbf{D}^s(G, \nu, R)$ is still injective (because the transpose has dense image) and compact when $s' > s$. We define the R-valued locally analytic distributions on G as the projective limit

$$\mathscr{D}(G, R) := \varprojlim_{s\to\infty} \mathbf{D}^s(G, \nu, R).$$

[15] As in "naïve" p-adic manifolds, as opposed to rigid analytic spaces, etc.

Notice that there is a natural R-bilinear pairing

$$\mathscr{D}(G,R)\otimes_R \mathscr{A}(G,R)\to R$$

which we write $(\mu,f)\mapsto \mu(f)$.

Remark 5.1.3. Each of $R\mapsto \mathbf{A}^s(G,\nu,R)$, $\mathscr{A}(G,R)$, and $\mathscr{D}(G,R)$ commute with completed tensor products; the distributions at a fixed radius do not. Compare with [11, Remark 3.1].

We now define the space of p-adic characters on G.

Definition 5.1.4. $\mathscr{X}(G)=\mathrm{Spf}(\mathbf{Z}_p[[G]])^{\mathrm{rig}}$.

Thus $\mathscr{X}(G)$ is a rigid analytic space over $\mathbf{Q}_p$ whose R-valued points are nothing but continuous characters $\chi: G\to R^\times$. It is well-known (see [37, Proposition 3.6.10] for example) that if $\chi\in\mathscr{X}(G)(\mathbf{Q}_p)$ then $g\mapsto\chi(g)$ defines an element of $\mathscr{A}(G,\mathbf{Q}_p)$. Further, if $\mu\in\mathscr{D}(G,\mathbf{Q}_p)$

$$\mathcal{A}_\mu(\chi):=\mu(g\mapsto\chi(g))$$

extends to a rigid analytic function on $\mathscr{X}(G)$. For instance, if $g\in G$ and $\delta_g\in\mathscr{D}(G,\mathbf{Q}_p)$ is the Dirac distribution then $\mathcal{A}_{\delta_g}$ is the rigid function ev_g on $\mathscr{X}(G)$ given by "evaluation at g". Further, $\mathcal{A}_{\mu_1*\mu_2}=\mathcal{A}_{\mu_1}\mathcal{A}_{\mu_2}$. See [68, Sections 1-2] for more details.

Definition 5.1.5. The Amice transform is the natural map

$$\begin{aligned}\mathscr{D}(G,R)&\xrightarrow{\mathcal{A}}\mathscr{O}(\mathscr{X}(G))\widehat{\otimes}_{\mathbf{Q}_p}R\\ \mu&\mapsto\mathcal{A}_\mu.\end{aligned}$$

Proposition 5.1.6. *The Amice transform is a topological isomorphism.*

Proof. By Remark 5.1.3, we can assume that $R=\mathbf{Q}_p$. Let H be an open (thus finite index) subgroup of G. Then, $\mathscr{D}(G,\mathbf{Q}_p)$ is finite free over $\mathscr{D}(H,\mathbf{Q}_p)$ with basis given by $\{\delta_g\}$ with g running over coset representatives of G/H and $\mathscr{O}(\mathscr{X}(G))$ is finite and free over $\mathscr{O}(\mathscr{X}(H))$ with basis given by $\{\mathrm{ev}_g\}$. Since $\mathcal{A}_{\delta_g}=\mathrm{ev}_g$, the result for G follows from the result for such an H. Since G is a CPA group, there exists an H topologically isomorphic to $\mathbf{Z}_p^n$, in which case the theorem is known by a multi-variable version of Amice's theorem [2] (see [68]). $\square$

5.2. **Locally analytic distributions on $\mathcal{O}_p$.** In this section we consider the CPA group $\mathcal{O}_p=\mathcal{O}_F\otimes_{\mathbf{Z}}\mathbf{Z}_p=\prod_{v|p}\mathcal{O}_v$. For $v\mid p$, we fix a uniformizer $\varpi_v\in\mathcal{O}_v$ and we write $\varpi_p\in\mathcal{O}_p$ for the corresponding tuple. Let e_v be the ramification index at $v\mid p$, and $\mathbf{e}=(e_v)_{v|p}\in\mathbf{Z}_{\geq1}^{\{v|p\}}$.

Start by choosing a $\mathbf{Z}_p$-linear isomorphism $\nu:\mathbf{Z}_p^d\simeq\mathcal{O}_p$ which we use as a chart. Using this we write $\mathbf{A}^\circ(\mathcal{O}_p,\mathbf{Q}_p)$ for the ring of functions $f:\mathcal{O}_p\to\mathbf{Q}_p$ such that $f\circ\nu$ is defined by an element of the Tate algebra $\mathbf{Z}_p\langle z_1,\dots,z_d\rangle$. The ring $\mathbf{A}(\mathcal{O}_p,\mathbf{Q}_p):=\mathbf{A}^\circ(\mathcal{O}_p,\mathbf{Q}_p)[1/p]$ is the ring we denoted $\mathbf{A}^0(\mathcal{O}_p,\nu,\mathbf{Q}_p)$ in Section 5.1, so $f\mapsto f\circ\nu$ defines an isomorphism $\mathbf{A}(\mathcal{O}_p,\mathbf{Q}_p)\simeq\mathbf{Q}_p\langle z_1,\dots,z_d\rangle$. The $\mathbf{Q}_p$-Banach structure on with the norm $\|f\|_{\mathbf{0}}$ on $\mathbf{A}(\mathcal{O}_p,\mathbf{Q}_p)$ defined by pulling back the supremum norm on $\mathbf{Q}_p\langle z_1,\dots,z_d\rangle$. It is independent of ν.

For $\mathbf{s}=(s_v)_{v|p}\in\mathbf{Z}_{\geq0}^{\{v|p\}}$ we now define

$$\begin{aligned}\mathbf{A}^{\mathbf{s},\circ}(\mathcal{O}_p,\mathbf{Q}_p)&:=\{f:\mathcal{O}_p\to\mathbf{Q}_p\mid z\mapsto f(a+\varpi_p^{\mathbf{s}}z)\text{ lies in }\mathbf{A}^\circ(\mathcal{O}_p,\mathbf{Q}_p)\text{ for all }a\in\mathcal{O}_p\};\\ \mathbf{A}^{\mathbf{s}}(\mathcal{O}_p,\mathbf{Q}_p)&=\mathbf{A}^{\mathbf{s},\circ}(\mathcal{O}_p,\mathbf{Q}_p)[1/p].\end{aligned}$$

If $f\in\mathbf{A}^{\mathbf{s}}(\mathcal{O}_p,\mathbf{Q}_p)$, then $f(a+\varpi_p^{\mathbf{s}}z)$ depends on a mod $\varpi_p^{\mathbf{s}}\mathcal{O}_p$ only up to translation in the z-variable. Thus we equip $\mathbf{A}^{\mathbf{s}}(\mathcal{O}_p,\mathbf{Q}_p)$ with a $\mathbf{Q}_p$-Banach norm by

$$\|f\|_{\mathbf{s}}:=\max_{a\in\mathcal{O}_p/\varpi_p^{\mathbf{s}}\mathcal{O}_p}\|f(a+\varpi_p^{\mathbf{s}}z)\|_{\mathbf{0}}.$$

If $\mathbf{s}' \geq \mathbf{s}$ (i.e. $s'_v \geq s_v$ for all $v \mid p$) then the natural map $\mathbf{A}^{\mathbf{s}}(\mathcal{O}_p, \mathbf{Q}_p) \to \mathbf{A}^{\mathbf{s}'}(\mathcal{O}_p, \mathbf{Q}_p)$ is continuous with dense image. If $\mathbf{s}' \geq \mathbf{s} + \mathbf{e}$ (i.e. $s'_v \geq s_v + e_v$ for each $v \mid p$) then it is compact. Furthermore, the $\mathbf{Q}_p$-Banach algebras $\mathbf{A}^{\mathbf{s}}(\mathcal{O}_p, \mathbf{Q}_p)$ are a co-final defining sequence for $\mathscr{A}(\mathcal{O}_p, \mathbf{Q}_p)$, as in Section 5.1, because if $s \in \mathbf{Z}_{\geq 0}$ and $\mathbf{s} := (se_v)_{v|p}$ then we have an obvious (topological) equality

$$\mathbf{A}^{\mathbf{s}}(\mathcal{O}_p, \mathbf{Q}_p) = \mathbf{A}^{s}(\mathcal{O}_p, m_{u_p} \circ \nu, \mathbf{Q}_p)$$

where m_{u_p} is multiplication by $\varpi_p^{\mathbf{e}} p^{-1}$ on $\mathcal{O}_p$. Thus we also have a topological isomorphism

$$\mathscr{A}(\mathcal{O}_p, \mathbf{Q}_p) = \varinjlim_{|\mathbf{s}| \to +\infty} \mathbf{A}^{\mathbf{s}}(\mathcal{O}_p, \mathbf{Q}_p) \tag{5.2.1}$$

where $|\mathbf{s}| = \min(s_v : v \mid p)$.

If R is a $\mathbf{Q}_p$-Banach algebra and $\mathbf{s} \in \mathbf{Z}_{\geq 0}^{\{v|p\}}$, we define $\mathbf{A}^{\mathbf{s}}(\mathcal{O}_p, R) := \mathbf{A}^{\mathbf{s}}(\mathcal{O}_p, \mathbf{Q}_p) \widehat{\otimes}_{\mathbf{Q}_p} R$ with its inductive tensor product topology. Any $\mathbf{Q}_p$-Banach space is potentially orthonormalizable ([70, Proposition 1]), so the R-Banach modules $\mathbf{A}^{\mathbf{s}}(\mathcal{O}_p, R)$ are potentially orthonormalizable as well ([26, Lemma 2.8]). If $\mathbf{s}' \geq \mathbf{s}$ then the natural map $\mathbf{A}^{\mathbf{s}}(\mathcal{O}_p, R) \to \mathbf{A}^{\mathbf{s}}(\mathcal{O}_p, R)$ is injective with dense image ([37, Corollary 1.1.27]) and if $\mathbf{s}' \geq \mathbf{s} + \mathbf{e}$ then the map is compact ([67, Lemma 18.12]). By (5.2.1) and [37, Proposition 1.1.32(i)] we deduce a topological identification

$$\mathscr{A}(\mathcal{O}_p, R) = \varinjlim_{|\mathbf{s}| \to +\infty} \mathbf{A}^{\mathbf{s}}(\mathcal{O}_p, R). \tag{5.2.2}$$

Finally, we write $\mathbf{D}^{\mathbf{s}}(\mathcal{O}_p, R)$ for R-Banach dual $\mathbf{A}^{\mathbf{s}}(\mathcal{O}_p, R)'$ equipped with its operator topology and convolution product. The R-Banach algebras $\mathbf{D}^{\mathbf{s}}(\mathcal{O}_p, R)$ are co-final with the Banach algebras in Section 5.1 (for the same reasons as above) and thus we have a topological identification

$$\mathscr{D}(\mathcal{O}_p, R) = \varprojlim_{|\mathbf{s}| \to +\infty} \mathbf{D}^{s}(\mathcal{O}_p, R).$$

Remark 5.2.1. The R-Banach modules $\mathbf{D}^{\mathbf{s}}(\mathcal{O}_p, R)$ are *not* the same as $\mathbf{D}^{\mathbf{s}}(\mathcal{O}_p, \mathbf{Q}_p) \widehat{\otimes}_{\mathbf{Q}_p} R$ and thus not obviously potentially orthornormalizable.

We now recall the following definition.

Definition 5.2.2. If R is a $\mathbf{Q}_p$-Banach algebra, a ring of definition R_0 for R is a subring $R_0 \subset R$ which is open and bounded.

We note that this implies as well that R_0 is p-adically separated and complete, and $R_0[1/p] = R$. After fixing $R_0 \subset R$ a ring of definition, we now define

$$\mathbf{A}^{\mathbf{s},\circ}(\mathcal{O}_p, R) := \mathbf{A}^{\mathbf{s},\circ}(\mathcal{O}_p, \mathbf{Q}_p) \widehat{\otimes}_{\mathbf{Z}_p} R_0.$$

The R_0-algebra $\mathbf{A}^{\mathbf{s},\circ}(\mathcal{O}_p, R)$ is naturally an open and bounded R_0-subalgebra $\mathbf{A}^{s}(\mathcal{O}_p, R)$ and we have an equality after inverting p. For the distributions, still with R_0 fixed, we define $\mathbf{D}^{s,\circ}(\mathcal{O}_p, R)$ as the R_0-linear dual

$$\mathbf{D}^{\mathbf{s},\circ}(\mathcal{O}_p, R) := \mathrm{Hom}_{R_0}(\mathbf{A}^{s,\circ}(\mathcal{O}_p, R), R_0).$$

Remark 5.2.3. The notations $\mathbf{A}^{\mathbf{s},\circ}$ and $\mathbf{D}^{\mathbf{s},\circ}$ are misleading in that they obviously depend on R_0. If R is reduced, then we may take R_0 to be the subring of power-bounded elements in R. In any case, the reader may also notice that we never make "natural use" of the lattices (as opposed to the functors $\mathbf{A}^{\mathbf{s}}(\mathcal{O}_p, -)$ and $\mathbf{D}^{\mathbf{s}}(\mathcal{O}_p, -)$).

5.3. **Actions by the monoid Δ.** We maintain the notations of the previous subsection and we also fix a $\mathbf{Q}_p$-Banach algebra R and a ring of definition $R_0 \subset R$. If $h(z)$ is a function on $\mathcal{O}_p^\times$ valued in a ring, then write $h(z)_!$ for its extension by zero to $\mathcal{O}_p$.

Lemma 5.3.1. *If $\chi : \mathcal{O}_p^\times \to R^\times$ is a continuous character, then there exists $\mathbf{s}(\chi) \in \mathbf{Z}_{\geq 0}^{\{v|p\}}$ such that $f \in \mathbf{A}^{\mathbf{s}(\chi)}(\mathcal{O}_p, R)$ when f is a function of either of the following two forms.*

(1) $f(z) = \chi(d + cz)$ with $c \in \varpi_p\mathcal{O}_p$ and $d \in \mathcal{O}_p^\times$.

(2) $f(z) = \chi(z)_!$.

If $\chi(\mathcal{O}_p^\times) \subset R_0^\times$, then there exists $\mathbf{s}^\circ(\chi) \in \mathbf{Z}_{\geq 0}^{\{v|p\}}$ depending on R_0 so that $f \in \mathbf{A}^{\mathbf{s}^\circ(\chi),\circ}(\mathcal{O}_p, R)$ for the same functions.

Proof. If $c \in \varpi_p\mathcal{O}_p$ and $d \in \mathcal{O}_p^\times$ then $\chi(d + cz) = \chi(d + cz)_!$. Since $z \mapsto d + cz$ is polynomial in z, we only need to prove the lemma where $f(z) = \chi(z)_!$. In the case where p is inverted, this is well-known. We now deduce the R_0-case from the R-case.

First, we observe that if $g \in \mathbf{A}(\mathcal{O}_p, R)$ and $g(0) \in R_0$, then there exist $\mathbf{s}(g)$ so that $g(\varpi_p^{\mathbf{s}(g)}z) \in \mathbf{A}^\circ(\mathcal{O}_p, R)$ (expand the series defining g). Now write $f(z) = \chi(z)_!$. For a running over a (finite) set of coset representatives for $\mathcal{O}_p/\varpi_p^{\mathbf{s}}\mathcal{O}_p$, there exists $g_a \in \mathbf{A}(\mathcal{O}_p, R)$ such that $f(a + \varpi_p^{\mathbf{s}(\chi)}z) = g_a(z)$. Since $g_a(0) = f(a) \in R_0$, the first sentence of this paragraph applies to each g_a and the lemma follows. □

Recall that $T \subset \mathrm{GL}_{2/\mathbf{Z}}$ denotes the diagonal torus. Thus $T(\mathcal{O}_p) \simeq (\mathcal{O}_p^\times)^2$ is naturally a CPA group.

Definition 5.3.2. The space of p-adic weights is $\mathscr{W} = \mathscr{X}(T(\mathcal{O}_p))$.

If $\Omega = \mathrm{Sp}(R)$ and $\lambda_\Omega : \Omega \to \mathscr{W}$ is a point then we often confuse it with the corresponding pair $\lambda_\Omega = (\lambda_{\Omega,1}, \lambda_{\Omega,2})$ where $\lambda_{\Omega,i} : \mathcal{O}_p^\times \to R^\times$ are continuous character. If R is a finite extension of $\mathbf{Q}_p$ we write just λ. In either case, we generally refer to both the point and the character as a p-adic weight.

Now consider the submonoid of $\mathrm{GL}_2(F_p)$ defined by

$$\Delta := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(F_p) \cap M_2(\mathcal{O}_p) \mid c \in \varpi_p\mathcal{O}_p \text{ and } d \in \mathcal{O}_p^\times \right\}.$$

If $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Delta$ then $cz + d \in \mathcal{O}_p^\times$ and so the left action $g \cdot z = \frac{az+b}{cz+d}$ of Δ on $\mathcal{O}_p$ is well-defined and it is clearly continuous.

Now consider $\Omega = \mathrm{Sp}(R)$ and let $\lambda_\Omega : \Omega \to \mathscr{W}$ be a p-adic weight. Set $\mathbf{s}(\Omega) := \max\{\mathbf{s}(\lambda_{\Omega,1}\lambda_{\Omega,2}^{-1}), \mathbf{s}(\lambda_{\Omega,2}^{-1})\}$ as above.[16] Then, for $\mathbf{s} \geq \mathbf{s}(\Omega)$ we may endow $\mathbf{A}^{\mathbf{s}}(\mathcal{O}_p, R)$ with a continuous R-linear right action of Δ via

$$f\big|_g(z) = \lambda_{\Omega,1}\lambda_{\Omega,2}^{-1}(cz + d)\lambda_{\Omega,2}\left(\det g \cdot \varpi_p^{-v(\det g)}\right) f(g \cdot z) \tag{5.3.1}$$

where $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Delta$, $f \in \mathbf{A}^{\mathbf{s}}(\mathcal{O}_p, R)$ and $z \in \mathcal{O}_p$.[17] This definition is well-posed by Lemma 5.3.1. We then equip $\mathbf{D}^{\mathbf{s}}(\mathcal{O}_p, R)$ with the dual left action: $(g \cdot \mu)(f) = \mu(f|_g)$. Either action is referred to as a "weight λ-action."

Remark 5.3.3. The monoid Δ and the action (5.3.1) differ from their definitions in [43, Section 2.2] by conjugation by $\left(\begin{smallmatrix} & 1 \\ \varpi_p & \end{smallmatrix}\right) \in \mathrm{GL}_2(F_p)$. Compare with Proposition 6.3.8(1).

The above action of Δ is compatible with the injective restriction map $\mathbf{A}^{\mathbf{s}}(\mathcal{O}_p, R) \to \mathbf{A}^{\mathbf{s}'}(\mathcal{O}_p, R)$ when $\mathbf{s}' \geq \mathbf{s}$, so we get a continuous action of Δ on $\mathscr{A}(\mathcal{O}_p, R)$. On the dual side, $\mathbf{D}^{\mathbf{s}}(\mathcal{O}_p, R)$ is equipped with a continuous R-linear left action by Δ and the compatibility extends this to a continuous action

[16] Inserting $\mathbf{s}(\lambda_{\Omega,2}^{-1})$ into the maximum is purely for convenience of notation later on (see Lemma 7.2.1).

[17] To be clear, we recall that $\varpi_p^{-v(\det g)}$ means $\prod_{v|p} \varpi_v^{-v(\det g_v)}$.

on $\mathscr{D}(\mathcal{O}_p, R)$. Finally, when the image of λ_Ω is contained in R_0, then (5.3.1) defines an action of Δ on $\mathbf{A}^{\mathbf{s},\circ}(\mathcal{O}_p, R)$ as well as a left action on $\mathbf{D}^{\mathbf{s},\circ}(\mathcal{O}_p, R)$ for all $\mathbf{s} \geq \mathbf{s}^\circ(\Omega) := \max\{\mathbf{s}^\circ(\lambda_{\Omega,1}\lambda_{\Omega,2}^{-1}), \mathbf{s}^\circ(\lambda_{\Omega,2}^{-1})\}$.

We summarize the notations presented above as follows.

Definition 5.3.4.

(1) If $\Omega = \mathrm{Sp}(R)$ is a $\mathbf{Q}_p$-affinoid space, $\lambda_\Omega : \Omega \to \mathscr{W}$ is a p-adic weight and $\mathbf{s} \geq \mathbf{s}(\Omega)$, then we write $\mathbf{A}^{\mathbf{s}}_\Omega := \mathbf{A}^{\mathbf{s}}(\mathcal{O}_p, R)$, $\mathbf{D}^{\mathbf{s}}_\Omega := \mathbf{D}^{\mathbf{s}}(\mathcal{O}_p, R)$, $\mathscr{A}_\Omega := \mathscr{A}(\mathcal{O}_p, R)$, and $\mathscr{D}_\Omega := \mathscr{D}(\mathcal{O}_p, R)$ for the above R-modules equipped with their continuous actions of Δ via λ_Ω above. When R_0 is a ring of definition containing the image of λ_Ω and $\mathbf{s} \geq \mathbf{s}^\circ(\Omega)$ then we write $\mathbf{A}^{\mathbf{s},\circ}_\Omega = \mathbf{A}^{\mathbf{s},\circ}(\mathcal{O}_p, R)$ and $\mathbf{D}^{\mathbf{s},\circ} = \mathbf{D}^{\mathbf{s},\circ}(\mathcal{O}_p, R)$ for the R_0-modules equipped with their action of Δ above.

(2) If $\lambda \in \mathscr{W}(\overline{\mathbf{Q}}_p)$ with residue field k_λ, we write $\mathbf{A}^{\mathbf{s}}_\lambda$, $\mathbf{D}^{\mathbf{s}}_\lambda$, $\mathscr{A}_\lambda$, and $\mathscr{D}_\lambda$ in place of $\mathbf{A}^{\mathbf{s}}_{\mathrm{Sp}\,k_\lambda}$, $\mathbf{D}^{\mathbf{s}}_{\mathrm{Sp}\,k_\lambda}$, $\mathscr{A}_{\mathrm{Sp}\,k_\lambda}$, and $\mathscr{D}_{\mathrm{Sp}\,k_\lambda}$.

5.4. **The integration map for cohomological weights.** Throughout this subsection we fix $L \subset \overline{\mathbf{Q}}_p$ and assume it contains the Galois closure of F inside $\overline{\mathbf{Q}}_p$. We also consider a fixed cohomological weight $\lambda = (\kappa, w)$. (The notations of the previous two subsections also remain in force.)

Recall we defined the L-vector space $\mathscr{L}_\lambda(L)$, equipped with a left action of $\mathrm{GL}_2(F_p)$ in (2.4.3). It thus inherits an action of the monoid $\Delta \subset \mathrm{GL}_2(F_p)$ from Section 5.3. We also view λ as a p-adic weight $\lambda = (\lambda_1, \lambda_2)$ where λ_i is given by

$$\lambda_i(z) = \prod_{v|p} \prod_{\sigma \in \Sigma_v} \sigma(z)^{e_i(\sigma)}$$

where $e_1(\sigma) = \frac{1}{2}(w + \kappa_\sigma)$ and $e_2(\sigma) = \frac{1}{2}(w - \kappa_\sigma)$. The residue field k_λ of $\lambda \in \mathscr{W}$ is contained in the Galois closure of F inside $\overline{\mathbf{Q}}_p$. Thus to a cohomological weight λ we also have a Δ-module of distributions $\mathscr{D}_\lambda \otimes_{k_\lambda} L$.

Definition 5.4.1. The integration map is the L-linear map $I_\lambda : \mathscr{D}_\lambda \otimes_{k_\lambda} L \to \mathscr{L}_\lambda(L)$ given by

$$I_\lambda(\mu)(X) = \mu((z + X)^\kappa) := \sum_{0 \leq j \leq \kappa} \binom{\kappa}{j} \mu(z^j) X^{\kappa - j}. \tag{5.4.1}$$

It is elementary to check the action of Δ has the following relationship to the integration map: if $g \in \Delta$ and $\mu \in \mathscr{D}_\lambda \otimes_{k_\lambda} L$, then

$$I_\lambda(g \cdot \mu) = \left(\varpi_p^{-v(\det g)}\right)^{\frac{w-\kappa}{2}} g \cdot I_\lambda(\mu). \tag{5.4.2}$$

Definition 5.4.2. $\mathscr{L}^\sharp_\lambda(L) := \mathscr{L}_\lambda(L) \otimes (\varpi_p^{-v(\det g)})^{\frac{w-\kappa}{2}}$ (as a left Δ-module).

Thus $\mathscr{L}^\sharp_\lambda$ is the same underlying L-vector space but the action of Δ has been twisted so that I_λ becomes equivariant (point (1) below). Before stating the next proposition, we note that any left Δ-module becomes a left $\mathcal{O}_p^\times$-module via the inclusion $\begin{pmatrix} \mathcal{O}_p^\times & \\ & 1 \end{pmatrix} \subset \Delta$.

Proposition 5.4.3.

(1) $I_\lambda : \mathscr{D}_\lambda \otimes_{k_\lambda} L \to \mathscr{L}^\sharp_\lambda(L)$ is Δ-equivariant.

(2) If $\mathcal{O}_L \subset L$ denotes the ring of integers and $\mathscr{L}^\sharp_\lambda(\mathcal{O}_L)$ are those polynomials with $\mathcal{O}_L$-coefficients then $\mathscr{L}^\sharp_\lambda(\mathcal{O}_L)$ is Δ-stable.

(3) The identity map $\mathscr{L}_\lambda(L) \to \mathscr{L}^\sharp_\lambda(L)$ is an isomorphism of left $\mathcal{O}_p^\times$-modules.

Proof. Point (1) is immediate from (5.4.2). The second point is straightforward from the definition. The third point is because if $x \in \mathcal{O}_p^\times$ and $g = \begin{pmatrix} x & \\ & 1 \end{pmatrix}$ then $\det(g) \in \mathcal{O}_p^\times$, so $\lambda_2(\varpi_p^{-v_p(\det g)}) = 1$. $\square$

5.5. **p-adic twisting.** In this subsection we consider two p-adic analogs of the twisting studied in Section 4.3. Recall that Γ_F is the Galois group of the maximal abelian extension of F unramified away from p and ∞. Global class field theory defines an isomorphism

$$\Gamma_F \simeq F^\times \backslash \mathbf{A}_F^\times / H$$

where H is the closure of the subgroup generated by $(F_\infty^\times)^\circ \widehat{\mathcal{O}}_F^{(p),\times}$. Thus there is a natural short exact sequence

$$1 \to \mathcal{O}_p^\times / \overline{\mathcal{O}_{F,+}^\times} \to \Gamma_F \to \mathrm{Cl}_F^+ \to 1, \tag{5.5.1}$$

where Cl_F^+ is the narrow class group, $\mathcal{O}_{F,+}^\times$ are the totally positive units, and the bar indicates the p-adic closure under the natural inclusion $\mathcal{O}_{F,+}^\times \subset \mathcal{O}_p^\times$. By Lemma 5.1.2 and (5.5.1), Γ_F is a CPA group. We write $\mathscr{X}(\Gamma_F)$ for the rigid analytic space parameterizing continuous p-adic characters on Γ_F.

Definition 5.5.1. Suppose that R is a $\mathbf{Q}_p$-Banach algebra and N is an R-module equipped with an R-linear left action $g \cdot n$ of the monoid Δ. If $\vartheta : \Gamma_F \to R^\times$ is an R-valued point of $\mathscr{X}(\Gamma_F)$ then we define a new left Δ-module by

$$N(\vartheta) = N \otimes \vartheta^{-1}|_{\mathcal{O}_p^\times}(\det g \cdot \varpi_p^{-v(\det g)}).$$

We note that $\mathscr{X}(\Gamma_F)$ also acts on $\mathscr{W}$ by central twists: if $\lambda = (\lambda_1, \lambda_2)$ is a character on $(\mathcal{O}_p^\times)^{\oplus 2}$ then we define we define $\vartheta \cdot \lambda := (\vartheta|_{\mathcal{O}_p^\times}\lambda_1, \vartheta|_{\mathcal{O}_p^\times}\lambda_2)$.

For the next three results, let $\Omega \to \mathscr{W}$ be a p-adic weight. The previous paragraph allows us to define a new p-adic weight $\vartheta^{-1} \cdot \Omega$ whenever $\vartheta \in \mathscr{X}(\Gamma_F)(\Omega)$.

Lemma 5.5.2. *If $\vartheta \in \mathscr{X}(\Gamma_F)(\Omega)$, then the identity map is an isomorphism $\mathscr{D}_\Omega(\vartheta) \simeq \mathscr{D}_{\vartheta^{-1}\cdot\Omega}$.*

Proof. This follows immediately from the definitions. □

Now consider a compact open subgroup K of $\mathrm{GL}_2(\mathbf{A}_{F,f})$ such that $K_p \subset \Delta$. If N is a left Δ-module then we define a local system on Y_K as in Section 2.2, with $\mathrm{GL}_2^+(F)$ acting trivially and $k \in K_p$ acting on the right as k^{-1} acts on the left. We view $\mathscr{O}(\Omega)$ as a trivial left Δ-module.

Lemma 5.5.3. *If $\vartheta \in \mathscr{X}(\Gamma_F)(\Omega)$, then $\vartheta_{\det} : \mathrm{GL}_2(\mathbf{A}_F) \to \mathscr{O}(\Omega)^\times$ given by $g \mapsto \vartheta(\det g)$ defines an element of $H^0(Y_K, \mathscr{O}(\Omega)(\vartheta))$.*

Proof. Since ϑ is trivial on $(F_\infty^\times)^\circ$, $\vartheta_{\det}$ is trivial on $\mathrm{GL}_2^+(F_\infty)$. So, $\vartheta_{\det}$ is a locally constant on $\mathrm{GL}_2(\mathbf{A}_F)$ and invariant under multiplication by K_∞°. Further, $\vartheta_{\det}$ trivial on $\mathrm{GL}_2(F)$ since ϑ is trivial on $F^\times$. Finally, if $k \in K$ then $\vartheta(\det k) = \vartheta(\det k_p)$ because ϑ vanishes on the units away from p. So finally, if $g \in \mathrm{GL}_2(\mathbf{A}_F)$ and $k \in K$, then

$$\vartheta_{\det}(gk) = \vartheta(\det k_p)\vartheta_{\det}(g) = \vartheta_{\det}(g)|_k.$$

This concludes the proof. □

Definition 5.5.4. If $\vartheta \in \mathscr{X}(\Gamma_F)(\Omega)$ and N is a left $\mathscr{O}(\Omega)[\Delta]$-module then we define the twisting map

$$\mathrm{tw}_\vartheta : H_c^*(Y_K, N) \to H_c^*(Y_K, N(\vartheta))$$

to be cup product with $\vartheta_{\det}$.

Finally we consider Hecke operators $[K\delta K]$ acting on the cohomology $H_c^*(Y_K, N)$.

Proposition 5.5.5. *Assume that $\begin{pmatrix} \mathcal{O}_p^\times & \\ & 1 \end{pmatrix} \subset K_p$. Then, for each finite place v of F we have*

$$[K \left(\begin{smallmatrix} \varpi_v & \\ & 1 \end{smallmatrix}\right) K] \circ \mathrm{tw}_\vartheta = \vartheta(\varpi_v)\, \mathrm{tw}_\vartheta \circ [K \left(\begin{smallmatrix} \varpi_v & \\ & 1 \end{smallmatrix}\right) K].$$

Proof. First, we claim that we can write $K\left(\begin{smallmatrix}\varpi_v & \\ & 1\end{smallmatrix}\right)K = \bigcup \delta_i K$ with $\delta_i \in \mathrm{GL}_2(F_v)$ such that $\det \delta_i = \varpi_v$ if $v \mid p$ and $\vartheta(\det \delta_i) = \vartheta(\varpi_v)$ in general. This is true for any δ_i if $v \nmid p$ since ϑ is trivial on $\det(K^p)$. But if $v \mid p$ and δ_i is any choice then $\det(\delta_i) = \varpi_v u_i^{-1}$ for some $u_i \in \mathcal{O}_v^\times$. By the assumption on K, we can replace δ_i by $\delta_i\left(\begin{smallmatrix}u_i & \\ & 1\end{smallmatrix}\right) \in \delta_i K$.

Now to prove the proposition we fix a choice of δ_i as above and compute using adelic cochains, freely using the notation from Section 2.2. For clarity, let us write $\delta \cdot n$ for the action of Δ on N and $\delta \star n$ for the action of Δ on $N(\vartheta)$. Let $\phi \in C^\bullet_{\mathrm{ad}}(K,N)$. Then, for all $\sigma \in C_\bullet(D_\infty)$ and $g_f \in \mathrm{GL}_2(\mathbf{A}_{F,f})$ we have

$$\begin{aligned}\mathrm{tw}_\vartheta([K\delta K]\phi)(\sigma \otimes [g_f]) &= \vartheta(\det g_f)([K\delta K]\phi)(\sigma\otimes[g_f])\\ &= \vartheta(\det g_f)\sum_i \delta_i\cdot\phi(\sigma\otimes[g_f\delta_i]).\end{aligned}$$

On the other hand, since $\vartheta(\det\delta_i) = \vartheta(\varpi_v)$, and $\det\delta_{i,p}\cdot\varpi_p^{-v(\det\delta_{i,p})} = 1$ we get

$$\begin{aligned}[K\delta K]\,\mathrm{tw}_\vartheta(\phi)(\sigma\otimes[g_f]) &= \sum_i \delta_i \star (\mathrm{tw}_\vartheta\,\phi)(\sigma\otimes[g_f\delta_i])\\ &= \sum_i \vartheta(\det g_f\det\delta_i)\vartheta(\det\delta_{i,p}\cdot\varpi_p^{-v(\det\delta_{i,p})})^{-1}\delta_i\cdot\phi(\sigma\otimes[g_f\delta_i])\\ &= \vartheta(\varpi_v)\vartheta(\det g_f)\sum_i\delta_i\cdot\phi(\sigma\otimes[g_f\delta_i]).\end{aligned}$$

This proves the proposition. ∎

To summarize, under the mild hypothesis of Proposition 5.5.5 (which is satisfied in practice), we can twist distribution-valued Hecke eigenclasses by p-adic characters of Γ_F and obtain new Hecke eigenclasses of a possibly different weight. But the twisting maps tw_ϑ do not preserve the cohomology of the finite-dimensional spaces $\mathscr{L}_\lambda$, so we also need a second kind of twisting that is a direct analog of Section 4.3.

As before, write $\theta : \mathbf{A}_F^\times \to \mathbf{C}^\times$ for a finite order Hecke character but we assume now that it is unramified away from p. Write $\mathfrak{f}$ for its conductor. Then $\theta^\iota := \iota\circ\theta$ defines a finite order character $\theta^\iota : \mathbf{A}_F^\times \to \overline{\mathbf{Q}}_p^\times$ which descends to a character of Γ_F. Suppose that L is a subfield of $\overline{\mathbf{Q}}_p$ containing the Galois closure of F and the values of θ^ι and also let $\mathfrak{n}$ be an integral ideal of $\mathcal{O}_F$. In analogy with Section 4.3 we define a linear map

$$\mathrm{tw}^{\mathrm{cl}}_{\theta^\iota} : H^*_c(Y_1(\mathfrak{n}),\mathscr{L}_\lambda(L)) \to H^*_c(Y_1(\mathfrak{n}\mathfrak{f}^2),\mathscr{L}_\lambda(L)) \tag{5.5.2}$$

by

$$\mathrm{tw}^{\mathrm{cl}}_{\theta^\iota} = \theta^\iota_{\det}\cup\sum_{t\in\Upsilon_{\mathfrak{f}}^\times}\theta^\iota(t)v^*_{t_0,p}.$$

Here the notation is just as in Section 4.3. Note, however, that because the local systems $\mathscr{L}_\lambda(L)$ are defined with respect to a right action of $\mathrm{GL}_2(F_p)$, we no longer have an isomorphism between $v_t^*\mathscr{L}_\lambda(L)$ and $\mathscr{L}_\lambda(L)$. In fact, the map written $v^*_{t,p}$ above is the map on cohomology fitting into the diagram

$$\begin{tikzcd} H^*_c(Y_1(\mathfrak{n}),\mathscr{L}_\lambda(L)) \arrow[r, "v^*_{t,p}"] \arrow[d, "\mathrm{pr}^*"'] & H^*_c(Y_{11}(\mathfrak{n}\mathfrak{f}^2),\mathscr{L}_\lambda(L)) \\ H^*_c(Y_{K_{11}(\mathfrak{n}\mathfrak{f}^2)_t},\mathscr{L}_\lambda(L)) \arrow[r, "r^*_{u_t}"'] & H^*_c(Y_{11}(\mathfrak{n}\mathfrak{f}^2),\mathscr{L}_\lambda(L)(u_t)), \arrow[u, "\simeq"'] \end{tikzcd} \tag{5.5.3}$$

where the right vertical arrow is induced by the isomorphism $P \mapsto u_t \cdot P$ of local systems $\mathscr{L}_\lambda(L)(u_t) \to \mathscr{L}_\lambda(L)$ in the opposite direction of the diagonal arrow in (2.4.4).

The image of $\mathrm{tw}^{\mathrm{cl}}_{\theta^\iota}$ is contained in $H^*_c(Y_1(\mathfrak{n}\mathfrak{f})^2, \mathscr{L}_\lambda(L))$ just as in the proof of Lemma 4.3.2. And, if E is a subfield of $\mathbf{C}$ containing the Galois closure of F and the values of θ and $L = \mathbf{Q}_p(\iota(E))$, then (2.4.4) implies that the diagram

$$\begin{tikzcd} H^d_c(Y_1(\mathfrak{n}), \mathscr{L}_\lambda(L)) \arrow[r, "\mathrm{tw}^{\mathrm{cl}}_{\theta^\iota}"] & H^d_c(Y_1(\mathfrak{n}\mathfrak{f}^2), \mathscr{L}_\lambda(L)) \\ H^d_c(Y_1(\mathfrak{n}), \mathscr{L}_\lambda(E)) \arrow[u, "\iota"] \arrow[r, "\mathrm{tw}_\theta"'] & H^d_c(Y_1(\mathfrak{n}\mathfrak{f}^2), \mathscr{L}_\lambda(E)). \arrow[u, "\iota"'] \end{tikzcd} \tag{5.5.4}$$

is commutative. We record here another adelic cochain computation.

Lemma 5.5.6. *If $\psi \in H^*_c(Y_1(\mathfrak{n}), \mathscr{L}_\lambda(L))$ is represented by $\widetilde{\psi} \in C^\bullet_{\mathrm{ad},c}(K_1(\mathfrak{n}), \mathscr{L}_\lambda(L))$, then $\mathrm{tw}^{\mathrm{cl}}_{\theta^\iota}(\psi) \in H^*_c(Y_1(\mathfrak{n}\mathfrak{f}^2), \mathscr{L}_\lambda(L))$ is represented by $\mathrm{tw}^{\mathrm{cl}}_{\theta^\iota}(\widetilde{\psi}) \in C^\bullet_{\mathrm{ad},c}(K_1(\mathfrak{n}\mathfrak{f}^2), \mathscr{L}_\lambda(L))$ whose value on a singular chain $\sigma = \sigma_\infty \otimes [g_f]$ is given by*

$$\mathrm{tw}^{\mathrm{cl}}_{\theta^\iota}(\widetilde{\psi})(\sigma) = \theta^\iota(\det g_f) \sum_{t \in \Upsilon_{\mathfrak{f}}^\times} \theta^\iota(t) \left(\begin{smallmatrix} 1 & t_0 \\ & 1 \end{smallmatrix}\right) \cdot \widetilde{\psi}(\sigma \left(\begin{smallmatrix} 1 & t_0 \\ & 1 \end{smallmatrix}\right)).$$

Proof. First, $\theta^\iota_{\det} \in H^0(Y_{11}(\mathfrak{n}f^2), L)$ is given by $g \mapsto \theta^\iota(\det g)$ and it is clearly represented on the level of adelic cochains by $\sigma_\infty \otimes [g_f] \mapsto \theta^\iota(\det g_f)$ (since θ^ι is trivial on $(F_\infty^\times)^\circ$). Comparing our claim with the definition of $\mathrm{tw}_{\theta^\iota}$, it is enough to show that $v^*_{t,p}(\psi)$ is represented by the adelic cochain

$$v^*_{t,p}(\widetilde{\psi})(\sigma) = \left(\begin{smallmatrix} 1 & t \\ & 1 \end{smallmatrix}\right) \cdot \widetilde{\psi}(\sigma \left(\begin{smallmatrix} 1 & t \\ & 1 \end{smallmatrix}\right)) \tag{5.5.5}$$

for any $t \in \mathbf{A}_{F,f}$. According to the definition (5.5.3) above, $v^*_{t,p}$ is the composition of three maps. The first map is the pullback of a projection. The second is the map induced by right multiplication by u_t. The third map is the map $P \mapsto u_t \cdot P$ on the level of local systems $\mathscr{L}_\lambda(L)(u_t) \mapsto \mathscr{L}_\lambda(L)$. Thus the computation (5.5.5) of $v^*_{t,p}(\widetilde{\psi})$ is immediate from the explanation following Proposition 2.2.1. □

Remark 5.5.7. The classical twisting (5.5.2) defined here compares directly with the twisting in Definition 5.5.4. Suppose that $\vartheta = \theta^\iota$ is a finite order p-adic Hecke character of Γ_F. We can apply the above discussion to $\mathfrak{n} \cap \mathbf{p}$ and then deduce is a commuting diagram

$$\begin{tikzcd} H^*_c(Y_1(\mathfrak{n}\cap\mathbf{p}), \mathscr{D}_\lambda) \arrow[r, "\mathrm{tw}_\vartheta"] \arrow[d, "I_\lambda"'] & H^*_c(Y_1(\mathfrak{n}\cap\mathbf{p}), \mathscr{D}_\lambda(\vartheta)) \arrow[r, "I_\lambda"] & H^*_c(Y_1(\mathfrak{n}\cap\mathbf{p}), \mathscr{L}^\sharp_\lambda(\vartheta)) \arrow[d, "\sum \theta^\iota(t) v^*_{t_0}"] \\ H^*_c(Y_1(\mathfrak{n}\cap\mathbf{p}), \mathscr{L}^\sharp_\lambda) \arrow[rr, "\mathrm{tw}^{\mathrm{cl}}_{\theta^\iota}"] & & H^*_c(Y_1((\mathfrak{n}\cap\mathbf{p})\mathfrak{f}^2), \mathscr{L}^\sharp_\lambda), \end{tikzcd}$$

where the right vertical arrow makes implicit use of the identity map inducing an isomorphism $\mathscr{L}_\lambda(\vartheta) \simeq \mathscr{L}_\lambda$ of local systems on $Y_{11}((\mathfrak{n}\cap\mathbf{p})\mathfrak{f}^2)$.

6. The eigenvariety

In this section we assume that $\mathfrak{n}$ is an integral ideal that is *co-prime* to p. Our goal is to define a certain eigenvariety of tame level $\mathfrak{n}$ and then show that reasonable classical points are smooth on this eigenvariety.

6.1. **A weight space.** Recall the notation from the start of Section 5.5. View $\overline{\mathcal{O}_{F,+}^\times} \subset T(\mathcal{O}_p)$ as a closed subgroup via the diagonal embedding.

Definition 6.1.1. $\mathscr{W}(1) := \mathscr{X}(T(\mathcal{O}_p)/\overline{\mathcal{O}_{F,+}^\times})$.

The dimension of $\mathscr{W}(1)$ as a rigid analytic space is $1 + d + \delta_{F,p}$ where $\delta_{F,p}$ is the Leopoldt defect, defined here to be one less than the dimension of $\mathcal{O}_p^\times/\overline{\mathcal{O}_F^\times}$ as a CPA group. There is a natural closed immersion $\mathscr{W}(1) \to \mathscr{W}$ and every cohomological weight defines a point in $\mathscr{W}(1)(\overline{\mathbf{Q}}_p)$.[18] There is also a natural action of $\mathscr{X}(\mathcal{O}_p^\times/\overline{\mathcal{O}_{F,+}^\times})$ on $\mathscr{W}(1)$ by central twisting (compare with Section 5.5). We denote this action by $\eta \cdot \lambda$ for $\eta \in \mathscr{X}(\mathcal{O}_p^\times/\overline{\mathcal{O}_{F,+}^\times})$ and $\lambda \in \mathscr{W}(1)$.

Definition 6.1.2. A weight $\lambda \in \mathscr{W}(1)(\overline{\mathbf{Q}}_p)$ is called twist cohomological if it is in the $\mathscr{X}(\mathcal{O}_p^\times/\overline{\mathcal{O}_{F,+}^\times})(\overline{\mathbf{Q}}_p)$-orbit of the cohomological weights.

The ambiguity in being simultaneously twist cohomological and cohomological is easy to control.

Lemma 6.1.3. *If $\lambda = (\kappa, w)$ and $\lambda' = (\kappa', w')$ are two cohomological weights and $\eta \in \mathscr{X}(\mathcal{O}_p^\times/\overline{\mathcal{O}_{F,+}^\times})(\overline{\mathbf{Q}}_p)$ such that $\lambda = \eta \cdot \lambda'$, then η is of the form $z \mapsto z^n$ for some $n \in \mathbf{Z}$, $\kappa = \kappa'$, and $w = w' + 2n$.*

We clarify before the proof that $z \mapsto z^n$ means the character on $\mathcal{O}_p^\times$ given by $z = (z_v) \mapsto \prod_{v|p}\prod_{\sigma\in\Sigma_v} \sigma(z_v)^n$.

Proof of Lemma 6.1.3. Write $\lambda = (\lambda_1, \lambda_2)$ and similarly for λ'. By assumption, we have $\lambda_i = \eta\lambda_i'$ for $i = 1, 2$. In particular $z^\kappa = \lambda_1\lambda_2^{-1} = \lambda_1'\lambda_2'^{-1} = z^{\kappa'}$, so $\kappa = \kappa'$. Since κ determines the parity of w (and the same for κ' and w') we conclude that $w - w'$ is an even integer, say $w - w' = 2n$. We finally deduce $\eta = \lambda_1\lambda_1'^{-1} = z^{\frac{w-w'}{2}} = z^n$, as claimed. □

Recall that if X is a rigid analytic space and $Z \subset X(\overline{\mathbf{Q}}_p)$ is a subset then Z is said to be accumulating if for each $z \in Z$ and U a connected admissible open neighborhood of z, $Z \cap U$ is Zariski-dense in U.

Lemma 6.1.4. *The twist cohomological weights in $\mathscr{W}(1)$ are Zariski-dense and accumulating.*

Proof. Clear. □

6.2. **Distribution-valued cohomology and eigenvarieties.** We write $I \subset \mathrm{GL}_2(\mathcal{O}_p)$ for the subgroup of matrices that are upper triangular modulo $p\mathcal{O}_p$. Since $I \subset \Delta$, each point $\Omega \to \mathscr{W}(1)$ defines a local system $\mathscr{D}_\Omega$ on $Y_{K_1(\mathfrak{n})I}$ and so we get associated $\mathscr{O}(\Omega)$-modules $H_c^*(\mathfrak{n}, \mathscr{D}_\Omega) := H_c^*(Y_{K_1(\mathfrak{n})I}, \mathscr{D}_\Omega)$ and $H_c^*(\mathfrak{n}, \mathbf{D}_\Omega^{\mathbf{s}}) := H_c^*(Y_{K_1(\mathfrak{n})I}, \mathbf{D}_\Omega^{\mathbf{s}})$ (for $\mathbf{s} \geq \mathbf{s}(\Omega)$). We define $H_*^{\mathrm{BM}}(\mathfrak{n}, \mathscr{A}_\Omega)$ and $H_*^{\mathrm{BM}}(\mathfrak{n}, \mathbf{A}_\Omega^{\mathbf{s}})$ similarly. Denote by $\mathbf{T}(\mathfrak{n}) \subset \mathbf{T}_{\mathbf{Q}_p}(K_1(\mathfrak{n})I)$ the $\mathbf{Q}_p$-subalgebra generated just by the operators T_v, S_v for $v \nmid \mathfrak{n}\mathbf{p}$ and U_v for $v \mid p$. Because Δ contains the elements $\left(\begin{smallmatrix}\varpi_v & \\ & 1\end{smallmatrix}\right)$ for $v \mid p$, the algebra $\mathbf{T}(\mathfrak{n})$ acts by $\mathscr{O}(\Omega)$-linear endomorphisms on $H_c^*(\mathfrak{n}, \mathscr{D}_\Omega)$, $H_*^{\mathrm{BM}}(\mathfrak{n}, \mathscr{A}_\Omega)$, $H_c^*(\mathfrak{n}, \mathbf{D}_\Omega^{\mathbf{s}})$, and $H_*^{\mathrm{BM}}(\mathfrak{n}, \mathbf{A}_\Omega^{\mathbf{s}})$. Finally we set $U_p := \prod_{v|p} U_v^{e_v} \in \mathbf{T}(\mathfrak{n})$.

Remark 6.2.1. Before moving forward, we acknowledge that we will reference many results from [43] below that are, strictly speaking, written with ordinary (co)homology rather than (co)homology with supports. The changes required in [43] are either explained there, implicit there, or they are inconsequential and transparent. We will directly reference [43] without further warning.

[18] We could have also considered a more general p-adic weight space. Namely, we could also take $\mathscr{W}(\mathfrak{n})$ defined to be those continuous characters of $T(\mathcal{O}_p)$ which vanish on the finite index subgroup $\Gamma(\mathfrak{n}) \subset \mathcal{O}_{F,+}^\times$ of units u which are congruent to 1 mod $\mathfrak{n}$. Then $\mathscr{W}(1) \subset \mathscr{W}(\mathfrak{n})$ is an open and closed embedding onto a union of connected components containing all the cohomological weights. But the local systems $\mathscr{D}_\lambda$ at level $\mathfrak{n}\mathbf{p}$ considered below are non-trivial exactly for $\lambda \in \mathscr{W}(\mathfrak{n})$.

For the rest of this subsection, Ω will denote an affinoid open subdomain in $\mathscr{W}(1)$ and $\mathbf{s}$ will implicitly mean $\mathbf{s} \geq \mathbf{s}(\Omega)$. Since $\mathbf{D}_\Omega^{\mathbf{s}}$ and $\mathbf{A}_\Omega^{\mathbf{s}}$ are $\mathbf{Q}_p$-vector spaces, the homology $H_*^{\mathrm{BM}}(\mathfrak{n}, \mathbf{A}_\Omega^{\mathbf{s}})$ is computed by a Borel–Serre complex $C_\bullet^{\mathrm{BM}}(\mathfrak{n}, \mathbf{A}_\Omega^{\mathbf{s}})$. The cohomology $H_c^*(\mathfrak{n}, \mathbf{D}_\Omega^{\mathbf{s}})$ is also computed by a Borel–Serre cochain complex $C_c^\bullet(\mathfrak{n}, \mathbf{D}_\Omega^{\mathbf{s}})$ (similarly for $\mathscr{A}_\Omega$ and $\mathscr{D}_\Omega$). These are complexes whose terms are finite direct sums of copies of the coefficients, or possibly the invariants of such a complex by the action of a finite group (see [43, Section 2.1]).

The operator U_p lifts to a compact operator (which we abusively write using the same symbol) on $C_\bullet^{\mathrm{BM}}(\mathfrak{n}, \mathbf{A}_\Omega^{\mathbf{s}})$. The Fredholm series $f_\Omega(t) = \det\left(1 - tU_p | C_\bullet^{\mathrm{BM}}(\mathfrak{n}, \mathbf{A}_\Omega^{\mathbf{s}})\right)$ is an entire function in t over $\mathscr{O}(\Omega)$, by [43, Proposition 3.1.1] it is independent of $\mathbf{s}$, and it behaves naturally under base change $\Omega \to \Omega'$. Write $f(t) \in \mathscr{O}(\mathscr{W}(1))\{\{t\}\}$ for the unique function whose restriction to each Ω is f_Ω. Following [43, Section 4.1], we say that a pair (Ω, h), with $h \geq 0$ a real number, is slope adapted if the series f_Ω admits a slope-$\leq h$ decomposition $f_\Omega = Q_{\Omega,h} R_{\Omega,h}$ (where $Q_{\Omega,h}$ is a polynomial; see [6, Section 4]). In that case, $\mathscr{Z}_{\Omega,h} := \mathrm{Sp}(\mathscr{O}(\Omega)[t]/Q_{\Omega,h}\mathscr{O}(\Omega)[t])$ is naturally an affinoid open subdomain of the spectral curve $\mathscr{Z} \subset \mathscr{W}(1) \times \mathbf{G}_m$ for f. By [43, Proposition 4.1.4], the $\mathscr{Z}_{\Omega,h}$ form an admissible covering of $\mathscr{Z}$, as (Ω, h) runs over slope adapted pairs. We summarize the facts we will need from [43, Section 3.1].

Proposition 6.2.2. *Suppose that (Ω, h) is slope adapted.*

(1) $C_\bullet^{\mathrm{BM}}(\mathfrak{n}, \mathscr{A}_\Omega)$ and $C_c^\bullet(\mathfrak{n}, \mathscr{D}_\Omega)$ admit slope-$\leq h$ decompositions

$$C_\bullet^{\mathrm{BM}}(\mathfrak{n}, \mathscr{A}_\Omega) \simeq C_\bullet^{\mathrm{BM}}(\mathfrak{n}, \mathscr{A}_\Omega)_{\leq h} \oplus C_\bullet^{\mathrm{BM}}(\mathfrak{n}, \mathscr{A}_\Omega)_{>h}$$
$$C_c^\bullet(\mathfrak{n}, \mathscr{D}_\Omega) \simeq C_c^\bullet(\mathfrak{n}, \mathscr{D}_\Omega)_{\leq h} \oplus C_c^\bullet(\mathfrak{n}, \mathscr{D}_\Omega)_{>h}.$$

(2) $C_c^\bullet(\mathfrak{n}, \mathscr{D}_\Omega)_{\leq h} \simeq \mathrm{Hom}_{\mathscr{O}(\Omega)}(C_\bullet^{\mathrm{BM}}(\mathfrak{n}, \mathscr{A}_\Omega)_{\leq h}, \mathscr{O}(\Omega))$.

(3) The homology $H_*^{\mathrm{BM}}(\mathfrak{n}, \mathscr{A}_\Omega)$ and cohomology $H_c^*(\mathfrak{n}, \mathscr{D}_\Omega)$ also admit slope-$\leq h$ decompositions

$$H_*^{\mathrm{BM}}(\mathfrak{n}, \mathscr{A}_\Omega) \simeq H_*^{\mathrm{BM}}(\mathfrak{n}, \mathscr{A}_\Omega)_{\leq h} \oplus H_*^{\mathrm{BM}}(\mathfrak{n}, \mathscr{A}_\Omega)_{>h}$$
$$H_c^*(\mathfrak{n}, \mathscr{D}_\Omega) \simeq H_c^*(\mathfrak{n}, \mathscr{D}_\Omega)_{\leq h} \oplus H_c^*(\mathfrak{n}, \mathscr{D}_\Omega)_{>h}.$$

(4) $H_*^{\mathrm{BM}}(\mathfrak{n}, \mathscr{A}_\Omega)_{\leq h} = H_*(C_\bullet^{\mathrm{BM}}(\mathfrak{n}, \mathscr{A}_\Omega)_{\leq h})$ and $H_c^*(\mathfrak{n}, \mathscr{D}_\Omega)_{\leq h} = H^*(C_c^\bullet(\mathfrak{n}, \mathscr{D}_\Omega)_{\leq h})$.

(5) If $\Omega' \subset \Omega$ is an affinoid subdomain, then the slope-$\leq h$ parts in (1) and (3) naturally commute with base change $\mathscr{O}(\Omega) \to \mathscr{O}(\Omega')$.

Proof. See the second through the fifth propositions of [43, Section 3.1]. □

The complexes $C_\bullet^{\mathrm{BM}}(\mathfrak{n}, \mathscr{A}_\Omega)_{\leq h}$ and $C_c^\bullet(\mathfrak{n}, \mathscr{D}_\Omega)_{\leq h}$ are naturally complexes $\mathscr{O}(\mathscr{Z}_{\Omega,h})$-modules where $t \in \mathscr{O}(\mathscr{Z}_{\Omega,h})$ acts via U_p^{-1}.

Proposition 6.2.3. *There exists complexes of coherent $\mathscr{O}_{\mathscr{Z}}$-modules $\mathscr{K}_\bullet^{\mathrm{BM}}$ and $\mathscr{K}_c^\bullet$ on $\mathscr{Z}$ uniquely determined by the property that*

$$\mathscr{K}_\bullet^{\mathrm{BM}}(\mathscr{Z}_{\Omega,h}) \simeq C_\bullet^{\mathrm{BM}}(\mathfrak{n}, \mathscr{A}_\Omega)_{\leq h}$$
$$\mathscr{K}_c^\bullet(\mathscr{Z}_{\Omega,h}) \simeq C_c^\bullet(\mathfrak{n}, \mathscr{D}_\Omega)_{\leq h}$$

for any slope adapted pair (Ω, h).

Proof. This is proven just like [43, Proposition 4.3.1] (the essential point is Proposition 6.2.2(5)). □

Definition 6.2.4. $\mathscr{M}_*^{\mathrm{BM}}$ (resp. $\mathscr{M}_c^*$) is the homology (resp. cohomology) sheaf of the complex $\mathscr{K}_\bullet^{\mathrm{BM}}$ (resp. $\mathscr{K}_c^\bullet$).

Thus, $\mathscr{M}_*^{\mathrm{BM}}$ and $\mathscr{M}_c^*$ are graded coherent $\mathscr{O}_{\mathscr{Z}}$-modules and if (Ω, h) is a slope adapted pair, then $\mathscr{M}_*^{\mathrm{BM}}(\mathscr{Z}_{\Omega,h}) \simeq H_*^{\mathrm{BM}}(\mathfrak{n}, \mathscr{A}_\Omega)_{\leq h}$ and $\mathscr{M}_c^*(\mathscr{Z}_{\Omega,h}) \simeq H_c^*(\mathfrak{n}, \mathscr{D}_\Omega)_{\leq h}$. We further have natural ring morphisms

$$\begin{tikzcd}[column sep=small] & \mathrm{End}_{\mathscr{O}(\mathscr{Z}_{\Omega,h})}\left(H_c^*(\mathfrak{n}, \mathscr{D}_\Omega)_{\leq h}\right) \\ \mathbf{T}(\mathfrak{n}) \arrow[ur, "\psi_{\Omega,h}"] \arrow[dr, "\psi'_{\Omega,h}"'] & \\ & \mathrm{End}_{\mathscr{O}(\mathscr{Z}_{\Omega,h})}\left(H_*^{\mathrm{BM}}(\mathfrak{n}, \mathscr{A}_\Omega)_{\leq h}\right), \end{tikzcd}$$

which glue to define morphisms $\psi : \mathbf{T}(\mathfrak{n}) \to \mathrm{End}_{\mathscr{O}_{\mathscr{Z}}}(\mathscr{M}_c^*)$ and $\psi' : \mathbf{T}(\mathfrak{n}) \to \mathrm{End}_{\mathscr{O}_{\mathscr{Z}}}(\mathscr{M}_*^{\mathrm{BM}})$.

Definition 6.2.5. The eigenvariety $\mathscr{E}(\mathfrak{n})$ (resp. $\mathscr{E}'(\mathfrak{n})$) is the $\mathbf{Q}_p$-rigid analytic space associated to the eigenvariety datum $(\mathscr{W}(1), \mathscr{Z}, \mathscr{M}_c^*, \mathbf{T}(\mathfrak{n}), \psi)$ (resp. $(\mathscr{W}(1), \mathscr{Z}, \mathscr{M}_*^{\mathrm{BM}}, \mathbf{T}(\mathfrak{n}), \psi')$) as in [43, Definition 4.3.2].

Remark 6.2.6. By calling one $\mathscr{E}(\mathfrak{n})$ and the other $\mathscr{E}'(\mathfrak{n})$, we indicate our focus on the distribution-valued cohomology. The function-valued homology is only a technical tool used later (see Section 6.4). Thus, in what follows, we will only indicate homology versions of results when strictly necessary (the reader should not infer a lack of truth from their lack of exposition).

The rest of this subsection concerns the basic properties of the eigenvariety $\mathscr{E}(\mathfrak{n})$. For instance, $\mathscr{E}(\mathfrak{n})$ comes equipped with a pair of maps $\upsilon : \mathscr{E}(\mathfrak{n}) \to \mathscr{Z}$, which is finite, and $\lambda : \mathscr{E}(\mathfrak{n}) \to \mathscr{W}(1)$ that factorize

(6.2.1)
$$\begin{tikzcd} \mathscr{E}(\mathfrak{n}) \arrow[r, "\upsilon"] \arrow[dr, "\lambda"'] & \mathscr{Z} \arrow[d, "\mathrm{pr}"] \\ & \mathscr{W}(1) \end{tikzcd}$$

where $\mathrm{pr} : \mathscr{Z} \subset \mathscr{W}(1) \times \mathbf{G}_m \to \mathscr{W}(1)$ is the projection. If $x \in \mathscr{E}(\mathfrak{n})$ we prefer to write $\lambda_x \in \mathscr{W}(1)$ for its weight, rather than $\lambda(x)$. By [43, Theorem 4.3.3], if $\lambda \in \mathscr{W}(1)$ is fixed, then the points $x \in \mathscr{E}(\mathfrak{n})(\overline{\mathbf{Q}}_p)$ with $\lambda_x = \lambda$ are in bijection with the ring morphisms $\psi_x : \mathbf{T}_\lambda(\mathfrak{n}) \to \overline{\mathbf{Q}}_p$ where

$$\mathbf{T}_\lambda(\mathfrak{n}) := \varprojlim_{h \to \infty} \mathrm{im}\left(\mathbf{T}(\mathfrak{n}) \to \mathrm{End}_{k_\lambda}(H_c^*(\mathfrak{n}, \mathscr{D}_\lambda)_{\leq h})\right).$$

Given $x \in \mathscr{E}(\mathfrak{n})(\overline{\mathbf{Q}}_p)$, we write $\mathfrak{m}_x \subset \mathbf{T}(\mathfrak{n})$ for the maximal ideal

$$\mathfrak{m}_x := \ker\left(\mathbf{T}(\mathfrak{n}) \to \mathbf{T}_\lambda(\mathfrak{n}) \xrightarrow{\psi_x} \overline{\mathbf{Q}}_p\right).$$

We also write k_x for the residue field of x.

The rigid analytic spaces $\mathscr{Z}$ and $\mathscr{W}(1)$ are both equidimensional of the same dimension. Since the map υ in (6.2.1) is finite, every irreducible component of $\mathscr{E}(\mathfrak{n})$ has dimension at most $\dim \mathscr{Z} = \dim \mathscr{W}(1) = 1 + d + \delta_{F,p}$. The space $\mathscr{E}(\mathfrak{n})$ is generally *not* equidimensional beyond the case $F = \mathbf{Q}$. For instance, if $d > 1$ there is always an Eisenstein component of $\mathscr{E}(\mathfrak{n})$ of dimension strictly smaller than $1 + d + \delta_{F,p}$.

Proposition 6.2.7. *If $X \subset \mathscr{E}(\mathfrak{n})$ is an irreducible component of (maximal) dimension $1 + d + \delta_{F,p}$, then $\lambda(X) \subset \mathscr{W}(1)$ is Zariski-open.*

Proof. The map υ is finite and X is closed in $\mathscr{E}(\mathfrak{n})$, so $\upsilon(X) \subset \mathscr{Z}$ is closed. Moreover, it is evidently irreducible of dimension $\dim \mathscr{Z}$. Thus $\upsilon(X)$ is an irreducible component of $\mathscr{Z}$ ([34, Corollary 2.2.7]). Since the irreducible components of the Fredholm variety $\mathscr{Z}$ are all defined by Fredholm hypersurfaces ([43, Proposition 4.1.2]), we deduce $\lambda(X) = \mathrm{pr}(\upsilon(X))$ is Zariski-open in $\mathscr{W}(1)$ from [43, Proposition 4.1.3]. $\square$

We will also need to briefly give atlases for our eigenvarieties. The eigenvariety $\mathscr{E}(\mathfrak{n})$ is admissibly covered by affinoid subdomains $\mathscr{E}_{\Omega,h} := \mathrm{Sp}(\mathbf{T}_{\Omega,h})$ where $\mathbf{T}_{\Omega,h}$ is the $\mathcal{O}(\Omega)$-algebra generated by the image of $\psi_{\Omega,h}$ inside $\mathrm{End}_{\mathcal{O}(\Omega)}(H^*_c(\mathfrak{n}, \mathscr{D}_\Omega)_{\leq h})$ and (Ω, h) runs over slope adapted pairs. Similarly, $\mathscr{E}'(\mathfrak{n})$ is covered by affinoid subdomains $\mathscr{E}'_{\Omega,h} := \mathrm{Sp}(\mathbf{T}'_{\Omega,h})$ where $\mathbf{T}'_{\Omega,h}$ is the $\mathcal{O}(\Omega)$-algebra generated by the image of $\psi'_{\Omega,h}$ inside $\mathrm{End}_{\mathscr{O}(\Omega)}(H^{\mathrm{BM}}_*(\mathscr{A}_\Omega)_{\leq h})$ and (Ω, h) is a slope adapted pair. The graded sheaf $\mathscr{M}^*_c$ on $\mathscr{Z}$ naturally gives rise to a graded sheaf of $\mathscr{O}_{\mathscr{E}(\mathfrak{n})}$-modules, for which we use the same notation, whose sections $\mathscr{M}^*_c(\mathscr{E}_{\Omega,h})$ are canonically identified with $H^*_c(\mathfrak{n}, \mathscr{D}_\Omega)_{\leq h}$. Similarly, there is a graded sheaf $\mathscr{M}^{\mathrm{BM}}_*$ on $\mathscr{E}'(\mathfrak{n})$ whose sections are given by $\mathscr{M}^{\mathrm{BM}}_*(\mathscr{E}'_{\Omega,h}) \simeq H^{\mathrm{BM}}_*(\mathfrak{n}, \mathscr{A}_\Omega)_{\leq h}$. (All of this follows from the construction of eigenvarieties as in the proof of [43, Theorem 4.2.2].)

Definition 6.2.8. Let $x \in \mathscr{E}(\mathfrak{n})(\overline{\mathbf{Q}}_p)$. A good neighborhood of x is a connected affinoid open U containing x with the property that there exists a slope adapted pair (Ω, h) such that U is a connected component of $\mathscr{E}_{\Omega,h}$.

If U is a good neighborhood of x and (Ω, h) is as in the definition thereof, denote by $e_U \in \mathbf{T}_{\Omega,h}$ the idempotent so that $\mathscr{O}(U) = e_U \mathbf{T}_{\Omega,h}$. Then, $\mathscr{M}^*_c(U) \cong e_U H^*_c(\mathfrak{n}, \mathscr{D}_\Omega)_{\leq h}$ is a Hecke-stable direct summand of $H^*_c(\mathfrak{n}, \mathscr{D}_\Omega)_{\leq h}$. The affinoid U is completely determined by the triple (Ω, h, e_U), and we say that U belongs to the slope adapted pair (Ω, h).

Proposition 6.2.9. *For any $x \in \mathscr{E}(\mathfrak{n})$, the collection of good neighborhoods of x are cofinal in the collection of admissible opens containing x.*

Proof. This proposition is a direct consequence of the construction of $\mathscr{E}(\mathfrak{n})$. $\square$

6.3. **Some special points.** In this subsection, we catalog certain important points on $\mathscr{E}(\mathfrak{n})$. Traditionally this would mean discussing "classical points." Here we discuss, as well, twists of classical points by p-adic Hecke characters, some of which do not exist if Leopoldt's conjecture is true for F.

For the moment, suppose that $\psi : \mathbf{T}(\mathfrak{n}) \to \overline{\mathbf{Q}}_p$ is a Hecke eigensystem and $\vartheta \in \mathscr{X}(\Gamma_F)(\overline{\mathbf{Q}}_p)$. Then we define a new Hecke eigensystem

$$\mathrm{tw}_\vartheta(\psi)(T) := \begin{cases} \vartheta(\varpi_v)\psi(T) & \text{if } T = T_v \text{ and } v \nmid \mathfrak{n}\mathbf{p} \text{ or } T = U_v \text{ and } v \mid p; \\ \vartheta(\varpi_v)^2\psi(T) & \text{if } T = S_v \text{ and } v \nmid \mathfrak{n}\mathbf{p}. \end{cases} \tag{6.3.1}$$

Let $\mathfrak{m}_\psi = \ker(\psi)$ and similarly set $\mathfrak{m}_{\mathrm{tw}_\vartheta(\psi)} = \ker(\mathrm{tw}_\vartheta(\psi))$. Recall that in Definition 5.5.4 we introduced a linear map tw_ϑ on the distribution-valued cohomology (see Lemma 5.5.2 also).

Lemma 6.3.1.

(1) $v_p(\psi(U_v)) = v_p(\mathrm{tw}_\vartheta(\psi)(U_v))$ for each $v \mid p$.

(2) The linear map tw_ϑ induces an isomorphism

$$\mathrm{tw}_\vartheta : H^*_c(\mathfrak{n}, \mathscr{D}_\lambda)_{\mathfrak{m}_\psi} \xrightarrow{\simeq} H^*_c(\mathfrak{n}, \mathscr{D}_{\vartheta^{-1}\cdot\lambda})_{\mathfrak{m}_{\mathrm{tw}_\vartheta(\psi)}}.$$

Proof. The group Γ_F is compact, so $\vartheta(\varpi_v)$ is a unit for all places v. That proves part (1). For part (2), tw_ϑ defines an isomorphism on the level of vector spaces (before localizing) because its inverse is $\mathrm{tw}_{\vartheta^{-1}}$. The compatibility with the Hecke action follows from Proposition 5.5.5. $\square$

Lemma 6.3.1 implies the following is well-posed.

Definition 6.3.2. If $x \in \mathscr{E}(\mathfrak{n})(\overline{\mathbf{Q}}_p)$ and $\vartheta \in \mathscr{X}(\Gamma_F)(\overline{\mathbf{Q}}_p)$, then we define $\mathrm{tw}_\vartheta(x) \in \mathscr{E}(\mathfrak{n})(\overline{\mathbf{Q}}_p)$ to be the point corresponding to the Hecke eigensystem $\mathrm{tw}_\vartheta(\psi_x)$.

One can view twisting by characters of Γ_F as giving a group action of $\mathscr{X}(\Gamma_F)(\overline{\mathbf{Q}}_p)$ on $\mathscr{E}(\mathfrak{n})(\overline{\mathbf{Q}}_p)$ compatible with the weight twisting in that

$$\begin{tikzcd} \mathscr{X}(\Gamma_F)(\overline{\mathbf{Q}}_p) \times \mathscr{E}(\mathfrak{n})(\overline{\mathbf{Q}}_p) \arrow[r, "(\vartheta, x)\mapsto \mathrm{tw}_\vartheta(x)"] \arrow[d, "\left(\vartheta|_{\mathcal{O}_p^\times}, \lambda\right)"'] & \mathscr{E}(\mathfrak{n})(\overline{\mathbf{Q}}_p) \arrow[d, "\lambda"] \\ \mathscr{X}(\mathcal{O}_p^\times/\overline{\mathcal{O}_{F,+}^\times})(\overline{\mathbf{Q}}_p) \times \mathscr{W}(1)(\overline{\mathbf{Q}}_p) \arrow[r, "(\eta,\lambda)\mapsto \eta^{-1}\cdot\lambda"'] & \mathscr{W}(1)(\overline{\mathbf{Q}}_p) \end{tikzcd} \tag{6.3.2}$$

is a commuting diagram. Of course, this is completely functorial and then gives actions on the level of rigid analytic groups.

Lemma 6.3.3. *For $x \in \mathscr{E}(\mathfrak{n})(\overline{\mathbf{Q}}_p)$, x is in the $\mathscr{X}(\Gamma_F)(\overline{\mathbf{Q}}_p)$-orbit of a point of cohomological weight if and only if λ_x is twist cohomological (Definition 6.1.2).*

Proof. By (6.3.2), if $x = \mathrm{tw}_\vartheta(x')$ and x' has cohomological weight, then x has twist cohomological weight. On the other hand, suppose that $\lambda_x = \eta \cdot \lambda$ where λ is a cohomological weight and $\eta \in \mathscr{X}(\mathcal{O}_p^\times/\overline{\mathcal{O}_{F,+}^\times})(\overline{\mathbf{Q}}_p)$. Then, choose any one of the finite number of extensions ϑ of η to a character of Γ_F and set $x' = \mathrm{tw}_\vartheta(x)$. By (6.3.2) again, x' has weight λ and thus $x = \mathrm{tw}_{\vartheta^{-1}}(x')$ is in the $\mathscr{X}(\Gamma_F)(\overline{\mathbf{Q}}_p)$-orbit of a point of cohomological weight. ☐

Now suppose that π is a cohomological cuspidal automorphic representation whose prime-to-p conductor divides $\mathfrak{n}$. Then, each choice of p-refinement α for π defines a Hecke eigensystem $\psi_{(\pi,\alpha)} : \mathbf{T}(\mathfrak{n}) \to \overline{\mathbf{Q}}_p$, depending on ι. Write $\mathfrak{m}_{(\pi,\alpha)} = \ker(\psi_{(\pi,\alpha)}) \subset \mathbf{T}(\mathfrak{n})$. If $L \subset \overline{\mathbf{Q}}_p$ denotes the residue field of $\psi_{(\pi,\alpha)}$ then $H^*_c(\mathfrak{n}, \mathscr{L}_\lambda(L))_{\mathfrak{m}_{(\pi,\alpha)}} \neq (0)$. Further, denote by $\psi^\sharp_{(\pi,\alpha)} : \mathbf{T}(\mathfrak{n}) \to \overline{\mathbf{Q}}_p$ the ring morphism where $\psi^\sharp_{(\pi,\alpha)}(T) = \psi_{(\pi,\alpha)}(T)$ for $T = T_v$ or $T = S_v$ with $v \nmid \mathfrak{n}\mathbf{p}$ and

$$\psi^\sharp_{(\pi,\alpha)}(U_v) = \alpha_v^\sharp = \varpi_v^{\frac{\kappa - w}{2}}\alpha_v = \varpi_v^{\frac{\kappa-w}{2}}\psi_{(\pi,\alpha)}(U_v) \quad (\text{if } v \mid p).$$

We write $\mathfrak{m}^\sharp_{(\pi,\alpha)} = \ker(\psi^\sharp_{(\pi,\alpha)})$. Thus, $H^*_c(\mathfrak{n}, \mathscr{L}^\sharp_\lambda(L))_{\mathfrak{m}^\sharp_{(\pi,\alpha)}} \neq (0)$ for $\mathscr{L}^\sharp_\lambda$ defined in Section 5.4.

Definition 6.3.4. Let $x \in \mathscr{E}(\mathfrak{n})(\overline{\mathbf{Q}}_p)$ be a point of cohomological weight $\lambda = (\kappa, w)$.

(1) $x := x(\pi, \alpha)$ is called classical if $\psi_x = \psi^\sharp_{(\pi,\alpha)}$ for some (unique) p-refined cuspidal automorphic representation (π, α) of weight λ and prime-to-p conductor dividing $\mathfrak{n}$. In this case we write $x = x(\pi, \alpha)$. We refer to the prime-to-p conductor of x as the prime-to-p conductor of π.

(2) x is called non-critical if x is classical and the integration map

$$I_\lambda : H^*_c(\mathfrak{n}, \mathscr{D}_\lambda \otimes_{k_\lambda} k_x)_{\mathfrak{m}_x} \to H^*_c(\mathfrak{n}, \mathscr{L}^\sharp_\lambda(k_x))_{\mathfrak{m}_x}$$

is an isomorphism.

We stress that (π, α) being p-refined, for us, includes the condition that π is either an unramified special representation or an unramified principal series.

We will extend these definitions below, and then we will also give numerical criteria for point to be non-critical. First, we check that being non-critical is stable (among classical points) under twisting.

Lemma 6.3.5. *Suppose that $x, x' \in \mathscr{E}(\mathfrak{n})(\overline{\mathbf{Q}}_p)$ are classical points and $x = \mathrm{tw}_\vartheta(x')$ for some $\vartheta \in \mathscr{X}(\Gamma_F)(\overline{\mathbf{Q}}_p)$. Then, the following conclusions hold.*

(1) $\vartheta = \mathbf{N}_p^n \vartheta'$ for ϑ' an unramified Artin character and $n \in \mathbf{Z}$.

(2) x is non-critical if and only if x' is non-critical.

Proof. We first prove (1). By Lemma 6.3.3 and Lemma 6.1.3, there exists an $n \in \mathbf{Z}$ such that $\vartheta|_{\mathcal{O}_p^\times}$ is $z \mapsto z^n$. Thus $\vartheta' := \vartheta \mathbf{N}_p^{-n}$ is trivial on $\mathcal{O}_p^\times$. We deduce from (5.5.1) that it factors through a character of the narrow class group, as promised.

For point (2) we use the notation of the previous paragraph, and we also write $\lambda_x = \lambda$ and $\lambda_{x'} = \lambda'$. We can write $\vartheta' = (\theta')^\iota$ where θ' is a finite order, unramified Hecke character. So, it follows from Remark 5.5.7 that the diagram

$$\begin{tikzcd} H_c^*(\mathfrak{n}, \mathscr{D}_{\lambda'}) \arrow[r, "\mathrm{tw}_\vartheta", "\simeq"'] \arrow[d, "I'_\lambda"'] & H_c^*(\mathfrak{n}, \mathscr{D}_\lambda) \arrow[d, "I_\lambda"] \\ H_c^*(\mathfrak{n}, \mathscr{L}_{\lambda'}^\sharp) \arrow[r, "\simeq", "\mathrm{tw}_{\mathbf{N}_p^n \theta^\iota}"'] & H_c^*(\mathfrak{n}, \mathscr{L}_\lambda^\sharp) \end{tikzcd}$$

is commutative (see Remark 4.3.3 for including twists by the adelic norm). Localizing at Hecke eigensystems, this proves the claim. $\square$

Now consider a twist cohomological weight λ. Thus there exists a cohomological weight $\lambda_0 = (\kappa_0, w_0)$ and $\lambda = \eta \cdot \lambda_0$ for some η. If $\lambda_1 = (\kappa_1, w_1)$ is another cohomological weight that can twisted to λ, then Lemma 6.1.3 implies that $\kappa_0 = \kappa_1$. Thus we can always write a twist cohomological weight $\lambda = (\kappa, *)$ to mean $\lambda = \eta \cdot (\kappa, w)$ for some w. This allows us to define numerical criteria at points $x \in \mathscr{E}(\mathfrak{n})(\overline{\mathbf{Q}}_p)$ of twist cohomological, not just cohomological, weight.

Definition 6.3.6. Let $x \in \mathscr{E}(\mathfrak{n})(\overline{\mathbf{Q}}_p)$ be of twist cohomological weight $\lambda_x = (\kappa, *)$. We say that:

(1) x is twist classical if there exists a classical point $x' \in \mathscr{E}(\mathfrak{n})(\overline{\mathbf{Q}}_p)$ and $\vartheta \in \mathscr{X}(\Gamma_F)(\overline{\mathbf{Q}}_p)$ such that $x = \mathrm{tw}_\vartheta(x')$.

(2) x is twist non-critical if $x = \mathrm{tw}_\vartheta(x')$ with x' a classical, non-critical point.

(3) x has non-critical slope if $v_p(\psi_x(U_p)) < \inf_\sigma(1 + \kappa_\sigma)$.

(4) x is extremely non-critical if $v_p(\psi_x(U_p)) < \frac{1}{2}\inf_\sigma(1 + \kappa_\sigma)$.

Note that Definition 6.3.6 applies in particular to points of cohomological weight. Further, Lemma 6.3.5 implies that whether or not x is twist non-critical is independent of the choice of classical point in the definition thereof. Finally, whether or not a point has non-critical slope (resp. is extremely non-critical) can be checked before or after twisting (by Lemma 6.3.1).

By definition a twist non-critical point is twist classical, but *a priori* the points (3) and (4) do not assume classicality. Proposition 6.3.8 below fills in the only non-trivial implication in the chain:

$$\text{extremely non-critical} \implies \text{non-critical slope} \implies \text{twist non-critical} \implies \text{twist classical}.$$

To prove this, we need a lemma.

Lemma 6.3.7. *If π is a cohomological cuspidal automorphic representation and α is a p-refinement, then $0 \leq v_p(\alpha_v^\sharp)$ for all $v \mid p$.*

Proof. If $L \subset \overline{\mathbf{Q}}_p$ is sufficiently large, then $H_c^d(\mathfrak{n}, \mathscr{L}_\lambda^\sharp(L))[\mathfrak{m}_{\pi,\alpha}^\sharp] \neq (0)$. But by Proposition 5.4.3, the U_v-operator acting on $H_c^d(\mathfrak{n}, \mathscr{L}_\lambda^\sharp(L))$ preserves the integral lattice $H_c^d(\mathfrak{n}, \mathscr{L}_\lambda^\sharp(\mathcal{O}_L))$. Thus $\alpha_v^\sharp$ must be integral. $\square$

Proposition 6.3.8. *Let $x \in \mathscr{E}(\mathfrak{n})(\overline{\mathbf{Q}}_p)$ be of twist cohomological weight λ.*

(1) If x has non-critical slope, then x is twist non-critical.
(2) If x is extremely non-critical, then the action of $\mathbf{T}_\lambda(\mathfrak{n})$ on $H^d_c(\mathfrak{n}, \mathscr{D}_\lambda)_{\mathfrak{m}_x}$ is semi-simple.

Proof. In case (1) (resp. (2)) we can write $x = \mathrm{tw}_\vartheta(x')$ where x' has cohomological weight and x' has non-critical slope (resp. is extremely non-critical). By Lemma 6.3.5 in case (1) and Lemma 6.3.1 in case (2), it suffices to replace x by x' and thus assume that x has cohomological weight. In that case, point (1) follows from [43, Theorem 3.2.5].[19]

We now prove (2) in the case x has cohomological weight. First, by definition an extremely non-critical point has non-critical slope and so is non-critical by point (1). Thus $H^d_c(\mathfrak{n}, \mathscr{D}_\lambda)_{\mathfrak{m}_x} \simeq H^d_c(\mathfrak{n}, \mathscr{L}^\sharp_\lambda(L))_{\mathfrak{m}_x}$. Now write $x = x(\pi, \alpha)$. It is known that the Hecke operators away from $\mathfrak{n}\mathbf{p}$ are semi-simple on the whole space $H^d_c(\mathfrak{n}, \mathscr{L}^\sharp_\lambda(L))$. If we localize at $\mathfrak{m}_x$ then the same is true for the operators U_v when π_v is Steinberg. Thus it remains to show that if π_v is unramified, then the U_v operator acts semi-simply. For that, it is sufficient to show that the two roots of $X^2 - a_v(\pi)X + \omega_\pi(\varpi_v)q_v$ are distinct. Here, $\omega_\pi(\varpi_v) = \zeta q_v^w$ where ζ is a root of unity, and $q_v = p^{f_v}$. In particular, it is enough to show that

$$v_p(\alpha_v) < \frac{f_v(1+w)}{2} = \frac{1}{e_v}\sum_{\sigma\in\Sigma_v}\frac{1+w}{2}. \tag{6.3.3}$$

But $\alpha_v^\sharp = \psi_x(U_v) = \alpha_v \varpi_v^{\frac{\kappa - w}{2}}$ satisfies $v_p(\alpha_v^\sharp) \geq 0$ (Lemma 6.3.7) and, since $\psi_x(U_p) = \prod_{v|p}(\alpha_v^\sharp)^{e_v}$ and x is extremely non-critical, we see that

$$v_p(\alpha_v^\sharp) < \frac{1}{e_v}\inf_{\sigma\in\Sigma_v}\frac{1+\kappa_\sigma}{2} < \frac{1}{e_v}\sum_{\sigma\in\Sigma_v}\frac{1+\kappa_\sigma}{2}.$$

The bound (6.3.3) follows immediately, completing the proof of (2). □

6.4. **The middle-degree eigenvariety.** We now return to the eigenvarieties $\mathscr{E}(\mathfrak{n})$. Recall the open affinoid charts $\mathscr{E}_{\Omega,h} = \mathrm{Sp}(\mathbf{T}_{\Omega,h})$ and $\mathscr{E}'_{\Omega,h} = \mathrm{Sp}(\mathbf{T}'_{\Omega,h})$ defined towards the end of Section 6.2. If A is a commutative ring we write A^{red} for its nilreduction, and if X is a rigid analytic space we write X^{red} for its nilreduction.

Proposition 6.4.1.

(1) If (Ω, h) is a slope adapted pair, then we have a natural commuting diagram

$$\begin{tikzcd} \mathbf{T}(\mathfrak{n}) \otimes_{\mathbf{Q}_p} \mathscr{O}(\Omega) \arrow[r, two heads, "\psi'_{\Omega,h}"] \arrow[d, two heads, "\psi_{\Omega,h}"'] & \mathbf{T}'_{\Omega,h} \arrow[d, dotted] \\ \mathbf{T}_{\Omega,h} \arrow[r, two heads] & \mathbf{T}^{\mathrm{red}}_{\Omega,h} \end{tikzcd}$$

(2) The morphisms $\mathbf{T}'_{\Omega,h} \to \mathbf{T}^{\mathrm{red}}_{\Omega,h}$ in part (1) glue to a canonical morphism $\tau : \mathscr{E}(\mathfrak{n})^{\mathrm{red}} \to \mathscr{E}'(\mathfrak{n})$.

Proof. By [43, Theorem 3.3.1] there is a first quadrant spectral sequence

$$E_2^{i,j} = \mathrm{Ext}^i_{\mathscr{O}(\Omega)}(H_j^{\mathrm{BM}}(\mathfrak{n}, \mathscr{A}_\Omega)_{\leq h}, \mathscr{O}(\Omega)) \Rightarrow H_c^{i+j}(\mathfrak{n}, \mathscr{D}_\Omega)_{\leq h} \tag{6.4.1}$$

---

[19] To make this calculation, one should take the Borel in [43] to be the upper-triangular Borel and the element t in [43, Theorem 3.2.5] to be $\begin{pmatrix} 1 & \\ & \varpi_p^{e_p} \end{pmatrix}$. Then, the U_t-operator in that reference is the U_p-operator in this paper (see Remark 5.3.3).

which is equivariant for the action of $\mathbf{T}(\mathfrak{n}) \otimes_{\mathbf{Q}_p} \mathscr{O}(\Omega)$. Thus, if $T \in \ker(\psi'_{\Omega,h})$, then acts trivially on every term in the E_2-page for the spectral sequence (6.4.1). In particular, that means that T acts nilpotently on the abutment $H_c^*(\mathfrak{n}, \mathscr{D}_\Omega)_{\leq h}$, which is what we wanted to show in (1).

The second part of the proposition is immediate from the construction of the eigenvariety and the local nature of the nilreduction. □

Now consider the graded sheaves $\mathscr{M}_*^{\mathrm{BM}} = \bigoplus_j \mathscr{M}_j^{\mathrm{BM}}$ on $\mathscr{E}'(\mathfrak{n})$. Le τ be as in Proposition 6.4.1(2). Since $\mathscr{M}_j^{\mathrm{BM}}$ is a coherent sheaf on $\mathscr{E}'(\mathfrak{n})$, its pullback $\tau^*\mathscr{M}_j^{\mathrm{BM}}$ to $\mathscr{E}(\mathfrak{n})^{\mathrm{red}}$ is also coherent. The natural map $i : \mathscr{E}(\mathfrak{n})^{\mathrm{red}} \to \mathscr{E}(\mathfrak{n})$ is a closed immersion, so $i_*\tau^*\mathscr{M}_j^{\mathrm{BM}}$ is thus a coherent sheaf on $\mathscr{E}(\mathfrak{n})$. In particular, its support is a closed analytic subset. In general, we write $\mathrm{supp}(\mathscr{M})$ for the support of a sheaf $\mathscr{M}$.

Definition 6.4.2.

$$\mathscr{E}(\mathfrak{n})_{\mathrm{mid}} := \mathscr{E}(\mathfrak{n}) - \left[\left(\bigcup_{j=d+1}^{2d} \mathrm{supp}(\mathscr{M}_c^j)\right) \cup \left(\bigcup_{j=0}^{d-1} \mathrm{supp}(i_*\tau^*\mathscr{M}_j^{\mathrm{BM}})\right)\right]$$

We immediately give a separate characterization of $\mathscr{E}(\mathfrak{n})_{\mathrm{mid}}$. The entire reason for introducing the homology-based eigenvariety was to give Definition 6.4.2 because it is not clear that condition (2) in the next proposition gives a well-defined affinoid open subspace.

Proposition 6.4.3. *If $x \in \mathscr{E}(\mathfrak{n})(\overline{\mathbf{Q}}_p)$, then the following conditions are equivalent.*

(1) $x \in \mathscr{E}(\mathfrak{n})_{\mathrm{mid}}(\overline{\mathbf{Q}}_p)$.
(2) $H_c^j(\mathfrak{n}, \mathscr{D}_{\lambda_x} \otimes_{k_{\lambda_x}} k_x)_{\mathfrak{m}_x} \neq (0)$ if and only if $j = d$.

Moreover, $\mathscr{E}(\mathfrak{n})^{\mathrm{mid}} \cap \mathrm{supp}(\mathscr{M}_c^j)$ is empty if $0 \leq j \leq d-1$ also.

Proof. This follows from [43, Proposition 4.5.2] and elementary manipulations of supports. □

We note that $\mathscr{E}(\mathfrak{n})_{\mathrm{mid}}$ is Zariski-open in $\mathscr{E}(\mathfrak{n})$. In particular, if $x \in \mathscr{E}(\mathfrak{n})_{\mathrm{mid}}$ then any sufficiently small good neighborhood U of x in $\mathscr{E}(\mathfrak{n})$ is actually contained in $\mathscr{E}(\mathfrak{n})_{\mathrm{mid}}$ (Proposition 6.2.9).

Proposition 6.4.4.

(1) The coherent sheaf $\mathscr{M}_c^d|_{\mathscr{E}(\mathfrak{n})_{\mathrm{mid}}}$ is flat over $\mathscr{W}(1)$.
(2) $\mathscr{E}(\mathfrak{n})_{\mathrm{mid}}$ is admissibly covered by good neighborhoods U belonging to slope adapated pairs (Ω, h) such that $\mathscr{O}(U)$ acts faithfully on the finite projective $\mathscr{O}(\Omega)$-module $\mathscr{M}_c^d(U) = e_U H_c^d(\mathfrak{n}, \mathscr{D}_\Omega)_{\leq h}$.

Proof. For (1), we want to show that if $x \in \mathscr{E}(\mathfrak{n})_{\mathrm{mid}}$ is of weight $\lambda = \lambda_x$, then for any slope adapated pair (Ω, h) the module $(H_c^d(\mathfrak{n}, \mathscr{D}_\Omega)_{\leq h})_{\mathfrak{m}_x} = \mathscr{M}_c^d(\mathscr{E}_{\Omega,h})_{\mathfrak{m}_x}$ is finite free over $\mathscr{O}(\Omega)_{\mathfrak{m}_\lambda}$. To do this, we consider a second quadrant spectral sequence ([43, Theorem 3.3.1])

$$E_2^{i,j} = \mathrm{Tor}_{-i}^{\mathscr{O}(\Omega)_{\mathfrak{m}_\lambda}}(\mathscr{M}_c^j(\mathscr{E}_{\Omega,h})_{\mathfrak{m}_x}, k_\lambda) \Rightarrow H_c^{i+j}(\mathfrak{n}, \mathscr{D}_\lambda)_{\mathfrak{m}_x}. \tag{6.4.2}$$

If $j \neq d$ then, since $x \in \mathscr{E}(\mathfrak{n})_{\mathrm{mid}}$, the $E_2^{i,j}$-term in (6.4.2) vanishes for all i. Thus we deduce canonical isomorphisms

$$\mathrm{Tor}_n^{\mathscr{O}(\Omega)_{\mathfrak{m}_\lambda}}(\mathscr{M}_c^d(\mathscr{E}_{\Omega,h})_{\mathfrak{m}_x}, k_\lambda) \simeq H_c^{d-n}(\mathfrak{n}, \mathscr{D}_\lambda)_{\mathfrak{m}_x} \tag{6.4.3}$$

for all $n \geq 0$. By Proposition 6.4.3 we further deduce that either side of (6.4.3) vanishes for $n > 0$. By the local criterion for flatness ([60, Section 22]), $\mathscr{M}_c^d(\mathscr{E}_{\Omega,h})_{\mathfrak{m}_x}$ is free over $\mathscr{O}(\Omega)_{\mathfrak{m}_\lambda}$. This proves (1).

Now we prove (2). First, it is immediate that $\mathscr{E}(\mathfrak{n})_{\mathrm{mid}}$ is admissibly covered by good neighborhoods U of $\mathscr{E}(\mathfrak{n})$. By definition, $\mathscr{O}(U) = e_U\mathbf{T}_{\Omega,h}$ acts faithfully on $\mathscr{M}_c^*(U) = e_U H_c^*(\mathfrak{n}, \mathscr{D}_\Omega)_{\leq h}$. But if $U \subset \mathscr{E}(\mathfrak{n})_{\mathrm{mid}}$ and $j \neq d$, then $\mathrm{Ann}_{\mathscr{O}(U)}(\mathscr{M}_c^j(U)) = \mathscr{O}(U)$ by Proposition 6.4.3. We thus deduce that

$\mathscr{O}(U)$ acts faithfully on $\mathscr{M}_c^d(U)$. Since $\mathscr{M}_c^d(U)$ is finite projective over $\mathscr{O}(\Omega)$ by part (1), we have completed the proof of (2). □

Lemma 6.4.5. *Every non-critical point on $\mathscr{E}(\mathfrak{n})$ belongs to $\mathscr{E}(\mathfrak{n})_{\mathrm{mid}}$.*

Proof. If x is non-critical of cohomological weight λ, then $H_c^*(\mathfrak{n}, \mathscr{D}_\lambda \otimes_{k_\lambda} k_x)_{\mathfrak{m}_x} \simeq H_c^*(\mathfrak{n}, \mathscr{L}_\lambda^\sharp(k_x))_{\mathfrak{m}_x}$. So, part (1) follows from Proposition 6.4.3 and knowing that cuspidal eigensystems in $H_c^*(\mathfrak{n}, \mathscr{L}_\lambda)$ appear only in middle degree (see [46]). □

Proposition 6.4.6.

(1) $\mathscr{E}(\mathfrak{n})_{\mathrm{mid}}$ is stable under twisting by $\mathscr{X}(\Gamma_F)$.
(2) Every twist non-critical point on $\mathscr{E}(\mathfrak{n})$ belongs to $\mathscr{E}(\mathfrak{n})_{\mathrm{mid}}$.
(3) If $X \subset \mathscr{E}(\mathfrak{n})_{\mathrm{mid}}$ is an irreducible component then $\dim X = \dim \mathscr{W}(1)$ and X is contained in a unique irreducible component of $\mathscr{E}(\mathfrak{n})$.
(4) The extremely non-critical points are a Zariski-dense accumulation subset of $\mathscr{E}(\mathfrak{n})_{\mathrm{mid}}$.

Proof. Part (1) follows immediately from Proposition 6.4.3 and Lemma 6.3.1. Part (2) then follows from part (1) and Lemma 6.4.5.

From [43, Theorem 1.1.6] and Proposition 6.4.3 we deduce that if x is a point on $\mathscr{E}(\mathfrak{n})_{\mathrm{mid}}$ then any irreducible component of $\mathscr{E}(\mathfrak{n})$ passing through x has dimension equal to $\dim \mathscr{W}(1)$. Thus the claim (3) follows from [34, Corollary 2.2.9].

Finally we prove (4). First, if $X \subset \mathscr{E}(\mathfrak{n})_{\mathrm{mid}}$ is an irreducible component then $\lambda(X)$ is Zariski-open in $\mathscr{W}(1)$ (by part (3) and Proposition 6.2.7). By Lemma 6.1.4 we deduce that X contains a point x_0 of twist cohomological weight. This reduces the statement of (4) to proving that extremely non-critical points are accumulating on a neighborhood near any point x_0 of twist cohomological weight.

Consider a good neighborhood $U \subset \mathscr{E}(\mathfrak{n})_{\mathrm{mid}}$ of x_0. Say U belongs to a slope adapted pair (Ω, h). First, U is the rigid analytic spectrum of $\mathscr{O}(U)$. Second, Proposition 6.4.4 implies $\mathscr{O}(U)$ acts faithfully on the finite projective $\mathscr{O}(\Omega)$-module $\mathscr{M}_c^d(U)$. So, by [31, Lemme 6.2.10], the irreducible components of U map surjectively onto Ω, and by [31, Lemme 6.2.8] we deduce that the pre-image $(\lambda|_U)^{-1}(Z) \subset U$ of any Zariski-dense subset $Z \subset \Omega$ is still Zariski-dense in U. Since x_0 has twist cohomological weight we conclude that U contains a Zariski-dense accumulating set of points of twist cohomological weight. On the other hand, we can easily shrink U so that $x \mapsto v_p(\psi_x(U_p))$ is constant on U as well, and thus see clearly that in fact we can take a Zariski-dense accumulating subset of extremely non-critical points as claimed. □

We now pause for a lemma of commutative algebra.

Lemma 6.4.7. *Suppose that A is a noetherian integral domain of characteristic zero and $A \to B$ is a finite morphism with B torsion free over A. Then, the following conditions are equivalent.*

(1) B is reduced.
(2) $A \to B$ is generically étale.
(3) The support of $\Omega^1_{B/A}$ in $\mathrm{Spec}(B)$ has positive codimension. (See the beginning of the proof.)

If furthermore M is a finite projective A-module and B is actually a commutative A-subalgebra of $\mathrm{End}_A(M)$ then these conditions are all equivalent to:

(4) There exists a Zariski-dense subset $X \subset \mathrm{Spec}(A)$ such that B has reduced image inside $\mathrm{End}_{A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}}(M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p})$ for all $\mathfrak{p} \in X$.

Here we say a finite map of Noetherian rings $A \to B$ is generically étale if it satisfies either of the following two equivalent conditions:

a. $B \otimes_A \mathrm{Frac}(A/\mathfrak{p})$ is a finite étale $\mathrm{Frac}(A/\mathfrak{p})$-algebra for all minimal primes $\mathfrak{p}$ of A;

b. there exists an open dense subsecheme $U \subset \mathrm{Spec}(A)$ such that $\mathrm{Spec}(B) \times_{\mathrm{Spec}(A)} U \to U$ is finite étale.

These conditions are equivalent because the locus where $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is not étale is closed in $\mathrm{Spec}(B)$ (see [61, Proposition 3.8] for instance) and this locus has closed image in $\mathrm{Spec}(A)$ because $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is proper (B being finite over A).

Proof of Lemma 6.4.7. If $\mathfrak{p} \in \mathrm{Spec}(A)$ write $k(\mathfrak{p})$ for its residue field. When $\mathfrak{p}$ is the generic point, we write $K = k(\mathfrak{p})$.

We note first that the hypotheses imply that B is equidimensional of the same dimension as A. This gives meaning to condition (3). Now we will show that (1) and (2) are equivalent. Since B is torsion free over A, B is reduced if and only if $B \otimes_A K$ is reduced. Thus it suffices to show that $B \otimes_A K$ is reduced if and only if $B \otimes_A K$ is a finite étale K-algebra. Since K has characteristic zero, this follows from Wedderburn's theorem (see [21, Prop. 3, Chap. VIII] for instance).

Our second claim is that (2) and (3) are equivalent. Since A is reduced, noetherian and $A \to B$ is finite we have that $A \to B$ is generically flat ([42, Theorem 6.9.1]). So being generically étale and generically unramified are equivalent, the latter being clearly equivalent to condition (3).

For the rest of the proof we will assume that B is as in the "furthermore". It is elementary to check that B is then a finite torsion free A-algebra, so that (1) through (3) are all equivalent. We will show that (2) implies (4) and (4) implies (1).

Begin by assuming (2) and choose a dense open subscheme $U \subset \mathrm{Spec}(A)$ that $A_{\mathfrak{p}} \to B \otimes_A A_{\mathfrak{p}}$ is finite étale for each $\mathfrak{p} \in U$. Then the fiber $B \otimes_A k(\mathfrak{p})$ is a finite étale $k(\mathfrak{p})$-algebra; in particular it is reduced. Since the natural map $B \to \mathrm{End}_{k(\mathfrak{p})}(M \otimes_A k(\mathfrak{p}))$ factors through $B \otimes_A k(\mathfrak{p})$ we see that B has reduced image as in (4) for all $\mathfrak{p} \in U$ meaning we can take $X = U$ to witness (4).

Finally assume that (4) holds and consider such a set X. Since M is projective over A and X is Zariski-dense in $\mathrm{Spec}(A)$, the natural map

$$\mathrm{End}_A(M) \to \prod_{\mathfrak{p} \in X} \mathrm{End}_{k(\mathfrak{p})}(M \otimes_A k(\mathfrak{p}))$$

is injective. Thus we deduce that

$$B \to \prod_{\mathfrak{p} \in X} \mathrm{End}_{k(\mathfrak{p})}(M \otimes_A k(\mathfrak{p})) \tag{6.4.4}$$

is also injective. On the other hand, B has reduced image in each coordinate of (6.4.4) by our assumption (4), so it follows that B is reduced. ☐

The previous lemma is applied to prove the following theorem.

Theorem 6.4.8. *$\mathscr{E}(\mathfrak{n})_{\mathrm{mid}}$ is reduced.*

Proof. We proved in Proposition 6.4.4 that $\mathscr{E}(\mathfrak{n})_{\mathrm{mid}}$ is admissibly covered by good affinoid opens U belonging to slope adapated pairs (Ω, h) such that $\mathscr{O}(U)$ is an $\mathscr{O}(\Omega)$-subalgebra of the endomorphism $\mathrm{End}_{\mathscr{O}(\Omega)}(\mathscr{M}_c^d(U))$, and $\mathscr{M}_c^d(U)$ is finite projective over $\mathscr{O}(\Omega)$. So, Lemma 6.4.7 provides criteria to check that each $\mathscr{O}(U)$ is reduced, which is what we will do.

First, if U contains an extremely non-critical point then condition (4) of Lemma 6.4.7 holds by Proposition 6.3.8. So $\mathscr{O}(U)$ is reduced in this case. Further, condition (3) of Lemma 6.4.7 implies that the support Z of $\Omega^1_{\mathscr{E}(\mathfrak{n})_{\mathrm{mid}}/\mathscr{W}(1)}$ meets U in a closed subspace of positive codimension.

By Proposition 6.4.6, good neighborhoods of extremely non-critical points are Zariski-dense and accumulating on each irreducible component of $\mathscr{E}(\mathfrak{n})_{\mathrm{mid}}$, so that Z does not contain any irreducible component of $\mathscr{E}(\mathfrak{n})_{\mathrm{mid}}$. This implies that Z must have positive codimension in $\mathscr{E}(\mathfrak{n})_{\mathrm{mid}}$ (see the argument in [34, Corollary 2.2.7] for instance) and *a fortiori* meets any good neighborhood U (all of which are equidimensional) in a closed subspace of positive codimension. Finally, the equivalence between conditions (1) and (3) in Lemma 6.4.7 prove that $\mathscr{O}(U)$ is reduced in general. $\square$

6.5. **Interlude on Galois representations.** If K is a field and $\overline{K}$ is a fixed algebraic closure we write G_K for the Galois group of $\overline{K}$ over K. Recall that if $K/\mathbf{Q}_\ell$ is finite extension, and if $\ell \neq p$, then any continuous representation $\rho : G_K \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ has a corresponding Weil–Deligne representation $\mathrm{WD}(\rho)$ ([75]). When $\ell = p$ we use the language (and standard notations like D_{dR}, D_{crys}, etc.) developed within the p-adic Hodge theory of Galois representations by Fontaine ([39]). In particular, if $\ell = p$ and ρ is potentially semistable then it too has an associated Weil–Deligne representation $\mathrm{WD}(\rho)$. For each embedding $\sigma : K \to \overline{\mathbf{Q}}_p$, we also write $\mathrm{HT}_\sigma(\rho)$ for σ-th Hodge–Tate weight which is defined to be the jumps in the Hodge filtration on the $\overline{\mathbf{Q}}_p$-vector space $D_{\mathrm{dR}}(\rho) \otimes_{K,\sigma} \overline{\mathbf{Q}}_p$.

Recall that we defined a normalized local Langlands correspondence r^ι over $\overline{\mathbf{Q}}_p$ (Section 1.10). If ρ is a representation of G_F then and v is a place of F then we write ρ_v for its restriction to a decomposition group at v. The previous paragraph then applies to the various ρ_v.

Theorem 6.5.1. *Let π be a cohomological cuspidal automorphic representation of conductor $\mathfrak{n}$. Then there exists a unique continuous and irreducible representation*

$$\rho_\pi : G_F \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$$

such that $\rho_{\pi,v}$ is potentially semi-stable at all $v \mid p$ and $\mathrm{WD}(\rho_{\pi,v}) = r^\iota(\pi_v)$ for all v.

Furthermore, if π has weight $\lambda = (\kappa, w)$ and $v \mid p$ then the following conclusions hold.

(1) If $\sigma \in \Sigma_v$, then $\mathrm{HT}_\sigma(\rho_{\pi,v}) = \{\frac{w-\kappa_\sigma}{2}, \frac{w+\kappa_\sigma}{2} + 1\}$.
(2) If π_v is an unramified special representation then $\rho_{\pi,v}$ is semistable non-crystalline.
(3) If π_v is an unramified principal series representation then $\rho_{\pi,v}$ is crystalline.

Proof. The construction of ρ_π and proving that it satisfies local-global compatibility away from p can be deduced from independent work of Carayol ([28]), Wiles ([81]), Blasius and Rogawski ([17]), and Taylor ([76]). The local-global compatibility at the p-adic places is due to Saito ([65, 66]), Blasius and Rogawski as before, and Skinner ([73]). $\square$

Remark 6.5.2. If π_v is an unramified principal series, then the characteristic polynomial of φ^{f_v} acting on $D_{\mathrm{crys}}(\rho_{\pi,v})$ is equal to the characteristic polynomial of $r^\iota(\pi_v)(\mathrm{Frob}_v)$ or, what is the same, the image of the v-th Hecke polynomial $X^2 - a_v(\pi)X + \omega_\pi(\varpi_v)q_v$ under ι.

We will now globalize the construction of Galois representations in Theorem 6.5.1 over $\mathscr{E}(\mathfrak{n})_{\mathrm{mid}}$. Write $\psi : \mathbf{T}(\mathfrak{n}) \to \mathscr{O}(\mathscr{E}(\mathfrak{n})_{\mathrm{mid}})$ to denote the universal Hecke eigensystem on $\mathscr{E}(\mathfrak{n})_{\mathrm{mid}}$.

Proposition 6.5.3. *There exists a unique two-dimensional pseudorepresentation*

$$T : G_{F,\mathfrak{np}} \to \mathscr{O}(\mathscr{E}(\mathfrak{n})_{\mathrm{mid}})$$

such that if $v \nmid \mathfrak{np}$ then $T(\mathrm{Frob}_v) = \psi(T_v)$.

Proof. First, Theorem 6.4.8 implies that $\mathscr{E}(\mathfrak{n})_{\mathrm{mid}}$ is reduced. Second, Theorem 6.5.1 and Proposition 6.4.6 implies that we have a Zariski-dense subset $Z \subset \mathscr{E}(\mathfrak{n})_{\mathrm{mid}}(\overline{\mathbf{Q}}_p)$ such that if $z \in Z$ then there is a Galois representations $\rho_z : G_{F,\mathfrak{np}} \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ with $\mathrm{tr}(\rho_z(\mathrm{Frob}_v)) = \psi_z(T_v)$ for all $v \nmid \mathfrak{np}$. Specifically, we take Z to be all those points which are twist classical and for $z \in Z$ of the form $z = \mathrm{tw}_\vartheta(x)$, with $x = x(\pi, \alpha)$ classical, we take $\rho_z = \rho_\pi \otimes \vartheta$ with ρ_π as in Theorem 6.5.1. This tautologically gives the Hecke eigensystem $\psi_{z'}$ away from $\mathfrak{np}$ by (6.3.1). The Zariski-density of these points follows from Propositions 6.3.8 and 6.4.6. Thus this proposition follows from a result of Chenevier ([31, Proposition 7.1.1]) once we check a boundedness condition. Specifically, the eigenvariety $\mathscr{E}(\mathfrak{n})_{\mathrm{mid}}$ is reduced and nested (in the sense of [12, Section 7.2]), so by [12, Lemma 7.2.11] the power bounded functions on $\mathscr{E}(\mathfrak{n})_{\mathrm{mid}}$ form a compact subring of $\mathscr{O}(\mathscr{E}(\mathfrak{n})_{\mathrm{mid}})$. The Lemma 6.5.4(2) below implies the Hecke eigenvalues away from $\mathfrak{n}\mathbf{p}$ lie in this compact subring, and so Chenevier's result applies for us. $\square$

To fill the gap in the previous proposition we need a small bit of notation. Define $\mathbf{T}^{\mathrm{nr}}_{\mathbf{Z}_p}(\mathfrak{n})$ as the $\mathbf{Z}_p$-span of the Hecke operators $(T_v)_{v\nmid \mathfrak{n}\mathbf{p}}$ inside $\mathbf{T}(\mathfrak{n})$. Let $\mathscr{E}_{\Omega,h}$ be an affinoid neighborhood on $\mathscr{E}(\mathfrak{n})$ with (Ω,h) slope adapted. Let $R=\mathscr{O}(\Omega)$ and suppose that $R_0\subset R$ is a ring of definition. We then define an R_0-module $H^d_c(\mathfrak{n},\mathbf{D}^{\mathbf{s},\circ}_\Omega)_{\leq h}$ by

$$H^d_c(\mathfrak{n},\mathbf{D}^{\mathbf{s},\circ}_\Omega)_{\leq h} := \mathrm{im}\left(H^d_c(\mathfrak{n},\mathbf{D}^{\mathbf{s},\circ}_\Omega)\to H^d_c(\mathfrak{n},\mathbf{D}^{\mathbf{s}}_\Omega)\twoheadrightarrow H^d_c(\mathfrak{n},\mathbf{D}^{\mathbf{s}}_\Omega)_{\leq h}\right).$$

Thus $H^d_c(\mathfrak{n},\mathbf{D}^{\mathbf{s},\circ}_\Omega)_{\leq h}$ is an R_0-submodule of the finite Banach R-module $H^d_c(\mathfrak{n},\mathbf{D}^{\mathbf{s}}_\Omega)_{\leq h}$.

Lemma 6.5.4. *Assume the notations of the previous paragraph.*

(1) $H^d_c(\mathfrak{n},\mathbf{D}^{\mathbf{s},\circ}_\Omega)_{\leq h}$ is bounded and stable under the natural action of $\mathbf{T}^{\mathrm{nr}}_{\mathbf{Z}_p}(\mathfrak{n})$.

(2) $\psi(\mathbf{T}^{\mathrm{nr}}_{\mathbf{Z}_p}(\mathfrak{n}))\subset\mathscr{O}(\mathscr{E}(\mathfrak{n}))$ consists of power bounded elements.

Proof. First, we will show that (2) follows from (1). Since, ψ is an algebra morphism, it is enough to check that $\psi(\mathbf{T}^{\mathrm{nr}}_{\mathbf{Z}_p}(\mathfrak{n}))$ is bounded. Part (1) of this lemma implies that the induced endomorphisms on $H^d_c(\mathfrak{n},\mathbf{D}^{\mathbf{s}}_\Omega)_{\leq h}$ are bounded and that is enough because the topology on $\mathscr{O}(\mathscr{E}(\mathfrak{n}))$ is the weakest topology making all of the natural maps $\mathscr{O}(\mathscr{E}(\mathfrak{n}))\to\mathscr{O}(\mathscr{E}_{\Omega,h})=\mathbf{T}_{\Omega,h}$ continuous.

Now we prove (1). Write $K=K_1(\mathfrak{n})I$. If $K'\subset K$ is an open and normal subgroup then we consider the diagram

$$\begin{tikzcd}
H^d_c(Y_K,\mathbf{D}^{\mathbf{s},\circ}_\Omega) \arrow[r] \arrow[dd] & H^d_c(Y_K,\mathbf{D}^{\mathbf{s}}_\Omega) \arrow[r, two heads] \arrow[d, equal] & H^d_c(Y_K,\mathbf{D}^{\mathbf{s}}_\Omega)_{\leq h} \arrow[d, equal] \\
 & H^d_c(Y_{K'},\mathbf{D}^{\mathbf{s}}_\Omega)^{K/K'} \arrow[r] \arrow[d, hook] & H^d_c(Y_{K'},\mathbf{D}^{\mathbf{s}}_\Omega)^{K/K'}_{\leq h} \arrow[d, hook] \\
H^d_c(Y_{K'},\mathbf{D}^{\mathbf{s},\circ}_\Omega) \arrow[r] & H^d_c(Y_{K'},\mathbf{D}^{\mathbf{s}}_\Omega) \arrow[r, two heads] & H^d_c(Y_{K'},\mathbf{D}^{\mathbf{s}}_\Omega)_{\leq h}.
\end{tikzcd}$$

The two equalities are because $\mathbf{D}^{\mathbf{s}}_\Omega$ is a $\mathbf{Q}$-vector space and K/K' is a finite group. The right-hand column consists of finite R-modules and thus the inclusion is continuous for the unique Banach R-module topologies. So, to check that the image of the top horizontal row is bounded, it is enough to check that the image of the bottom horizontal row is bounded. Replacing $\mathfrak{n}$ by a smaller ideal we can assume Y_K is a neat level (Proposition 2.3.3). In that case, the cohomology $H^d_c(Y_K,M)$ is computed by Borel–Serre complexes $C^\bullet_c(Y_K,M)$ for $M=\mathbf{D}^{\mathbf{s},\circ}_\Omega$ or $M=\mathbf{D}^{\mathbf{s}}_\Omega$ (see the start of Section 6.2 or [43, p.15-16]). In that case, the image of $H^d_c(Y_K,\mathbf{D}^{\mathbf{s},\circ}_\Omega)\to H^d_c(Y_K,\mathbf{D}^{\mathbf{s}}_\Omega)_{\leq h}$ is obviously bounded as it is the image, in cohomology, of the bounded subcomplex $C^*_c(K,\mathbf{D}^{\mathbf{s},\circ}_\Omega)\subset C^*_c(K,\mathbf{D}^{\mathbf{s}}_\Omega)$ under the quotient map $C^*_c(K,\mathbf{D}^{\mathbf{s}}_\Omega)\to C^*_c(K,\mathbf{D}^{\mathbf{s}}_\Omega)_{\leq h}$. $\square$

The lemma completes the proof of Proposition 6.5.3. So now, for $x\in\mathscr{E}(\mathfrak{n})_{\mathrm{mid}}(\overline{\mathbf{Q}}_p)$, we write T_x for the specialization of the pseudorepresentation in Proposition 6.5.3 to the residue field k_x. A theorem of Taylor ([77, Theorem 1(2)]) implies that for each x there exists a unique continuous and semi-simple representation $\rho_x: G_F\to\mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ so that $\mathrm{tr}(\rho_x)=T_x$. Note that if x is a classical point then in fact ρ_x may be defined over k_x by the unicity, and the construction of the classical ρ_x (as in the proofs of Theorem 6.5.1).

We now turn towards the important properties of ρ_x at the p-adic places. If $\lambda = (\lambda_1, \lambda_2) \in \mathscr{W}$ is any p-adic weight then we can restrict to each λ_i to $\lambda_{i,v}$ along $\mathcal{O}_v^\times \hookrightarrow \mathcal{O}_p^\times$. We then define characters $\eta_{i,v}$ on $\mathcal{O}_v^\times$ by

$$\eta_{1,v}(\lambda) := \lambda_{2,v}^{-1}$$
$$\eta_{2,v}(\lambda) := (\lambda_{1,v} \prod_{\sigma \in \Sigma_v} \sigma)^{-1}.$$

For any character η on $\mathcal{O}_v^\times$, we write $\mathrm{LT}_{\varpi_v}(\eta)$ for the extension of η to $F_v^\times$ by forcing $\varpi_v \mapsto 1$. This is unitary, so we use the same notation to denote its continuous extension to a Galois character on G_{F_v}. These normalizations are designed so that $\lambda = (\kappa, w)$ is a cohomological weight, then $\mathrm{HT}_\sigma(\mathrm{LT}_{\varpi_v}(\eta_{1,v}(\lambda))) = \frac{w-\kappa_\sigma}{2}$ and $\mathrm{HT}_\sigma(\mathrm{LT}_{\varpi_v}(\eta_{2,v}(\lambda))) = \frac{w+\kappa_\sigma}{2} + 1$ for all $\sigma \in \Sigma_v$ (compare with Theorem 6.5.1).

The next lemma will only be used later (see the proof of Theorem 6.6.3).

Lemma 6.5.5. *If $x \in \mathscr{E}(\mathfrak{n})_{\mathrm{mid}}(\overline{\mathbf{Q}}_p)$, then $\det(\rho_{x,v})|_{\mathcal{O}_v^\times} = \eta_{1,v}(\lambda_x)\eta_{2,v}(\lambda_x)$. In particular, the kernel of the composition*

$$\mathcal{O}_F^\times \to \mathcal{O}_p^\times \xrightarrow{\det(\rho_{x,v})_{v|p}} k_x^\times$$

contains a subgroup of finite index in $\mathcal{O}_F^\times$.

Proof. This is true at classical x by Theorem 6.5.1, twist classical x by the definition of twisting, and all x by interpolation. □

Lemma 6.5.6. *Suppose that $x \in \mathscr{E}(\mathfrak{n})_{\mathrm{mid}}(\overline{\mathbf{Q}}_p)$ is a classical point. Then, there exists a good affinoid neighborhood $x \in U \subset \mathscr{E}(\mathfrak{n})_{\mathrm{mid}}$ and a continuous linear representation $\rho_U : G_{F,\mathfrak{n}\mathbf{p}} \to \mathrm{GL}_2(\mathscr{O}(U))$ such that $\rho_U \otimes_{\mathscr{O}(U)} k_u = \rho_u$ for each $u \in U$.*

Proof. Write $x = x(\pi, \alpha)$. Since π is cuspidal the Galois representation $\rho_x = \rho_\pi$ is absolutely irreducible. Write $\mathscr{O}_x$ for the rigid local ring of x on $\mathscr{E}(\mathfrak{n})_{\mathrm{mid}}$. Then $\mathscr{O}_x$ is a Henselian local ring ([16, Theorem 2.1.5]), so by [64, Corollarie 5.2] there exists a continuous lift $\rho_{\mathscr{O}_x}$ of ρ_x to $\mathscr{O}_x$ such that $\mathrm{tr}(\rho_{\mathscr{O}_x})$ is equal to the specialization of the pseudorepresentation T as in Theorem 6.5.3 to the ring $\mathscr{O}_x$. By [12, Lemma 4.3.7] we can extend $\rho_{\mathscr{O}_x}$ to a continuous representation ρ_U over some affinoid neighborhood of U in a manner compatible with the pseudorepresentation T. Being absolutely irreducible is a Zariski-open condition on U ([31, Section 7.2.1]) and so we may, if necessary, shrink U and assume that ρ_u is absolutely irreducible at each $u \in U$. At that point the equality $\mathrm{tr}(\rho_u) = \mathrm{tr}(\rho_U \otimes_{\mathcal{O}(U)} k_u)$ becomes an equality of true representations by the theorem of Brauer and Nesbitt. This proves the lemma. □

Lemma 6.5.7. *Suppose that $x \in \mathscr{E}(\mathfrak{n})_{\mathrm{mid}}(\overline{\mathbf{Q}}_p)$ is a classical point of prime-to-p conductor $\mathfrak{n}$. Then, if U is a good neighborhood of x in $\mathscr{E}(\mathfrak{n})_{\mathrm{mid}}$, then U contains a Zariski-dense and accumulating subset of points x' which are twist classical of the form $y = \mathrm{tw}_\vartheta(x')$ where x' is classical and also has prime-to-p conductor $\mathfrak{n}$.*

Proof. For $\mathfrak{n} \subsetneq \mathfrak{n}'$, write $\widetilde{\mathscr{E}}(\mathfrak{n}')$ for the eigenvariety constructed out of the finite slope subspaces $H^*(\mathfrak{n}', \mathscr{D}_\lambda)_{\leq h}$ except only with endomorphisms by $\mathbf{T}(\mathfrak{n})$ (i.e. ignore the Hecke operators at primes dividing $\mathfrak{n}/\mathfrak{n}'$). Then the construction we outlined gives a natural closed immersion $\widetilde{\mathscr{E}}(\mathfrak{n}') \hookrightarrow \mathscr{E}(\mathfrak{n})$. If x is as in the statement of the lemma, it is not in the image of any of the finitely many such embeddings by the same argument as [11, Lemma 2.7]. So, the lemma follows from the further observation that if $y = \mathrm{tw}_\vartheta(x')$ where $x' = (\pi', \alpha')$ then the quantity "prime-to-p conductor of x'" is actually independent of choosing x' (since ϑ is unramified away from p). □

Proposition 6.5.8. *Let $x = x(\pi,\alpha) \in \mathscr{E}(\mathfrak{n})_{\mathrm{mid}}(\overline{\mathbf{Q}}_p)$ be a classical point with prime-to-p conductor $\mathfrak{n}$. Choose U and ρ_U as in Lemma 6.5.6. Write $\mathscr{O}_x$ for the rigid local ring on $\mathscr{E}(\mathfrak{n})_{\mathrm{mid}}$ at x and $\rho_{\mathscr{O}_x}$ for the specializiation of ρ_U along $\mathscr{O}(U) \to \mathscr{O}_x$.*

1. *If $w \nmid p$ and I_w is the choice of an inertia subgroup at w then $\rho_{\mathscr{O}_x}|_{I_w} = \rho_x|_{I_w} \otimes_{k_x} \mathscr{O}_x$.*
2. *Assume further that if $v \mid p$ and π_v is an unramified principal series then the v-th Hecke polynomial has distinct roots.*[20] *Then, if $v \mid p$, then*
$$D^{+}_{\mathrm{crys}}(\rho_{U,v} \otimes \mathrm{LT}_{\varpi_K}(\eta_{1,v}(\lambda_U))^{-1})^{\varphi^{f_v} = \psi(U_v)}$$
is locally free of rank one over $F_v^{\mathrm{nr}} \otimes_{\mathbf{Q}_p} \mathscr{O}(U)$ and commutes with base change on U.

In part (b), $F_v^{\mathrm{nr}} \subset F_v$ means the maximal unramified extension of $\mathbf{Q}_p$ inside F_v; if ρ is an R-linear representation of G_{F_v}, then $D_{\mathrm{crys}}(\rho)$ is an $(F_v^{\mathrm{nr}} \otimes_{\mathbf{Q}_p} R)$-module.

Proof of Proposition 6.5.8. The argument for part (1) follows exactly as in the proof of "property (iii)" in [11, Theorem 2.1.6] (with Lemma 6.5.7 replacing [11, Lemma 2.7]).

To prove part (2), we fix $v \mid p$. It is straightforward to see that the family $\rho_U \otimes \mathrm{LT}_{\varpi_K}(\eta_{1,v}(\lambda_U))^{-1}$ of Galois representations over the reduced rigid analytic space U is a weakly-refined family in the sense of [58, Definition 1.5]. Namely one takes, in the notation of [58], the $\kappa_{i,v}$ to be the logarithms of our multiplicative Hodge–Tate weights $\eta_{i,v}$ (after the trivial shift caused by twisting), $\psi(U_v)$ for the function F, and for the Zariski-dense subset Z we take the set of all extremely non-critical points. Thus once the axioms in [58, Definition 1.5] are verified, part (2) of this proposition follows from [58, Proposition 5.13], where the hypothesis at the fixed point x follows from the regularity assumption on x (it needs to be assumed the crystalline eigenvalues are distinct; see Remark 6.5.2).

The verification of the axioms in [58, Definition 1.5] is routine. We will go through the most crucial axiom ([58, Definition 1.5(d)]) in order to illustrate the consistency of our normaliziations. We need to check the space in (2) is non-zero after specializing U to any extremely non-critical point z. For that write $z = \mathrm{tw}_{\vartheta}(x)$ where $x = x(\pi,\alpha)$. Then, $\rho_z = \rho_x \otimes \vartheta$ and $\lambda_z = \lambda_x \otimes \vartheta^{-1}$. So,
$$\rho_z \otimes \mathrm{LT}_{\varpi_v}(\eta_{1,v,z})^{-1} \simeq \left(\rho_x \otimes \mathrm{LT}_{\varpi_v}(\eta_{1,v,x})^{-1}\right) \otimes \mathrm{LT}_{\varpi_v}(\vartheta_v^{-1})\vartheta_v.$$
The second tensorand here is the unramified (hence crystalline) character of $\mathcal{O}_v^{\times}$ sending ϖ_v to $\vartheta_v(\varpi_v)$. Since we also have $\psi_z(U_v) = \vartheta(\varpi_v)\psi_x(U_v)$, we see that checking (2) holds for z is equivalent to checking that (2) holds for x, i.e. without loss of generality we can assume that $z = x$ is classical. But then (2) follows immediately from Theorem 6.5.1. □

6.6. **Smoothness at some decent classical points.** We now generalize the definition of non-critical.

Definition 6.6.1. A classical point $x = x(\pi,\alpha) \in \mathscr{E}(\mathfrak{n})(\overline{\mathbf{Q}}_p)$ is decent if either:

1. It is non-critical as in Definition 6.3.4, or
2. The following three conditions hold.
   - (a) $H^*_c(\mathfrak{n}, \mathscr{D}_\lambda)_{\mathfrak{m}_x}$ is concentrated only in degree d,
   - (b) The Selmer group $H^1_f(G_F, \mathrm{ad}\,\rho_\pi)$ vanishes, and
   - (c) For each $v \mid p$, α_v is a simple root of $X^2 - a_v(\pi)X + \omega_\pi(\varpi_v)q_v$.

In condition 2(b) of Definition 6.6.1, $\mathrm{ad}\,\rho_\pi$ is the adjoint representation $\rho_\pi \otimes \rho_\pi^\vee \simeq \mathrm{End}(\rho_\pi)$.

Lemma 6.6.2. *If $x \in \mathscr{E}(\mathfrak{n})(\overline{\mathbf{Q}}_p)$ is decent, then $x \in \mathscr{E}(\mathfrak{n})_{\mathrm{mid}}(\overline{\mathbf{Q}}_p)$.*

Proof. If x is non-critical, then this follows from Lemma 6.4.5. Otherwise, see Proposition 6.4.3. □

[20] Compare with condition 2(c) in Definition 6.6.1 below.

We will see later (Theorem 8.1.4) that the Hecke eigensystem corresponding to a decent point x has multiplicity one in the distribution-valued cohomology. When x is a non-critical point, this is a classical automorphic fact. But if x satisfies condition (2) of Definition 6.6.1, we deduce it from the following geometric theorem on the eigenvariety. The proof occupies the rest of this subsection.

Theorem 6.6.3. *Suppose that $x \in \mathscr{E}(\mathfrak{n})_{\mathrm{mid}}(\overline{\mathbf{Q}}_p)$ is decent, the prime-to-p conductor of x is $\mathfrak{n}$, and condition 2(c) in Definition 6.6.1 is satisfied. Then, $\mathscr{E}(\mathfrak{n})_{\mathrm{mid}}$ is smooth at x.*

To be clear, the assumption on x in Theorem 6.6.3 is that either x satisfies condition (2) of Definition 6.6.1 or x is non-critical and further satisfies condition 2(c) of Definition 6.6.1. The proof in case x satisfies (2) is at the end of the subsection. In case x is non-critical, the proof is in Proposition 6.6.4 below.

We now fix some notation that will remain in force for the rest of this section. We will write $x \in \mathscr{E}(\mathfrak{n})_{\mathrm{mid}}(\overline{\mathbf{Q}}_p)$ and $\lambda = \lambda_x$ for its weight. Write $L = k_x$ for the residue field at x. We write $\mathscr{O}_x$ for the rigid local ring on $\mathscr{E}(\mathfrak{n})_{\mathrm{mid}}$ at x and $\mathscr{O}_\lambda$ for the rigid local ring on $\mathscr{W}(1)$ at λ.

We first prove Theorem 6.6.3 in the non-critical case.

Proposition 6.6.4. *If $x \in \mathscr{E}(\mathfrak{n})_{\mathrm{mid}}(\overline{\mathbf{Q}}_p)$ is as in Theorem 6.6.3 and non-critical, then $\lambda : \mathscr{E}(\mathfrak{n})_{\mathrm{mid}} \to \mathscr{W}(1)$ is étale at x.*

Proof. This argument is essentially due to Chenevier ([32, Theorem 4.8]).[21]

Let U be a sufficiently small good neighborhood of x, belonging to a slope adapted pair (Ω, h), such that x is the unique reduced point of U lying above $\lambda \in \Omega$. Set $M = \mathscr{M}_c^d(U)$. For each $\epsilon \in \{\pm 1\}^{\Sigma_F}$, let M^ϵ be the ϵ-component, so $M = \bigoplus_\epsilon M^\epsilon$. Since these are $\mathscr{O}(\Omega)$-direct summands of M they are each finite projective over $\mathscr{O}(\Omega)$ (see Proposition 6.4.4) and U is the rigid analytic spectrum of the image of $\mathbf{T}(\mathfrak{n}) \otimes_{\mathbf{Q}_p} \mathscr{O}(\Omega) \to \mathrm{End}_{\mathscr{O}(\Omega)}(M^\epsilon)$ (for any ϵ). Further, if $\lambda' \in \Omega$ is any weight then

$$M^\epsilon / \mathfrak{m}_{\lambda'} M^\epsilon = \bigoplus_{\substack{y \in U \\ \lambda_y = \lambda'}} H_c^d(\mathfrak{n}, \mathscr{D}_{\lambda'})^\epsilon_{\mathfrak{m}_{y'}}. \tag{6.6.1}$$

Remember that we have assumed x is the unique point above λ. So, since x is assumed to be non-critical, the prime-to-p conductor of π is $\mathfrak{n}$, and part (c) of Definition 6.6.1 holds, we deduce that (6.6.1) is in fact 1-dimensional. If λ' is any other weight near to λ over which all the points y' are extremely non-critical with prime-to-p conductor $\mathfrak{n}$ (such weights are accumulating at λ) then $H_c^d(\mathfrak{n}, \mathscr{D}_{\lambda'})^\epsilon_{\mathfrak{m}_{y'}}$ is also 1-dimensional. Since the dimension of (6.6.1) is constant with respect to λ' we deduce M^ϵ is projective of rank one over $\mathscr{O}(\Omega)$. So, the composition $\mathscr{O}(\Omega) \to \mathscr{O}(U) \to \mathrm{End}_{\mathscr{O}(\Omega)}(M^\epsilon)$ becomes an isomorphism after a finite field extension, meaning $\mathscr{O}(\Omega) \to \mathscr{O}(U)$ is étale. □

For the remainder of this subsection we fix a decent classical point $x \in \mathscr{E}(\mathfrak{n})_{\mathrm{mid}}(\overline{\mathbf{Q}}_p)$ of weight λ as in Theorem 6.6.3. Because Proposition 6.6.4 deals with the non-critical case of Theorem 6.6.3, we will further assume that x satisfies condition (2) of Definition 6.6.1. Write $\rho = \rho_x$ for the global Galois representation and ρ_v for its restriction to a place v.

If $v \mid p$ then we write $\mathfrak{X}_v$ for the deformation functor on local Artin L-algebras with residue field L parameterizing deformations $\widetilde{\rho}_v$ of ρ_v. Since the Hodge–Tate weights of ρ_v are distinct at each embedding $\sigma \in \Sigma_v$ (by direct inspection in Theorem 6.5.1) to such a deformation $\widetilde{\rho}_v$ we may naturally associate characters $\widetilde{\eta}_{i,v}$ of $\mathcal{O}_v^\times$ whose Hodge–Tate–Sen weights $\mathrm{HT}_\sigma(\widetilde{\eta}_{i,v})$ are lifts of the Hodge–Tate weights $\mathrm{HT}_\sigma(\eta_{i,v}(\lambda))$ of ρ_v.

[21] In the case of $F = \mathbf{Q}$ there is also an argument given by Bellaïche ([11, Lemma 2.8]) which relies on *a priori* knowing that the weight map is flat. In general, this is only observed at decent points and only after the arguments in this section. See Section 8.1.

Recall that $\alpha_v^\sharp = \phi_x(U_v)$ is an eigenvalue for φ^{f_v} acting on $D^+_{\mathrm{crys}}(\rho_v \otimes \mathrm{LT}_{\varpi_v}(\eta_{1,v}(\lambda))^{-1})$, and $\alpha_v^\sharp$ is simple by the assumption 2(c) in Definition 6.6.1. Thus we have a relatively representable subfunctor $\mathfrak{X}_v^{\mathrm{Ref}} \subset \mathfrak{X}_v$ by declaring a deformation $\widetilde{\rho}_v$ lies in $\mathfrak{X}_v^{\mathrm{Ref}}$ if and only if $D^+_{\mathrm{crys}}(\widetilde{\rho}_v \otimes \mathrm{LT}_{\varpi_v}(\widetilde{\eta}_{1,v})^{-1})^{\varphi=\widetilde{\Phi}}$ is free of rank one for some lift $\widetilde{\Phi}$ of $\alpha_v^\sharp$ ([14, Definition 3.5]). Write $\mathfrak{t}_v^{\mathrm{Ref}} = \mathfrak{X}_v^{\mathrm{Ref}}(L[\varepsilon])$ for the Zariski tangent space to $\mathfrak{X}_v^{\mathrm{Ref}}$ ($L[\varepsilon]$ is the ring of dual numbers $L[u]/(u^2)$). Then, $\mathfrak{t}_v^{\mathrm{Ref}}$ is a subspace of $H^1(G_{F_v}, \mathrm{ad}\,\rho_v)$, the Zariski tangent space to $\mathfrak{X}_v$. Inside $H^1(G_{F_v}, \mathrm{ad}\,\rho_v)$, we also have a local Bloch–Kato Selmer group $H^1_f(G_{F_v}, \mathrm{ad}\,\rho_v)$ (see [18] and the text prior to Lemma A.0.11). The relationship between $\mathfrak{X}_v^{\mathrm{Ref}}$ and the H^1_f is studied in [14, Section 3] when ρ_v is crystalline. We treat the semistable, but non-crystalline, case in Appendix A.

Proposition 6.6.5.

(1) $H^1_f(G_{F_v}, \mathrm{ad}\,\rho_v) \subset \mathfrak{t}_v^{\mathrm{Ref}}$.
(2) $\dim_L \mathfrak{t}_v^{\mathrm{Ref}}/H^1_f(G_{F_v}, \mathrm{ad}\,\rho_v) \leq 2(F_v : \mathbf{Q}_p)$.

Proof. If ρ_v is crystalline then part (1) is clear and part (2) is proven in [14, Corollary 3.16].[22] If ρ_v is semistable but non-crystalline, see Lemma A.0.11 for part (1) and Corollary A.0.16 for part (2). ☐

If $v \nmid p$ then write $\mathfrak{X}_{v,f}$ for the minimally ramified deformations of ρ_v, i.e. deformations $\widetilde{\rho}_v$ to local Artin L-algebras so that $\widetilde{\rho}_v \simeq \rho_v \otimes_L A$ as representations of an inertia group at v. The Zariski tangent space to this deformation problem is the local Bloch–Kato Selmer group $H^1_f(G_{F_v}, \mathrm{ad}\,\rho_v) \subset H^1(G_{F_v}, \mathrm{ad}\,\rho_v)$.

Finally, denote by $\mathfrak{X}_\rho^{\mathrm{Ref}}$ the deformations of the global representation ρ which are weakly-refined at $v \mid p$ and minimally ramified at $v \nmid p$. The arrow $\mathfrak{X}_\rho^{\mathrm{Ref}} \to \mathfrak{X}_\rho$ is relatively representable (it is a fiber product of relatively representable functors) and, since ρ is absolutely irreducible, we deduce that there is a universal deformation ring R_ρ^{Ref} representing $\mathfrak{X}_\rho^{\mathrm{Ref}}$.

From now on, write $H^1_{/f}$ for the quotient H^1/H^1_f. Then, the tangent space $\mathfrak{t}_\rho^{\mathrm{Ref}}$ to $\mathfrak{X}_\rho^{\mathrm{Ref}}$ apparently sits in a an exact sequence

$$0 \to \mathfrak{t}_\rho^{\mathrm{Ref}} \to H^1(G_F, \mathrm{ad}\,\rho) \to \left(\prod_{v\mid p} H^1(G_{F_v}, \mathrm{ad}\,\rho_v)/\mathfrak{t}_v^{\mathrm{Ref}}\right) \oplus \left(\prod_{v\nmid p} H^1_{/f}(G_{F_v}, \mathrm{ad}\,\rho_v)\right).$$

The global Bloch–Kato Selmer group $H^1_f(G_F, \mathrm{ad}\,\rho)$ is contained in $\mathfrak{t}_\rho^{\mathrm{Ref}}$ (Proposition 6.6.5(a)), but by assumption 2(b) of Definition 6.6.1, it vanishes. So we deduce that there is a canonical containment

$$\mathfrak{t}_\rho^{\mathrm{Ref}} \subset \bigoplus_{v\mid p} \mathfrak{t}_v^{\mathrm{Ref}}/H^1_f(G_{F_v}, \mathrm{ad}\,\rho_v). \tag{6.6.2}$$

We note that we have upper bounds for the dimensions of the spaces in the sum (6.6.2) by Proposition 6.6.5.

Recall that $H^1(G_F, L)$ parameterizes infinitesimal deformations of any one-dimensional $L^\times$-valued character of G_F. Precisely, if $c \in H^1(G_F, L) = \mathrm{Hom}(G_F, L)$ and $\chi : G_F \to L$ is a character then the deformation $\chi_c : G_F \to L[\varepsilon]^\times$ is given by $\chi_c(\sigma) = \chi(\sigma)(1 + c(\sigma)\varepsilon)$.

[22] The hypothesis on a "critical-type" in [14] is vacuously satisfied in dimension 2. This kind of miracle is under-emphasized in our arguments for space reasons. In fact, in higher dimensions the smoothness statement we are after is provably false sometimes! See [24].

Definition 6.6.6. $H^1_{\mathrm{rel}}(G_F, L)$ is the subspace of characters χ_c whose restriction $\chi_{c,v}$ to G_{F_v} (for $v \mid p$) makes the composition

$$\mathscr{O}_F^\times \to \prod_{v|p} \mathscr{O}_v^\times \xrightarrow{(\mathrm{Art}_{F_v})_v} \prod_{v|p} G_{F_v}^{\mathrm{ab}} \xrightarrow{(\chi_{c,v})_v} L[\varepsilon]^\times$$

vanish in a subgroup of finite index in $\mathscr{O}_F^\times$.

The subscript "rel" means "relevant" as in "relevant to an eigenvariety".

Lemma 6.6.7. *The image of $H^1_{\mathrm{rel}}(G_F, L) \to \bigoplus_{v|p} H^1_{/f}(G_{F_v}, L)$ has co-dimension $d - 1 - \delta_{F,p}$.*

Proof. By local class field theory, $\bigoplus_{v|p} H^1_{/f}(G_{F_v}, L)$ is naturally identified with $\mathrm{Hom}(\mathscr{O}_p^\times, L)$ which has dimension $\dim \mathscr{O}_p^\times = d$ (dimension of $\mathscr{O}_p^\times$ as a CPA group). The image of the relevant deformations are those morphism $\mathscr{O}_p^\times \to L$ factoring through some quotient with the same dimension as $\mathscr{O}_p^\times/\overline{\mathscr{O}_F^\times}$. Since $\mathscr{O}_p^\times/\overline{\mathscr{O}_F^\times}$ has dimension $d - (d - 1 - \delta_{F,p}) = 1 + \delta_{F,p}$, the lemma follows. $\square$

We are now ready to give the proof of Theorem 6.6.3. We recall that $x \in \mathscr{E}(\mathfrak{n})_{\mathrm{mid}}(\overline{\mathbf{Q}}_p)$ is a classical point of prime-to-p conductor $\mathfrak{n}$ and satisfies condition (2) in Definition 6.6.1 (we have already used all these assumptions in the discussion above).

Proof of Theorem 6.6.3 when x satisfies condition (2) in Definition 6.6.1. Let $\mathfrak{t}_x$ be the tangent space to $\mathscr{E}(\mathfrak{n})_{\mathrm{mid}}$ at x. Since $\mathscr{E}(\mathfrak{n})_{\mathrm{mid}}$ is equidimensional of dimension $1 + d + \delta_{F,p}$ (Proposition 6.4.6), we have a lower bound $1 + d + \delta_{F,p} \leq \dim \mathfrak{t}_x$. To prove the theorem we need to show the reverse inequality holds.

Set

$$\mathfrak{t}_\rho^{\mathrm{Ref,rel}} := \mathfrak{t}_\rho^{\mathrm{Ref}} \cap \ker\left(\det : H^1(G_F, \mathrm{ad}\,\rho) \to H^1(G_F, L)/H^1_{\mathrm{rel}}(G_F, L)\right).$$

Lemma 6.5.6 defines a lift $\rho_{\mathscr{O}_x}$ of ρ to $\mathscr{O}_x$ and Proposition 6.5.8 defines, by universality, a canonical point $R_\rho^{\mathrm{Ref}} \to \widehat{\mathscr{O}}_x$. A standard argument (see [14, Proposition 4.3] for instance) shows that R_ρ^{Ref} surjects onto $\widehat{\mathscr{O}}_x$. Thus, there is an induced inclusion $\mathfrak{t}_x \subset \mathfrak{t}_\rho^{\mathrm{Ref}}$ on tangent spaces. In fact, $\mathfrak{t}_x \subset \mathfrak{t}_\rho^{\mathrm{Ref,rel}}$ by Lemma 6.5.5 and Proposition 6.5.8. We claim that $\dim \mathfrak{t}_\rho^{\mathrm{Ref,rel}} \leq 1 + d + \delta_{F,p}$, from which the inequality we want for $\dim \mathfrak{t}_x$ follows.

To prove the claim, consider the determinant $\det_v : \mathfrak{t}_v^{\mathrm{Ref}} \to H^1(G_{F_v}, L)$ (for $v \mid p$). We observe that it is surjective. Indeed, if $\widetilde{d} : F^\times \to L[\varepsilon]^\times$ is a deformation of $\det \rho_v$ then $\widetilde{d}$ is unitary and write $\widetilde{d}(x) = \det \rho_v(x)(1 + a(x)\varepsilon)$ where $a : F^\times \to L$ is a homomorphism which extends to G_F^{ab}. Write $\rho_{v,L[\varepsilon]}$ for the trivial deformation of ρ_v to $L[\varepsilon]$ (evidently an element of $\mathfrak{t}_v^{\mathrm{Ref}}$) and then set $\widetilde{\rho}_v := \rho_{v,L[\varepsilon]}(1 + \frac{a(x)}{2}\varepsilon)$. One checks immediately that $\det \widetilde{\rho}_v$ is equal to $\widetilde{d}$ and that moreover $\widetilde{\rho}_v$ is still an element of $\mathfrak{t}_v^{\mathrm{Ref}}$ (since twisting does not effect membership, as the definition contains a twist already).

Thus we see that the map

$$\bigoplus_{v|p} \mathfrak{t}_v^{\mathrm{Ref}}/H^1_f(G_{F_v}, \mathrm{ad}\,\rho_v) \xrightarrow{(\det_v)_{v|p}} \frac{\left(\bigoplus_{v|p} H^1_{/f}(G_{F_v}, L)\right)}{\mathrm{im}\left(H^1_{\mathrm{rel}}(G_F, L) \to \bigoplus_{v|p} H^1_{/f}(G_{F_v}, L)\right)}$$

is also surjective. By Proposition 6.6.5 and Lemma 6.6.7, we deduce that

$$\dim \ker((\det_v)_{v|p}) \leq \left(\sum_{v|p} 2(F_v : \mathbf{Q}_p)\right) - (d - 1 - \delta_{F,p}) = d + 1 + \delta_{F,p}.$$

On the other hand, under the natural inclusion $\mathfrak{t}_\rho^{\mathrm{Ref}} \hookrightarrow \bigoplus_{v|p} \mathfrak{t}_v^{\mathrm{Ref}}/H^1_f(G_{F_v}, \mathrm{ad}\,\rho_v)$ we have that $\mathfrak{t}_\rho^{\mathrm{Ref,rel}} \subset \ker((\det_v)_{v|p})$ so we have shown that $\dim \mathfrak{t}_\rho^{\mathrm{Ref,rel}} \le 1 + d + \delta_{F,p}$. This completes the proof. □

7. Period maps

Recall that we write Γ_F for the maximal abelian extension of F unramified away from p and ∞. This is a CPA group and hence we have R-valued distributions $\mathscr{D}(\Gamma_F, R)$ for any affinoid point $\mathrm{Sp}(R) = \Omega \to \mathscr{W}$. The goal of this section is to define, and study, canonical morphisms

$$\mathscr{P}_\Omega : H^d_c(\mathfrak{n}, \mathscr{D}_\Omega) \to \mathscr{D}(\Gamma_F, R)$$

which we call period maps. Amice's theorem then links the period maps to p-adic L-functions.

7.1. Analytic distributions on Γ_F.

Consider the canonical exact sequence

$$1 \to \overline{\mathcal{O}_{F,+}^\times} \to \mathcal{O}_p^\times \xrightarrow{j_p} \Gamma_F \to \mathrm{Cl}_F^+ \to 1 \tag{7.1.1}$$

where Cl_F^+ is the narrow class group, and the map j_p is induced by the natural inclusion $\mathcal{O}_p^\times \hookrightarrow \mathbf{A}_F^\times$. We will need to make explicit some LB-structures on rings of analytic functions.

We begin with $\mathcal{O}_p^\times$. In Section 5.3 we defined, for $f \in \mathscr{A}(\mathcal{O}_p^\times, \mathbf{Q}_p)$, the "extension by zero" function $f_! : \mathcal{O}_p \to \mathbf{Q}_p$

$$f_!(a) = \begin{cases} f(a) & \text{if } a \in \mathcal{O}_p^\times, \\ 0 & \text{otherwise.} \end{cases}$$

The map $f \mapsto f_!$ defines a closed embedding $\mathscr{A}(\mathcal{O}_p^\times, \mathbf{Q}_p) \hookrightarrow \mathscr{A}(\mathcal{O}_p, \mathbf{Q}_p)$. For $\mathbf{s} \in \mathbf{Z}_{\ge 0}^{\{v|p\}}$ we set $\mathbf{A}^{\mathbf{s},\circ}(\mathcal{O}_p^\times, \mathbf{Q}_p) := \mathscr{A}(\mathcal{O}_p^\times, \mathbf{Q}_p) \cap \mathbf{A}^{\mathbf{s},\circ}(\mathcal{O}_p, \mathbf{Q}_p)$ and

$$\mathbf{A}^{\mathbf{s}}(\mathcal{O}_p^\times, \mathbf{Q}_p) := \mathbf{A}^{\mathbf{s},\circ}(\mathcal{O}_p^\times, \mathbf{Q}_p)[1/p] = \mathscr{A}(\mathcal{O}_p^\times, \mathbf{Q}_p) \cap \mathbf{A}^{\mathbf{s}}(\mathcal{O}_p, \mathbf{Q}_p),$$

all the intersections happening within $\mathscr{A}(\mathcal{O}_p^\times, \mathbf{Q}_p)$. By (5.2.1), and because $\mathscr{A}(\mathcal{O}_p^\times, \mathbf{Q}_p)$ is closed inside $\mathscr{A}(\mathcal{O}_p, \mathbf{Q}_p)$, we deduce from [37, Proposition 1.1.41] that there is a natural topological identification

$$\mathscr{A}(\mathcal{O}_p^\times, \mathbf{Q}_p) \simeq \varinjlim_{|\mathbf{s}| \to +\infty} \mathbf{A}^{\mathbf{s}}(\mathcal{O}_p^\times, \mathbf{Q}_p). \tag{7.1.2}$$

Now consider Γ_F. If $\gamma \in \Gamma_F$ write $r_\gamma : \Gamma_F \to \Gamma_F$ for multiplication by γ. Then, if $\gamma \in \Gamma_F$ and $f \in \mathscr{A}(\Gamma_F, \mathbf{Q}_p)$ we define

$$f|_{\gamma\mathcal{O}_p^\times} := f \circ r_\gamma \circ j_p$$

which is an element of $\mathscr{A}(\mathcal{O}_p^\times, \mathbf{Q}_p)$. For each $\mathbf{s} \in \mathbf{Z}_{\ge 0}^{\{v|p\}}$ we define

$$\mathbf{A}^{\mathbf{s},\circ}(\Gamma_F, \mathbf{Q}_p) := \{f \in \mathscr{A}(\Gamma_F, \mathbf{Q}_p) \mid f|_{\gamma\mathcal{O}_p^\times} \in \mathbf{A}^{\mathbf{s},\circ}(\mathcal{O}_p^\times, \mathbf{Q}_p) \text{ for each } \gamma \in \Gamma_F\}, \tag{7.1.3}$$

and

$$\mathbf{A}^{\mathbf{s}}(\Gamma_F, \mathbf{Q}_p) := \mathbf{A}^{\mathbf{s},\circ}(\Gamma_F, \mathbf{Q}_p)[1/p] = \{f \in \mathscr{A}(\Gamma_F, \mathbf{Q}_p) \mid f|_{\gamma\mathcal{O}_p^\times} \in \mathbf{A}^{\mathbf{s}}(\mathcal{O}_p^\times, \mathbf{Q}_p) \text{ for each } \gamma \in \Gamma_F\}.$$

Lemma 7.1.1. *The natural map*

$$\varinjlim_{|\mathbf{s}| \to +\infty} \mathbf{A}^{\mathbf{s}}(\Gamma_F, \mathbf{Q}_p) \to \mathscr{A}(\Gamma_F, \mathbf{Q}_p) \tag{7.1.4}$$

is a topological isomorphism.

Proof. Note that $H = \mathrm{im}(j_p)$ is a CPA group and the natural map $\mathscr{A}(H, \mathbf{Q}_p) \to \mathscr{A}(\mathcal{O}_p^\times, \mathbf{Q}_p)$ is closed embedding. By the same argument above (especially (7.1.2) and [37, Proposition 1.1.41]) we deduce that $\mathbf{A}^{\mathbf{s}}(H, \mathbf{Q}_p) := \mathscr{A}(H, \mathbf{Q}_p) \cap \mathbf{A}^{\mathbf{s}}(\mathcal{O}_p^\times, \mathbf{Q}_p)$ presents $\mathscr{A}(H, \mathbf{Q}_p)$ topologically as a locally convex inductive limit

$$\mathscr{A}(H, \mathbf{Q}_p) \simeq \varinjlim_{|\mathbf{s}| \to +\infty} \mathbf{A}^{\mathbf{s}}(H, \mathbf{Q}_p). \tag{7.1.5}$$

Choose coset representatives $\gamma_1, \dots, \gamma_h$ for Γ_F/H. Then, the natural topological isomorphism

$$\begin{aligned} \mathscr{A}(\Gamma_F, \mathbf{Q}_p) &\xrightarrow{\simeq} \bigoplus_{i=1}^{h} \mathscr{A}(H, \mathbf{Q}_p) \\ f &\mapsto (h \mapsto f(\gamma_i h)) \end{aligned}$$

identifies the subspace $\mathbf{A}^{\mathbf{s}}(\Gamma_F, \mathbf{Q}_p)$ defined above with the direct sum of the subspaces $\mathbf{A}^{\mathbf{s}}(H, \mathbf{Q}_p)$ we just defined. So the map (7.1.4) being a topological isomorphism is a consequence of the same fact for (7.1.5) and the fact that locally convex inductive limits commute with finite products. This completes the proof. $\square$

Now suppose that R is a $\mathbf{Q}_p$-Banach algebra and R_0 is a ring of definition. Then, for any of the CPA groups G which appear above, we set $\mathbf{A}^{\mathbf{s},\circ}(G, R) := \mathbf{A}^{\mathbf{s},\circ}(G, \mathbf{Q}_p)\widehat{\otimes}_{\mathbf{Z}_p} R_0$ and $\mathbf{A}^{\mathbf{s}}(G, R) := \mathbf{A}^{\mathbf{s},\circ}(G, R)[1/p] = \mathbf{A}^{\mathbf{s}}(G, \mathbf{Q}_p)\widehat{\otimes}_{\mathbf{Q}_p} R$. We define distribution algebras $\mathbf{D}^{\mathbf{s}}(G, R) = \mathbf{A}^{\mathbf{s}}(G, R)'$ and $\mathbf{D}^{\mathbf{s},\circ}(G, R) = \mathrm{Hom}_{R_0}(\mathbf{A}^{\mathbf{s},\circ}(*, R), R_0)$, with the same caveat as in Remark 5.2.3.

We note the following analogue of Lemma 5.3.1, which illustrates the compatibility of our notations of $\mathbf{s}$-analytic.

Lemma 7.1.2. *Suppose that $\chi : \mathcal{O}_p^\times \to R$ is a continuous character and $R_0 \subset R$ is a ring of definition containing the image of χ. Then for $\mathbf{s}^\circ(\chi)$ as in Lemma 5.3.1, we have $\chi \in \mathbf{A}^{\mathbf{s}^\circ(\chi)+1,\circ}(\mathcal{O}_p^\times, R)$ (similarly for $\mathbf{s}(\chi)$).*

Proof. This follows immediately from the following observation whose proof we omit: if $f : \mathcal{O}_p \to R$ is a function and $z \mapsto f(a + \varpi_p z)$ defines an element of $\mathbf{A}^{\mathbf{s},\circ}(\mathcal{O}_p, R)$ for each $a \in \mathcal{O}_p$, then f itself defines an element of $\mathbf{A}^{\mathbf{s}+\mathbf{1},\circ}(\mathcal{O}_p, R)$. $\square$

7.2. **Definition of period maps.** Recall (Section 2.3) that C_∞ denotes the Shintani cone. If $\Omega = \mathrm{Sp}(R) \to \mathscr{W}$ is a $\mathbf{Q}_p$-affinoid with corresponding weight λ_Ω, then we write $\mathrm{t}^*\mathbf{A}_\Omega^{\mathbf{s}}$ for the local system on C_∞ induced by the right action of $\mathcal{O}_p^\times$

$$f\big|_{u_p}(z) := f\big|_{\left(\begin{smallmatrix} u_p & \\ & 1 \end{smallmatrix}\right)}(z) = \lambda_{\Omega,2}(u_p) f(u_p z)$$

for each $f \in \mathbf{A}^{\mathbf{s}}(\mathcal{O}_p, R)$, $u_p \in \mathcal{O}_p^\times$, and $z \in \mathcal{O}_p$ (here $\mathbf{s} \geq \mathbf{s}(\Omega)$). The action is compatible with changing $\mathbf{s} \geq \mathbf{s}(\Omega)$, and if $R_0 \subset R$ is a ring of definition for R containing the values of λ_Ω and $\mathbf{s} \geq \mathbf{s}^\circ(\Omega)$ then it preserves the R_0-submodule $\mathrm{t}^*\mathbf{A}_\Omega^{\mathbf{s},\circ}$.

Lemma 7.2.1. *Fix a ring of definition $R_0 \subset R$ and $\mathbf{s} \geq \mathbf{s}^\circ(\Omega)$. For $f \in \mathbf{A}^{\mathbf{s},\circ}(\Gamma_F, R)$, $x \in \mathbf{A}_F^\times$, and $z \in \mathcal{O}_p$ define*

$$Q_\Omega^{\mathbf{s},\circ}(f)(x)(z) = \begin{cases} \lambda_{\Omega,2}^{-1}(z) \cdot f(xz) & \textit{if } z \in \mathcal{O}_p^\times; \\ 0 & \textit{otherwise.} \end{cases} \tag{7.2.1}$$

Then, $f \mapsto Q_\Omega^{\mathbf{s},\circ}$ defines an R_0-module morphism

$$Q_\Omega^{\mathbf{s},\circ} : \mathbf{A}^{\mathbf{s},\circ}(\Gamma_F, R) \to H^0(\mathrm{C}_\infty, \mathrm{t}^*\mathbf{A}_\Omega^{\mathbf{s},\circ}).$$

Moreover, the induced map $Q^{\mathbf{s}}_{\Omega} : \mathbf{A}^{\mathbf{s}}(\Gamma_F, R) \to H^0(\mathrm{C}_\infty, \mathrm{t}^*\mathbf{A}^{\mathbf{s}})$ *is independent of* R_0 *and if* $\mathbf{s}' \geq \mathbf{s}$ *then fits naturally into a commuting diagram*

$$\begin{tikzcd} \mathbf{A}^{\mathbf{s}}(\Gamma_F, R) \arrow[r, "Q^{\mathbf{s}}_{\Omega}"] \arrow[d] & H^0(\mathrm{C}_\infty, \mathrm{t}^*\mathbf{A}^{\mathbf{s}}_{\Omega}) \arrow[d] \\ \mathbf{A}^{\mathbf{s}'}(\Gamma_F, R) \arrow[r, "Q^{\mathbf{s}'}_{\Omega}"] & H^0(\mathrm{C}_\infty, \mathrm{t}^*\mathbf{A}^{\mathbf{s}'}_{\Omega}) \end{tikzcd}$$

and these extend to a natural map

$$Q_\Omega : \mathscr{A}(\Gamma_F, R) \to H^0(\mathrm{C}_\infty, \mathrm{t}^*\mathscr{A}_\Omega).$$

Proof. All the claims after inverting p are clear, so we just prove the first statement.

Let $f \in \mathbf{A}^{\mathbf{s},\circ}(\Gamma_F, R)$ and set $q = Q^{\mathbf{s},\circ}_{\Omega}(f)$ defined in (7.2.1). It follows from Lemma 7.1.2 and the precise definitions of the radii that $q(x) \in \mathbf{A}^{\mathbf{s},\circ}_{\Omega}$ for each $x \in \mathbf{A}^\times_F$, giving us a continuous function $q : \mathbf{A}^\times_F \to \mathbf{A}^{\mathbf{s},\circ}_{\Omega}$ which we want to show it is a section in $H^0(\mathrm{C}_\infty, \mathrm{t}^*\mathbf{A}^{\mathbf{s},\circ}_{\Omega})$.

First, q is locally constant on $F^\times_\infty$ because the function f itself factors through $F^\times_{\infty,+}$. It remains to show that $q(\xi x u) = q(x)|_{u_p}$ for all $\xi \in F^\times$, $x \in \mathbf{A}^\times_F$ and $u \in \widehat{\mathcal{O}}^\times_F$. If $z \in \mathcal{O}_p - \mathcal{O}^\times_p$ then both $q(\xi x u)$ and $q(x)|_{u_p}$ vanish on z. If $z \in \mathcal{O}^\times_p$ though, then

$$q(x)|_{u_p}(z) = \lambda_{\Omega,2}(u_p) q(x)(u_p z) = \lambda^{-1}_{\Omega,2}(z) f(x u_p z) = \lambda^{-1}_{\Omega,2}(z) f(\xi x u z) = q(\xi x u)(z).$$

For the second to last equality we used that f is a function on Γ_F. This completes the proof. ☐

Remark 7.2.2. The use of the word natural at the end of the statement of Lemma 7.2.1 refers to the apparent compatibility with change of ring. Namely, if $\mathrm{Sp}(R) = \Omega \to \mathscr{W}$ factors through $\mathrm{Sp}(R') = \Omega'$ then we have a commuting diagram

$$\begin{tikzcd} \mathscr{A}(\Gamma_F, R) \arrow[r, "Q_\Omega"] & H^0(\mathrm{C}_\infty, \mathrm{t}^*\mathscr{A}_\Omega) \\ \mathscr{A}(\Gamma_F, R') \arrow[r, "Q_{\Omega'}"'] \arrow[u] & H^0(\mathrm{C}_\infty, \mathrm{t}^*\mathscr{A}_{\Omega'}). \arrow[u] \end{tikzcd}$$

Throughout the rest of this subsection we consider an integral ideal $\mathfrak{m} \subset \mathcal{O}_F$ and we assume that $\mathfrak{m} \subset \mathbf{p}$. Since $K_1(\mathfrak{m})$ is t-good, we have a proper embedding $\mathrm{t} : \mathrm{C}_\infty \hookrightarrow Y_1(\mathfrak{m})$ as in (2.3.5).

For each Ω as above, $\mathrm{t}^*\mathbf{A}^{\mathbf{s},\circ}_{\Omega}$ is the pullback of the local system $\mathbf{A}^{\mathbf{s},\circ}_{\Omega}$ on $Y_1(\mathfrak{m})$ (which is well-posed because $\mathfrak{m} \subset \mathbf{p}$). There are similar obvious comments regarding $\mathbf{A}^{\mathbf{s}}_{\Omega}$ and $\mathscr{A}_\Omega$. Thus, by Lemma 7.2.1, we get a composition $\mathscr{Q}_\Omega = \mathrm{t}_* \circ \mathrm{PD} \circ Q_\Omega$

$$\begin{tikzcd} \mathscr{A}(\Gamma_F, R) \arrow[r, "Q_\Omega"] \arrow[rrr, bend right, "\mathscr{Q}_\Omega"'] & H^0(\mathrm{C}_\infty, \mathrm{t}^*\mathscr{A}_\Omega) \arrow[r, "\mathrm{PD}"] & H^{\mathrm{BM}}_d(\mathrm{C}_\infty, \mathrm{t}^*\mathscr{A}_\Omega) \arrow[r, "\mathrm{t}_*"] & H^{\mathrm{BM}}_d(Y_1(\mathfrak{m}), \mathscr{A}_\Omega). \end{tikzcd} \tag{7.2.2}$$

We also have natural analogs $\mathscr{Q}^{\mathbf{s}}_{\Omega}$ and $\mathscr{Q}^{\mathbf{s},\circ}_{\Omega}$.

Now recall that the natural pairing $\mathscr{D}_\Omega \otimes_R \mathscr{A}_\Omega \to R$ together with the cap product defines a canonical R-bilinear pairing

$$\langle -, - \rangle : H^d_c(Y_1(\mathfrak{m}), \mathscr{D}_\Omega) \otimes_R H^{\mathrm{BM}}_d(Y_1(\mathfrak{m}), \mathscr{A}_\Omega) \to R.$$

Thus we define $\mathscr{P}_\Omega : H^d_c(Y_1(\mathfrak{m}), \mathscr{D}_\Omega) \to \mathrm{Hom}_R(\mathscr{A}(\Gamma_F, R), R)$ to be given by

$$\mathscr{P}_\Omega(\Psi)(f) = \langle \Psi, \mathscr{Q}_\Omega(f) \rangle. \tag{7.2.3}$$

Replacing $\mathscr{Q}_\Omega$ with $\mathscr{Q}_\Omega^{\mathbf{s}}$ or $\mathscr{Q}_\Omega^{\mathbf{s},\circ}$, we also get analogous morphisms $\mathscr{P}_\Omega^{\mathbf{s}}$ and $\mathscr{P}_\Omega^{\mathbf{s},\circ}$. The rest of this subsection is devoted to proving the following theorem.

Theorem 7.2.3. *The image of $\mathscr{P}_\Omega$ is contained in $\mathscr{D}(\Gamma_F, R) \subset \mathrm{Hom}_R(\mathscr{A}(\Gamma_F, R), R)$.*

Omitting the proof, we record precisely the definition of the period map(s).

Definition 7.2.4. If $\Omega = \mathrm{Sp}(R) \to \mathscr{W}$ is a point, then the period map $\mathscr{P}_\Omega$ is the R-linear map

$$\begin{aligned} \mathscr{P}_\Omega : H_c^d(Y_1(\mathfrak{m}), \mathscr{D}_\Omega) &\to \mathscr{D}(\Gamma_F, R) \\ \mathscr{P}_\Omega(\Psi)(f) &= \langle \Psi, \mathscr{Q}_\Omega(f)\rangle \end{aligned}$$

defined above.

To prove Theorem 7.2.3, we note the following lemma on recognizing when certain linear functions are continuous.

Lemma 7.2.5. *Suppose that R is a $\mathbf{Q}_p$-Banach algebra and R_0 a ring of definition for R. If M is a potentially orthonormalizable R-Banach module with R-Banach dual M', and M_0 is any open and bounded R_0-submodule of M, then the natural map $\mathrm{Hom}_{R_0}(M_0, R_0)[1/p] \to \mathrm{Hom}_R(M, R)$ factors through an isomorphism*

$$\mathrm{Hom}_{R_0}(M_0, R_0)[1/p] \simeq M',$$

and the topology on M' is the gauge topology defined by the R_0-submodule $\mathrm{Hom}_{R_0}(M_0, R_0)$.

Proof. We first set some notation. If I is a set we write $c(I, R)$ for the set of sequences $(r_i)_{i \in I}$ with $r_i \in R$ and such that for each $\varepsilon > 0$, $|r_i| < \varepsilon$ for all but finitely many i (cf. [70, Section 1]). We let $c(I, R_0)$ be those sequences with $r_i \in R_0$ for each i. Finally, we let $b(I, R)$ but those sequences r_i which are bounded. Note that $c(I, R)' \simeq b(I, R)$.

By definition, we can choose a R-Banach module isomorphism $f : c(I, R) \simeq M$ for some set I. Then $c(I, R_0) \subset c(I, R)$ is open and bounded, and $M_0 = f(c(I, R_0))$ is then an open and bounded R_0-submodule of M (boundedness is clear, and openness follows from the open mapping theorem). For this particular choice of M_0, the lemma follows by direct inspection, since f induces compatible isomorphisms $\mathrm{Hom}_{R_0}(M_0, R_0) \simeq \prod_I R_0$ and $M' \simeq b(I, R) \simeq (\prod_I R_0)[1/p]$. The case of a general M_0 then reduces to this special case upon noting that any two open bounded R_0-submodules $M_{0,1}, M_{0,2}$ satisfy $p^N M_{0,1} \subset M_{0,2} \subset p^{-N} M_{0,1}$ for $N \gg 0$. $\square$

Proof of Theorem 7.2.3. By Lemma 7.1.1 we have

$$\mathrm{Hom}_R(\mathscr{A}(\Gamma_F, R), R) \cong \varprojlim_{|\mathbf{s}| \to \infty} \mathrm{Hom}_R(\mathbf{A}^{\mathbf{s}}(\Gamma_F, R), R) \tag{7.2.4}$$

and

$$\mathscr{D}(\Gamma_F, R) = \varprojlim_{|\mathbf{s}| \to +\infty} \mathbf{D}^{\mathbf{s}}(\Gamma_F, R). \tag{7.2.5}$$

Choose now a ring of definition $R_0 \subset R$ containing the image of λ_Ω. By definition, R_0 is open and bounded in R and $\mathbf{A}^{\mathbf{s},\circ}(\Gamma_F, R) \subset \mathbf{A}^{\mathbf{s}}(\Gamma_F, R)$ is also open and bounded. Furthermore, $\mathbf{A}^{\mathbf{s}}(\Gamma_F, R)$ is potentially orthonormalizable for each $\mathbf{s}$ since it is the completed scalar extension of a $\mathbf{Q}_p$-Banach space, which is always potentially orthornormalizable (see [70, Proposition 1] and [26, Lemma 2.8]). Thus, Lemma 7.2.5, together with (7.2.4) and (7.2.5), implies that

$$\mathscr{D}(\Gamma_F, R) \simeq \varprojlim_{|\mathbf{s}| \to +\infty} \mathrm{Hom}_{R_0}(\mathbf{A}^{\mathbf{s},\circ}(\Gamma_F, R), R_0)[1/p] \subset \mathrm{Hom}_R(\mathscr{A}(\Gamma_F, R), R). \tag{7.2.6}$$

Now consider the commuting diagram

$$\begin{tikzcd}
\mathscr{A}(\Gamma_F, R) \arrow[r, "\mathscr{Q}_\Omega"] & H_d^{\mathrm{BM}}(Y_1(\mathfrak{m}), \mathscr{A}_\Omega) \\
\mathbf{A}^{\mathbf{s}}(\Gamma_F, R) \arrow[r, "\mathscr{Q}_\Omega^{\mathbf{s}}"] \arrow[u] & H_d^{\mathrm{BM}}(Y_1(\mathfrak{m}), \mathbf{A}_\Omega^{\mathbf{s}}) \arrow[u] \\
\mathbf{A}^{\mathbf{s},\circ}(\Gamma_F, R) \arrow[r, "\mathscr{Q}_\Omega^{\mathbf{s},\circ}"] \arrow[u] & H_d^{\mathrm{BM}}(Y_1(\mathfrak{m}), \mathbf{A}_\Omega^{\mathbf{s},\circ}). \arrow[u]
\end{tikzcd}$$

Since $\mathbf{D}_\Omega^{\mathbf{s}}$ is the R-Banach dual of $\mathbf{A}_\Omega^{\mathbf{s}}$ and $\mathbf{D}_\Omega^{\mathbf{s},\circ} \subset \mathbf{D}_\Omega^{\mathbf{s}}$ is the R_0-linear dual of $\mathbf{A}_\Omega^{\mathbf{s},\circ}$ (similarly for Γ_F), the naturality of the pairings $\langle -, - \rangle$ implies that

$$\begin{tikzcd}
H_c^d(Y_1(\mathfrak{m}), \mathscr{D}_\Omega) \arrow[r, "\mathscr{P}_\Omega"] \arrow[d] & \mathrm{Hom}_R(\mathscr{A}(\Gamma_F, R), R) \arrow[d] \\
H_c^d(Y_1(\mathfrak{m}), \mathbf{D}_\Omega^{\mathbf{s}}) \arrow[r, "\mathscr{P}_\Omega^{\mathbf{s}}"] & \mathrm{Hom}_R(\mathbf{A}^{\mathbf{s}}(\Gamma_F, R), R) \\
H_c^d(Y_1(\mathfrak{m}), \mathbf{D}_\Omega^{\mathbf{s},\circ}) \arrow[r, "\mathscr{P}_\Omega^{\mathbf{s},\circ}"] \arrow[u] & \mathrm{Hom}_{R_0}(\mathbf{A}^{\mathbf{s},\circ}(\Gamma_F, R), R_0) \arrow[u]
\end{tikzcd} \tag{7.2.7}$$

is also a commuting diagram.

Finally, consider $\Psi \in H_c^d(Y_1(\mathfrak{m}), \mathscr{D}_\Omega)$ and write $\Psi^{\mathbf{s}} \in H_c^d(Y_1(\mathfrak{m}), \mathbf{D}_\Omega^{\mathbf{s}})$ for its restriction to $\mathbf{D}_\Omega^{\mathbf{s}}$. Since sheaf cohomology commutes with flat scalar extension in the coefficients, and $\mathbf{D}_\Omega^{\mathbf{s},\circ}[1/p] = \mathbf{D}_\Omega^{\mathbf{s}}$, the bottom left vertical arrow in (7.2.7) is an isomorphism after inverting p. Following the diagram (7.2.7) around, we deduce that

$$\mathscr{P}_\Omega^{\mathbf{s}}(\Psi^{\mathbf{s}}) \in \mathrm{Hom}_{R_0}(\mathbf{A}^{\mathbf{s},\circ}(\Gamma_F, R), R_0)[1/p] \subset \mathrm{Hom}_R(\mathbf{A}^{\mathbf{s}}(\Gamma_F, R), R).$$

Since $\mathbf{s}$ is arbitrary, (7.2.6) shows that $\mathscr{P}_\Omega(\Psi) \in \mathscr{D}(\Gamma_F, R)$ by (7.2.6). $\square$

7.3. **Compatibilities.** In this brief subsection we catalog some straightforward features of the period maps. We let $\mathfrak{m} \subset \mathbf{p}$ be an integral ideal and we generally let $\Omega = \mathrm{Sp}(R) \to \mathscr{W}$ be an affinoid point of weight space.

Lemma 7.3.1. *If $\Omega \to \mathscr{W}$ factors through $\mathrm{Sp}(R') = \Omega'$ then the natural diagram*

$$\begin{tikzcd}
H_c^d(Y_1(\mathfrak{m}), \mathscr{D}_\Omega) \arrow[r, "\mathscr{P}_\Omega"] & \mathscr{D}(\Gamma_F, R) \\
H_c^d(Y_1(\mathfrak{m}), \mathscr{D}_{\Omega'}) \arrow[r, "\mathscr{P}_{\Omega'}"'] \arrow[u] & \mathscr{D}(\Gamma_F, R') \arrow[u]
\end{tikzcd}$$

is commutative.

Proof. This is clear (see Remark 7.2.2). $\square$

Lemma 7.3.2. *If $\mathfrak{m}' \subset \mathfrak{m}$ and $\mathrm{pr} : Y_1(\mathfrak{m}') \to Y_1(\mathfrak{m})$ is the projection map, then $\mathscr{P}_\Omega(\Psi) = \mathscr{P}_\Omega(\mathrm{pr}^* \Psi)$ for all $\Psi \in H_c^d(Y_1(\mathfrak{m}), \mathscr{D}_\Omega)$.*

Proof. Temporarily denote $\mathscr{P}_\Omega^{\mathfrak{m}}$ and $\mathscr{Q}_\Omega^{\mathfrak{m}}$ for the maps defined above with the level specified. We want to show $\mathscr{P}_\Omega^{\mathfrak{m}}(\Psi) = \mathscr{P}_\Omega^{\mathfrak{m}'}(\mathrm{pr}^* \Psi)$ for all $\Psi \in H_c^d(Y_1(\mathfrak{m}), \mathscr{D}_\Omega)$.

What is clear is that pr is compatible with the two possible embeddings t. So, it follows from the definition (7.2.2) that $\mathrm{pr}_*(\mathscr{Q}_\Omega^{\mathfrak{m}'}(f)) = \mathscr{Q}_\Omega^{\mathfrak{m}}(f)$ for all $f \in \mathscr{A}(\Gamma_F, R)$. And now if $f \in \mathscr{A}(\Gamma_F, R)$ and $\Psi \in H_c^d(Y_1(\mathfrak{m}), \mathscr{D}_\Omega)$ then we see that

$$\langle \mathrm{pr}^* \Psi, \mathscr{Q}_\Omega^{\mathfrak{m}'}(f)\rangle = \langle \Psi, \mathrm{pr}_*(\mathscr{Q}_\Omega^{\mathfrak{m}'}(f))\rangle = \langle \Psi, \mathscr{Q}_\Omega^{\mathfrak{m}}(f)\rangle.$$

This proves the lemma. $\square$

We also note the following truly tautological relationship between the period map and the Amice transform (Proposition 5.1.6).

Lemma 7.3.3. *If $\chi : \Gamma_F \to R^\times$ is a continuous character and $\Psi \in H_c^d(Y_1(\mathfrak{m}), \mathscr{D}_\Omega)$, then*

$$\mathscr{P}_\Omega(\Psi)(\chi) = \mathcal{A}_{\mathscr{P}_\Omega(\Psi)}(\chi). \tag{7.3.1}$$

Finally, it will be helpful to note the interaction between the period map and the Archimedean Hecke operators (a more involved calculation with the U_v-operators is the subject of Section 7.6 below).

Proposition 7.3.4. *Let $\Psi \in H_c^d(Y_1(\mathfrak{m}), \mathscr{D}_\Omega)$. Then,*

(1) *If $\chi : \Gamma_F \to R^\times$ is a continuous character and $\zeta \in \pi_0(F_\infty^\times)$, then $\mathscr{P}_\Omega(T_\zeta \Psi)(\chi) = \chi(\zeta)\mathscr{P}_\Omega(\Psi)(\chi)$.*
(2) *If $\varepsilon \in \{\pm 1\}^{\Sigma_F}$ and $\Psi \in H_c^d(Y_1(\mathfrak{m}), \mathscr{D}_\Omega)^\epsilon$ then $\mathscr{P}_\Omega(\Psi)(\chi) = 0$ unless $\chi(\zeta) = \epsilon(\zeta)$ for all $\zeta \in \pi_0(F_\infty^\times)$.*

Proof. $\mathscr{P}_\Omega(\Psi)$ is linear in Ψ. In particular, part (2) clearly follows from part (1). To prove (1), we set some notation. Write $\rho_\zeta : Y_1(\mathfrak{m}) \to Y_1(\mathfrak{m})$ for right multiplication by $\begin{pmatrix} \zeta & \\ & 1 \end{pmatrix}$, so T_ζ is the pullback ρ_ζ^*. On the other hand, write $r_\zeta : \mathrm{C}_\infty \to \mathrm{C}_\infty$ for right multiplication by ζ.

It is trivial to check from the definition in Lemma 7.2.1 that $r_\zeta^*(Q_\Omega(\chi)) = \chi(\zeta)Q_\Omega(\chi)$. Since $(r_\zeta)_* \circ \mathrm{PD} \circ r_\zeta^* = \mathrm{PD}$ (see Proposition 2.3.1 and (2.1.7)) we deduce that

$$((r_\zeta)_* \circ \mathrm{PD})(Q_\Omega(\chi)) = \chi(\zeta)\,\mathrm{PD}(Q_\Omega(\chi)). \tag{7.3.2}$$

But $\mathscr{Q}_\Omega = \mathrm{t}_* \circ \mathrm{PD} \circ Q_\Omega$, and $(\rho_\zeta)_* \circ \mathrm{t}_* = \mathrm{t}_* \circ (r_\zeta)_*$, so we get

$$\mathscr{P}_\Omega(T_\zeta\Psi)(\chi) = \langle \rho_\zeta^*\Psi, \mathscr{Q}_\Omega(\chi)\rangle = \langle \Psi, (\rho_\zeta)_*\mathscr{Q}_\Omega(\chi)\rangle = \chi(\zeta)\langle \Psi, \mathscr{Q}_\Omega(\chi)\rangle,$$

as we promised in part (1). $\square$

Remark 7.3.5. If $\epsilon \in \{\pm 1\}^{\Sigma_F}$ then write $\mathscr{X}(\Gamma_F)^\epsilon$ for those characters χ on Γ_F such that $\chi(\zeta) = \epsilon(\zeta)$ for all $\zeta \in \pi_0(F_\infty^\times)$. Then, $\mathscr{X}(\Gamma_F)$ is a disjoint union

$$\mathscr{X}(\Gamma_F) = \bigcup_\epsilon \mathscr{X}(\Gamma_F)^\epsilon$$

and so $\mathscr{O}(\mathscr{X}(\Gamma_F)) = \bigoplus_\epsilon \mathscr{O}(\mathscr{X}(\Gamma_F)^\epsilon)$. The previous two lemmas say that $\mathcal{A} \circ \mathscr{P}_\Omega$ respects the direct sum decompositions in the following diagram

$$\begin{tikzcd}
H_c^d(Y_1(\mathfrak{m}), \mathscr{D}_\Omega) \arrow[r, "\mathscr{P}_\Omega"] \arrow[d, equal] & \mathscr{D}(\Gamma_F, R) \arrow[r, "\mathcal{A}"] & \mathscr{O}(\mathscr{X}(\Gamma_F))\widehat{\otimes}_{\mathbf{Q}_p} R \arrow[d, equal] \\
\bigoplus_\epsilon H_c^d(Y_1(\mathfrak{m}), \mathscr{D}_\Omega)^\epsilon \arrow[rr] & & \bigoplus_\epsilon \mathscr{O}(\mathscr{X}(\Gamma_F)^\epsilon)\widehat{\otimes}_{\mathbf{Q}_p} R.
\end{tikzcd}$$

7.4. **Growth properties.** In this subsection we analyze the growth properties of our period morphisms $\mathscr{P}_\lambda$ (over a field). If $L/\mathbf{Q}_p$ is a finite extension then we always take the ring of integers $\mathcal{O}_L \subset L$ to be a ring of definition. We also fix an integral ideal $\mathfrak{m} \subset \mathbf{p}$ as in the previous subsection.

Definition 7.4.1. Let $L/\mathbf{Q}_p$ be a finite extension and $h \geq 0$. If $\mu \in \mathscr{D}(\Gamma_F, L)$, then we say that μ has growth *of order* $\leq h$ if

$$\sup_{\mathbf{s}} \left(\sup_{f \in A^{\mathbf{s},\circ}(\Gamma_F, L)} p^{-|\mathbf{s}|h} |\mu(f)| \right) < +\infty.$$

Proposition 7.4.2. *If $\Psi \in H^d_c(Y_1(\mathfrak{m}), \mathscr{D}_\lambda \otimes_{k_\lambda} L)_{\leq h}$ then $\mathscr{P}_\lambda(\Psi)$ is a distribution with growth of order $\leq h$.*

Proof. By Lemma 7.3.2 and Lemma 2.3.3 we may assume that $Y_1(\mathfrak{m})$ is a manifold (compare with the proof of Lemma 6.5.4).

With h fixed, this means that for some $\mathbf{s}_0$, the slope-$\leq h$ part

$$H^d_c(Y_1(\mathfrak{m}), \mathscr{D}_\lambda \otimes_{k_\lambda} L)_{\leq h} \simeq H^d_c(Y_1(\mathfrak{m}), \mathbf{D}^{\mathbf{s}_0}_\lambda \otimes_{k_\lambda} L)_{\leq h}$$

is equal to the slope-$\leq h$ part of the d-th cohomology of a Borel–Serre complex

$$C^\bullet_c(\mathbf{D}^{\mathbf{s}_0}_\lambda \otimes_{k_\lambda} L) \simeq C^\bullet_c(\mathbf{D}^{\mathbf{s}_0}_\lambda \otimes_{k_\lambda} L)_{\leq h} \oplus C^\bullet_c(\mathbf{D}^{\mathbf{s}_0}_\lambda \otimes_{k_\lambda} L)_{>h}.$$

The terms which make up the complex $C^\bullet_c(\mathbf{D}^{\mathbf{s}_0}_\lambda \otimes_{k_\lambda} L)$ are finite direct sums of the Banach space $D^{\mathbf{s}_0}_\lambda \otimes_{k_\lambda} L$. Thus, the family of operators $\{p^{|\mathbf{s}|h} U_p^{-|\mathbf{s}|}\}$ on $C^\bullet_c(\mathbf{D}^{\mathbf{s}_0}_\lambda \otimes_{k_\lambda} L)_{\leq h}$ is a family whose operator norms are bounded independent of $\mathbf{s}$.

Now choose $\Psi \in H^d_c(Y_1(\mathfrak{m}), \mathscr{D}_\lambda \otimes_{k_\lambda} L)_{\leq h}$, $\mathbf{s}_0$ as in the previous paragraph and write $\Psi^{\mathbf{s}_0}$ in $H^d_c(Y_1(\mathfrak{m}), \mathbf{D}^{\mathbf{s}_0}_\lambda \otimes_{k_\lambda} L)_{\leq h}$ for the restriction of Ψ to radius $\mathbf{s}_0$. Given $\mathbf{s}$, write

$$\Psi_{\mathbf{s}} := (p^h U_p^{-1})^{|\mathbf{s}|}(\Psi^{\mathbf{s}_0}) \in H^d_c(Y_1(\mathfrak{m}), \mathbf{D}^{\mathbf{s}_0}_\lambda \otimes_{k_\lambda} L).$$

By the boundedness discussion in the previous paragraph, we may choose a single $C > 0$ such that $p^C \Psi_{\mathbf{s}} \in H^d_c(Y_1(\mathfrak{m}), \mathbf{D}^{\mathbf{s}_0,\circ}_\lambda \otimes_{k^\circ_\lambda} \mathcal{O}_L)$ for all $\mathbf{s} \geq \mathbf{s}_0$. Here we are using the reduction in the first sentence of this proof so that $H^d_c(Y_1(\mathfrak{m}), \mathbf{D}^{\mathbf{s}_0,\circ}_\lambda \otimes_{\mathcal{O}_\lambda} \mathcal{O}_L)$ is the cohomology in degree d of the bounded sub-complex $C^\bullet_c(\mathbf{D}^{\mathbf{s}_0,\circ}_\lambda \otimes_{k^\circ_\lambda} \mathcal{O}_L) \subset C^\bullet_c(\mathbf{D}^{\mathbf{s}_0}_\lambda \otimes_{k_\lambda} L)$.

Now let $\mathbf{s} \geq \mathbf{s}_0$ and $f \in \mathbf{A}^{\mathbf{s},\circ}(\Gamma_F, L)$. Then we compute

$$\mathscr{P}_\lambda(\Psi)(f) = \mathscr{P}^{\mathbf{s}_0}_\lambda(\Psi^{\mathbf{s}_0})(f) = p^{-C} p^{-|\mathbf{s}|h} \mathscr{P}^{\mathbf{s}_0}_\lambda(U_p^{|\mathbf{s}|} p^C \Psi_{\mathbf{s}})(f). \tag{7.4.1}$$

Now note that the Hecke operator U_p is self-adjoint under $\langle -, - \rangle$, and so

$$\mathscr{P}^{\mathbf{s}_0}_\lambda(U_p^{|\mathbf{s}|} p^C \Psi_{\mathbf{s}})(f) = \langle U_p^{|\mathbf{s}|} p^C \Psi_{\mathbf{s}}, \mathscr{Q}^{\mathbf{s},\circ}_\lambda(f) \rangle = \langle p^C \Psi_{\mathbf{s}}, U_p^{|\mathbf{s}|} \mathscr{Q}^{\mathbf{s},\circ}_\lambda(f) \rangle. \tag{7.4.2}$$

Since (7.4.2) is the pairing between the element $U_p^{|\mathbf{s}|} \mathscr{Q}^{\mathbf{s},\circ}_\lambda(f) \in H_d^{\mathrm{BM}}(Y_1(\mathfrak{m}), \mathbf{A}^{\mathbf{s},\circ} \otimes_{k^\circ_\lambda} \mathcal{O}_L)$ and the image of $p^C \Psi_{\mathbf{s}}$ in $H^d_c(Y_1(\mathfrak{m}), \mathbf{D}^{\mathbf{s},\circ} \otimes_{k^\circ_\lambda} \mathcal{O}_L)$, it is necessarily an element of $\mathcal{O}_L$. And so (7.4.1) shows that

$$\left| p^{-|\mathbf{s}|h} \mathscr{P}_\lambda(\Psi)(f) \right| < p^C,$$

independent of $\mathbf{s}$ and f, completing the proof. ∎

7.5. **The p-adic evaluation class.** In this subsection we consider $L \subset \overline{\mathbf{Q}}_p$ finite over $\mathbf{Q}_p$ and containing the Galois closure of F. We also use $\lambda = (\kappa, w)$ to denote a cohomological weight, which we view as a p-adic weight as in Section 5.4.

Definition 7.5.1. If m is an integer critical with respect to λ, then we define $\delta^\star_{m,p} \in \mathscr{L}_\lambda(L)^\vee$ by

$$\delta^\star_{m,p}(X^j) = \begin{cases} \binom{\kappa}{j}^{-1} & \text{if } j = \frac{\kappa+w}{2} - m, \\ 0 & \text{otherwise.} \end{cases}$$

Now write $\mathbf{N}_p : \mathbf{A}_F^\times \to L^\times$ for the p-adic realization of the adelic norm $|\cdot|_{\mathbf{A}_F}$ via ι. That is, $\mathbf{N}_p$ is given by the following formula

$$\mathbf{N}_p(x) = |x_f|_{\mathbf{A}_F} \left(\prod_{v|\infty} \operatorname{sgn}(x_v)\right) \cdot \left(\prod_{v|p} \prod_{\sigma\in\Sigma_v} \sigma(x_v)\right). \tag{7.5.1}$$

The character $\mathbf{N}_p$ is the adelic version of the cyclotomic character on Γ_F, but we also write $\mathbf{N}_p$ for the induced element of $\mathscr{X}(\Gamma_F)$. We also consider the local system $\mathrm{t}^*\mathscr{L}_\lambda(L)^\vee$ on C_∞ corresponding to the right $\mathcal{O}_p^\times$-module structure on $\mathscr{L}_\lambda(L)^\vee$ gotten by restricting to $\begin{pmatrix} \mathcal{O}_p^\times & \\ & 1 \end{pmatrix} \hookrightarrow \mathrm{GL}_2(F_p)$.

Lemma 7.5.2. *If $x_p \in F_p^\times$ then $\delta^\star_{m,p}|_{\begin{pmatrix} x_p & \\ & 1\end{pmatrix}} = \left(\prod_{v|p}\prod_{\sigma\in\Sigma_v}\sigma(x_v)\right)^m \cdot \delta^\star_{m,p}$. Thus,*

1. *If $x_p \in \mathcal{O}_p^\times$ then $\delta^\star_{m,p}|_{\begin{pmatrix} x_p & \\ & 1\end{pmatrix}} = \mathbf{N}_p^m(x_p)\delta^\star_{m,p}$.*
2. *The formula $\delta_{m,p}(x) = \mathbf{N}_p^m(x)\delta^\star_{m,p}$ defines an element of $H^0(\mathrm{C}_\infty, \mathrm{t}^*\mathscr{L}_\lambda(L)^\vee)$.*

Proof. By definition,

$$\delta^\star_{m,p}|_{\begin{pmatrix} x_p & \\ & 1\end{pmatrix}}(X^j) = \prod_{v|p}\prod_{\sigma\in\Sigma_v} \sigma(x_v)^{\frac{w-\kappa_\sigma}{2}}\sigma(x_v)^{\kappa_\sigma}\delta^\star_{m,p}\left((X_\sigma/\sigma(x_v))^{j_\sigma}\right).$$

The final term in the product is only non-zero if $j = \frac{\kappa+w}{2} - m$, in which case what we get is $\delta^\star_{m,p}(X^j)$ times the coefficient

$$\prod_{v|p}\prod_{\sigma\in\Sigma_v}\sigma(x_v)^{\frac{w-\kappa_\sigma}{2}}\sigma(x_v)^{\kappa_\sigma}\sigma(x_v)^{m-\frac{\kappa_\sigma+w}{2}} = \prod_{v|p}\prod_{\sigma\in\Sigma_v}\sigma(x_v)^m.$$

This completes the proof point (1). To prove point (2) we first note that $\mathbf{N}_p$ is locally constant on $F_\infty^\times$ and thus to check $\delta_{m,p}$ actually defines a section we need to check that $\delta_{m,p}(\xi x u) = \delta_{m,p}(x)|_{u_p}$ if $\xi \in F^\times$, $x \in \mathbf{A}_F^\times$ and $u \in \widehat{\mathcal{O}}_F^\times$. But that follows immediately from point (1). ☐

Recall from Section 5.4 that we have the dual integration map $I_\lambda^\vee : \mathscr{L}_\lambda(L)^\vee \to \mathscr{A}_\lambda \otimes_{k_\lambda} L$.

Lemma 7.5.3. *If m is an integer critical with respect to λ, then*

$$I_\lambda^\vee(\delta^\star_{m,p}) = \prod_{v|p}\prod_{\sigma\in\Sigma_v}\sigma(-)^{\frac{\kappa_\sigma-w}{2}+m}.$$

In particular, if $z \in \mathcal{O}_p^\times$ then $I_\lambda^\vee(\delta^\star_{m,p})(z) = \mathbf{N}_p^m(z)\lambda_2^{-1}(z)$.

Proof. Recall that $\delta^\star_{m,p}(X^j)$ is zero except if $j = \frac{\kappa+w}{2} - m$, in which case it takes the value $\binom{\kappa}{j}^{-1}$. Thus, if $\mu \in \mathscr{D}_\lambda(L)$, then

$$\mu\left(I_\lambda^\vee(\delta^\star_{m,p})\right) = \delta^\star_{m,p}\left(I_\lambda(\mu)\right) = \delta^\star_{m,p}\left(\sum_j \binom{\kappa}{j}\mu(z^j)X^{\kappa-j}\right) = \mu\left(z^{\frac{\kappa-w}{2}+m}\right)$$

Since μ is arbitrary, we are finished. $\square$

It convenient here to calculate the interaction between $\delta_{m,p}$ and Q_λ as defined in Section 7.2.

Lemma 7.5.4. *Let m be an integer critical with respect to λ.*

(1) If $x \in \mathbf{A}_F^\times$, then $Q_\lambda(\mathbf{N}_p^m)(x)|_{\mathcal{O}_p^\times} = I_\lambda^\vee(\delta_{m,p}(x))|_{\mathcal{O}_p^\times}$.

(2) If $f = (f_v) \in \mathbf{Z}_{\geq 1}^{\{v|p\}}$ and $a \in \mathcal{O}_p^\times$, then $Q_\lambda(\mathbf{N}_p^m)(x)\big| \begin{pmatrix} \varpi_p^f & a \\ & 1 \end{pmatrix} = I_\lambda^\vee(\delta_{m,p}(x))\big| \begin{pmatrix} \varpi_p^f & a \\ & 1 \end{pmatrix}$.

Proof. (2) follows from (1) because if $a \in \mathcal{O}_p^\times$ and $f_v \geq 1$ for all $v \mid p$, then $a + \varpi_p^f z \in \mathcal{O}_p^\times$ for all $z \in \mathcal{O}_p$. It remains to prove (1). By definition, in Lemma 7.5.2, $\delta_{m,p}(x) = \mathbf{N}_p^m(x)\delta_{m,p}^\star$. Let $u \in \mathcal{O}_p^\times$. By Lemma 7.5.3, we have $I_\lambda^\vee(\delta_{m,p}^\star)(u) = \mathbf{N}_p(u)^m\lambda_2^{-1}(u)$. Thus, $I_\lambda^\vee(\delta_{m,p}(x))(u) = \mathbf{N}_p^m(x)\mathbf{N}_p^m(u)\lambda_2^{-1}(u)$. Since $\mathbf{N}_p(-)$ is multiplicitive and $u \in \mathcal{O}_p^\times$, this is also the value of $Q_\lambda(\mathbf{N}_p^m)(x)(u)$ (see (7.2.1)). $\square$

In analogy with Definition 4.4.6 we make the following definition.

Definition 7.5.5. If $K \subset \mathrm{GL}_2(\mathbf{A}_{F,f})$ is a t-good subgroup, then we define $\mathrm{cl}_p(m) := \mathrm{t}_*(\mathrm{PD}(\delta_{m,p})) \in H_d^{\mathrm{BM}}(Y_K, \mathscr{L}_\lambda(L)^\vee)$ where $\delta_{m,p}$ is as in Lemma 7.5.2.

This p-adic evaluation class is completely analogous to the Archimedean one previously defined in Definition 4.4.6 and Definition 7.5.5. Namely, suppose that $E \subset \mathbf{C}$ is a subfield containing the Galois closure of F in $\mathbf{C}$ and let $L = \mathbf{Q}_p(\iota(E))$. Then for any compact open subgroup $K \subset \mathrm{GL}_2(\mathbf{A}_{F,f})$ containing $\begin{pmatrix} \widehat{\mathcal{O}}_F^\times & \\ & 1 \end{pmatrix}$ we have a natural commuting diagram

$$\begin{array}{ccc} H_d^{\mathrm{BM}}(Y_K, \mathscr{L}_\lambda(E)^\vee) & \xrightarrow[\simeq]{\iota} & H_d^{\mathrm{BM}}(Y_K, \mathscr{L}_\lambda(L)^\vee) \\ \uparrow{\scriptstyle \mathrm{t}_*} & & \uparrow{\scriptstyle \mathrm{t}_*} \\ H_d^{\mathrm{BM}}(\mathrm{C}_\infty, \mathrm{t}^*\mathscr{L}_\lambda(E)^\vee) & \xrightarrow[\simeq]{\iota} & H_d^{\mathrm{BM}}(\mathrm{C}_\infty, \mathrm{t}^*\mathscr{L}_\lambda(L)^\vee) \\ \uparrow{\scriptstyle \mathrm{PD}} & & \uparrow{\scriptstyle \mathrm{PD}} \\ H^0(\mathrm{C}_\infty, \mathrm{t}^*\mathscr{L}_\lambda(E)^\vee) & \xrightarrow[\simeq]{\iota} & H^0(\mathrm{C}_\infty, \mathrm{t}^*\mathscr{L}_\lambda(L)^\vee) \end{array} \tag{7.5.2}$$

The horizontal maps are all isomorphisms as indicated (Proposition 2.2.2).

Proposition 7.5.6. *If m is an integral critical with respect to λ, then $\iota(\mathrm{cl}_\infty(m)) = \mathrm{cl}_p(m)$.*

Proof. By (7.5.2) and the definitions it is enough to check that $\iota(\delta_m) = \delta_{m,p}$ (where δ_m is as in Proposition 4.4.5 and $\delta_{m,p}$ is as in Lemma 7.5.2).

To be clear, by the construction in Proposition 2.2.2, $\iota(\delta_m)$ is the section $x \mapsto \iota(\delta_m(x))|_{x_p}$ where the ι on the right-hand side is the natural way of turning an element of $\mathscr{L}_\lambda(E)^\vee$ into an element of $\mathscr{L}_\lambda(L)^\vee$ via scalar extension along ι. In particular, $\iota(\delta_m^\star) = \delta_{m,p}^\star$. Thus we can compute

$$\iota(\delta_m(x)) = \iota((|x_f|_{\mathbf{A}_F} \prod_{v|\infty} \mathrm{sgn}(x_v))^m \delta_m^\star) = (\mathbf{N}_p(x) \prod_{v|p} \prod_{\sigma \in \Sigma_v} \sigma(x_v)^{-1})^m \delta_{m,p}^\star$$

(compare with (7.5.1)). And now Lemma 7.5.2 tells us that $\iota(\delta_m(x))|_{x_p} = \mathbf{N}_p(x)\delta_{m,p}^\star = \delta_{m,p}(x)$. This completes the proof. $\square$

We will finally make a computation regarding the p-adic evaluation class that is used later in Corollary 7.6.7. (One could also give an analogous Archimedean computation and use Proposition 7.5.6.)

Let $v \mid p$ and denote $V_v^+ = \left(\begin{smallmatrix} \varpi_v & \\ & 1 \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbf{A}_{F,f})$. Suppose that we fix a t-good subgroup K. Write $K_{\varpi_v} := K \cap V_v^+ K (V_v^+)^{-1}$ and similarly $K_{\varpi_v^{-1}} = K \cap (V_v^+)^{-1} K V_v^+$. Then right multiplication by V_v^+ induces a map $V_v^+ : Y_{K_{\varpi_v}} \to Y_{K_{\varpi_v^{-1}}}$ that lifts to a map of local systems $\mathscr{L}_\lambda(L)^\vee \to \mathscr{L}_\lambda(L)^\vee$ given by $\delta \mapsto \delta|_{V_v^+}$. More precisely we are considering the composition of two maps on the level of local systems. The first is the map on the base given by V_v^+ and the identity map on the local system where $K_{\varpi_v^{-1}}$ acts on $\mathscr{L}_\lambda(L)$ by the twisted action $\mathscr{L}_\lambda(L)((V_v^+)^{-1})$ of $(V_v^+)^{-1} K V_v^+$. The second map is the identity on the base $Y_{K_{\varpi_v^{-1}}}$ and the right translation on the level of local systems. (Compare with (2.2.5).)

In any case, we thus have a pushforward map

$$(V_v^+)_* : H_d^{\mathrm{BM}}(Y_{K_{\varpi_v}}, \mathscr{L}_\lambda^\vee(L)) \to H_d^{\mathrm{BM}}(Y_{K_{\varpi_v^{-1}}}, \mathscr{L}_\lambda^\vee(L)). \tag{7.5.3}$$

Note that both K_{ϖ_v} and $K_{\varpi_v^{-1}}$ are still t-good because K is. Thus there is a p-adic evaluation class $\mathrm{cl}_p(m)$ on either side of (7.5.3).

Lemma 7.5.7. *$(V_v^+)_* \,\mathrm{cl}_p(m) = q_v^m \,\mathrm{cl}_p(m)$.*

Proof. Consider the diagram

$$\begin{tikzcd} H_d^{\mathrm{BM}}(Y_{K_{\varpi_v}}, \mathscr{L}_\lambda(L)^\vee) \arrow[r, "(V_v^+)_*"] & H_d^{\mathrm{BM}}(Y_{K_{\varpi_v^{-1}}}, \mathscr{L}_\lambda(L)^\vee) \\ H_d^{\mathrm{BM}}(\mathrm{C}_\infty, \mathrm{t}^*\mathscr{L}_\lambda(L)^\vee) \arrow[u, "\mathrm{t}_*"] \arrow[r, "(r_{\varpi_v})_*"] & H_d^{\mathrm{BM}}(\mathrm{C}_\infty, \mathrm{t}^*\mathscr{L}_\lambda(L)^\vee) \arrow[u, "\mathrm{t}_*"'] \\ H^0(\mathrm{C}_\infty, \mathrm{t}^*\mathscr{L}_\lambda(L)^\vee) \arrow[u, "\mathrm{PD}"] & H^0(\mathrm{C}_\infty, \mathrm{t}^*\mathscr{L}_\lambda(L)^\vee). \arrow[l, "r_{\varpi_v}^*"'] \arrow[u, "\mathrm{PD}"'] \end{tikzcd} \tag{7.5.4}$$

Here we write r_{ϖ_v} for the map on C_∞ which is right multiplication by ϖ_v and with a non-trivial action to the level of local systems as above. The pullback map $r_{\varpi_v}^*$ is the map given by $(r_{\varpi_v}^* s)(x) = s(x\varpi_v)|_{(V_v^+)^{-1}}$ for all $s \in H^0(\mathrm{C}_\infty, \mathrm{t}^*\mathscr{L}_\lambda(L)^\vee)$ and $x \in \mathbf{A}_F^\times$. Taking $s = \delta_{m,p}$ we get

$$r_{\varpi_v}^*(\delta_{m,p})(x) = \delta_{m,p}(x\varpi_v)|_{(V_v^+)^{-1}} = \mathbf{N}_p^m(x\varpi_v)\delta_{m,p}^\star|_{(V_v^+)^{-1}} = |\varpi_v|_{\mathbf{A}_F}^m \mathbf{N}_p^m(x)\delta_{m,p}^\star = q_v^{-m}\delta_{m,p}(x).$$

(The third equality used (7.5.1) and Lemma 7.5.2.) Thus, $r_{\varpi_v}^*\delta_{m,p} = q_v^{-m}\delta_{m,p}$. The conclusion now follows from Proposition 2.3.1 and (2.1.7). $\square$

7.6. **Abstract interpolation.** The main result in this subsection (Theorem 7.6.4 below) is an "abstract" equality of functionals on a certain overconvergent cohomology group. It relates the Hecke action at p to the p-adic evaluation classes via the period maps.

In the remainder of this section we fix a finite order Hecke character θ of conductor $\mathfrak{f} = \prod_v \mathfrak{p}_v^{f_v}$ where $f_v = 0$ if $v \nmid p$. We write θ^ι for $\iota \circ \theta$, which is thus a $\overline{\mathbf{Q}}_p$-valued Hecke character. We also fix a field $L \subset \overline{\mathbf{Q}}_p$ containing the Galois closure of F in $\overline{\mathbf{Q}}_p$ and the values of θ^ι. Thus θ^ι is an element of $\mathscr{A}(\Gamma_F, L)$. Set $f_{+,v} = \max(f_v, 1)$ and let $f_+ = (f_{+,v}) \in \mathbf{Z}_{\geq 1}^{\{v|p\}}$. We also fix a cohomological weight λ.

Recall the definition of

$$Q_\lambda : \mathscr{A}(\Gamma_F, L) \to H^0(\mathrm{C}_\infty, \mathrm{t}^*\mathscr{A}_\lambda \otimes_{k_\lambda} L)$$

from Lemma 7.2.1. In particular, if $g \in \mathscr{A}(\Gamma_F, L)$ and $x \in \mathbf{A}_F^\times$ then $Q_\lambda(g)(x)$ is an element of $\mathscr{A}_\lambda \otimes_{k_\lambda} L$ and $\mathscr{A}_\lambda \otimes_{k_\lambda} L$ has a right action of Δ.

Lemma 7.6.1. *If $a \in \mathcal{O}_p$, $g \in \mathscr{A}(\Gamma_F, L)$ and $x \in \mathbf{A}_F^\times$ then*

$$Q_\lambda(g\theta^\iota)(x)\big| \begin{pmatrix} \varpi_p^{f_+} & a \\ & 1 \end{pmatrix} = \begin{cases} \theta^\iota(ax)\cdot Q_\lambda(g)(x)\big| \begin{pmatrix} \varpi_p^{f_+} & a \\ & 1 \end{pmatrix} & \text{if } a \in \mathcal{O}_p^\times, \\ 0 & \text{if } a \notin \mathcal{O}_p^\times. \end{cases}$$

Proof. If $z \in \mathcal{O}_p$, then $a + \varpi_p^{f_+} z \in \mathcal{O}_p^\times$ if and only if $a \in \mathcal{O}_p^\times$. Thus, by (5.3.1) and the definition of Q_λ we deduce

$$(7.6.1)\quad Q_\lambda(g\theta^\iota)(x)\big| \begin{pmatrix} \varpi_p^{f_+} & a \\ & 1 \end{pmatrix}(z)$$
$$= Q_\lambda(g\theta^\iota)(x)(a + z\varpi_p^{f_+}) = \begin{cases} (g\theta^\iota)(x(a+\varpi_p^{f_+}z))\lambda_2^{-1}(a+z\varpi_p^{f_+}) & \text{if } a \in \mathcal{O}_p^\times, \\ 0 & \text{if } a \notin \mathcal{O}_p^\times. \end{cases}$$

This already proves the case $a \notin \mathcal{O}_p^\times$. When $a \in \mathcal{O}_p^\times$, $\theta^\iota(a + \varpi_p^{f_+} z) = \theta^\iota(a)$ by definition of the conductor of θ and thus the case $a \in \mathcal{O}_p^\times$ follows from multiplicativity of θ. $\square$

We now fix further notation. Set

$$R_0^\times := \prod_{f_v = 0} (\mathcal{O}_v/\varpi_v\mathcal{O}_v)^\times$$
$$R_1^\times := \prod_{f_v > 0} (\mathcal{O}_v/\varpi_v^{f_v}\mathcal{O}_v)^\times.$$

If $b \in R_0^\times$ and $c \in R_1^\times$ then we write $b + c$ for the natural element of $(\mathcal{O}_p/\varpi_p^{f_+}\mathcal{O}_p)^\times \simeq R_0^\times \times R_1^\times$. Implicit in the notation below is that any choices of lifts are irrelevant. For instance, $\theta^\iota(b + c)$ makes perfect sense for $b \in R_0^\times$ and $c \in R_1^\times$.

As before, let $v \mid p$ and let V_v^+ be the matrix $\left(\begin{smallmatrix} \varpi_v & \\ & 1 \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbf{A}_{F,f})$. In general, if $K \subset \mathrm{GL}_2(\mathbf{A}_{F,f})$ is a compact open subgroup and $\mathfrak{m}$ is an ideal then we have a natural map

$$\begin{pmatrix} \varpi_v^{f_{v,+}} & \\ & 1 \end{pmatrix} = (V_v^+)^{f_{v,+}} : H_c^*(Y_1(\mathfrak{m}), \mathscr{L}_\lambda(L)) \to H_c^*(Y_1^0(\mathfrak{m}; \mathfrak{p}_v^{f_v^+}), \mathscr{L}_\lambda(L))$$

where $Y_1^0(\mathfrak{m}; \mathfrak{p}_v^{f_{v,+}}) = Y_{K_1^0(\mathfrak{m}; \mathfrak{p}_v^{f_{v,+}})}$ and

$$K_1^0(\mathfrak{m}; \mathfrak{p}_v^{f_{v,+}}) = \{g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in K_1(\mathfrak{m}) \mid b \in \mathfrak{p}_v^{f_{v,+}}\widehat{\mathcal{O}}_F\}.$$

It is clear that this morphism is independent of the choice of uniformizer ϖ_v. Furthermore, since $K_1(\mathfrak{m}) \supset K_1^0(\mathfrak{m}; \mathfrak{p}_v^{f_{v,+}})$ if $\mathfrak{p}_v \mid \mathfrak{m}$ then we can also take the endomorphism U_v of $H_c^*(Y_1(\mathfrak{m}), \mathscr{L}_\lambda(L))$ and post-compose it with pullback along $Y_1^0(\mathfrak{m}; \mathfrak{p}_v^{f_{v,+}}) \to Y_1(\mathfrak{m})$. This discussion gives meaning to the following lemma.

Lemma 7.6.2. *Let $\mathfrak{m} \subset \mathbf{p}$. If $\psi \in H_c^*(Y_1(\mathfrak{m}), \mathscr{L}_\lambda(L))$ is represented by an adelic cochain $\widetilde{\psi}$ and $W := \prod_{f_v=0}(U_v - V_v^+)\prod_{f_v>0}(V_v^+)^{f_v}$, then $W(\psi) \in H_c^*(Y_1^0(\mathfrak{m}; \mathbf{p}^{f_+}), \mathscr{L}_\lambda(L))$ is represented by the adelic cochain*

$$W(\widetilde{\psi})(\sigma) = \sum_{b \in R_0^\times} \begin{pmatrix} \varpi_p^{f_+} & b \\ & 1 \end{pmatrix} \cdot \widetilde{\psi}\left(\sigma \begin{pmatrix} \varpi_p^{f_+} & b \\ & 1 \end{pmatrix}\right).$$

Proof. According to the definitions (Section 2.2) we have

$$((V_v^+)^{f_v}\widetilde{\psi})(\sigma) = \begin{pmatrix} \varpi_v^{f_v} & \\ & 1 \end{pmatrix} \cdot \widetilde{\psi}\left(\sigma \begin{pmatrix} \varpi_v^{f_v} & \\ & 1 \end{pmatrix}\right)$$
$$((U_v - V_v^+)\widetilde{\psi})(\sigma) = \sum_{b_v \in (\mathcal{O}_v/\varpi_v\mathcal{O}_v)^\times} \left(\begin{smallmatrix} \varpi_v & b_v \\ & 1 \end{smallmatrix}\right) \cdot \widetilde{\psi}\left(\sigma \left(\begin{smallmatrix} \varpi_v & b_v \\ & 1 \end{smallmatrix}\right)\right).$$

Here we are using $\mathfrak{m} \subset \mathbf{p}$ to use the given description of the U_v-operator. In the second formula, we are free to choose coset representatives in $\widehat{\mathcal{O}}_F$ for $(\mathcal{O}_v/\varpi_v\mathcal{O}_v)^\times$ that are supported only on v. But then the matrices in the two formulas above, as one ranges over all $v \mid p$, necessarily commute and the formula for $W(\widetilde{\psi})$ is clear. $\square$

We make a similar calculation for the next lemma. But note that we do not specify the level at which the result ends up (it is not "pretty"). This omission is harmless because we will apply Lemma 7.6.3 only through Lemma 7.6.2 at which point we know precisely the resulting level subgroup.

Lemma 7.6.3. *Let $\mathfrak{m} \subset \mathbf{p}$. If $\psi \in H^*_c(Y_1(\mathfrak{m}), \mathscr{L}_\lambda(L))$ is represented by an adelic cochain $\widetilde{\psi}$ and $b \in R_0^\times$, then $\left(\begin{smallmatrix} \varpi_p^{f_+} & b \\ & 1 \end{smallmatrix}\right) \cdot \mathrm{tw}^{\mathrm{cl}}_{\theta^\iota}(\psi)$ is represented by the adelic cochain*

$$\sigma = \sigma_\infty \otimes [g_f] \mapsto (\prod_{f_v=0} \theta^\iota(\varpi_v))\theta^\iota(\det g_f) \sum_{c \in R_1^\times} \theta^\iota(c+b) \left(\begin{smallmatrix} \varpi_p^{f_+} & b+c \\ & 1 \end{smallmatrix}\right) \cdot \widetilde{\psi}\left(\sigma \left(\begin{smallmatrix} \varpi_p^{f_+} & b+c \\ & 1 \end{smallmatrix}\right)\right).$$

Proof. First, by definition we have

$$\left(\left(\begin{smallmatrix} \varpi_p^{f_+} & b \\ & 1 \end{smallmatrix}\right) \cdot \mathrm{tw}^{\mathrm{cl}}_{\theta^\iota}(\widetilde{\psi})\right)(\sigma) = \left(\begin{smallmatrix} \varpi_p^{f_+} & b \\ & 1 \end{smallmatrix}\right) \cdot \mathrm{tw}^{\mathrm{cl}}_{\theta^\iota}(\widetilde{\psi})\left(\sigma \left(\begin{smallmatrix} \varpi_p^{f_+} & b \\ & 1 \end{smallmatrix}\right)\right). \tag{7.6.2}$$

Set $\varpi^{(0)} = \prod_{f_v=0} \varpi_v$ and $\varpi^{(1)} := \prod_{f_v>0} \varpi_v^{f_v}$, so that $\mathfrak{f}\widehat{\mathcal{O}}_F = \varpi^{(1)}\widehat{\mathcal{O}}_F$. If $c \in R_1^\times$ then choose a lift $\widehat{c}$ to $\widehat{\mathcal{O}}_F^\times$ so that $\widehat{c} \mapsto c$ in $R_1^\times$ but $\widehat{c} \mapsto b$ in $R_0^\times$. Then, $\{\widehat{c}/\varpi^{(1)}\}_{c \in R_1^\times}$ is a set of representatives for $\Upsilon_{\mathfrak{f}}^\times$, so Lemma 5.5.6 implies that

$$\begin{aligned} &\mathrm{tw}^{\mathrm{cl}}_{\theta^\iota}(\widetilde{\psi})\left(\sigma \left(\begin{smallmatrix} \varpi_p^{f_+} & b \\ & 1 \end{smallmatrix}\right)\right) \\ &\qquad = \theta^\iota(\varpi_p^{f_+} \det g_f) \sum_{c \in R_1^\times} \theta^\iota(\widehat{c}/\varpi^{(1)}) \left(\begin{smallmatrix} 1 & \widehat{c}_0/\varpi^{(1)} \\ & 1 \end{smallmatrix}\right) \widetilde{\psi}\left(\sigma \left(\begin{smallmatrix} \varpi_p^{f_+} & b \\ & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & \widehat{c}_0/\varpi^{(1)} \\ & 1 \end{smallmatrix}\right)\right), \end{aligned} \tag{7.6.3}$$

where as before $\widehat{c}_0$ is zero at places $v \nmid \mathfrak{f}$. In particular, $\varpi_p^{f_+}\widehat{c}_0/\varpi^{(1)} = \widehat{c}_0$ and so

$$\left(\begin{smallmatrix} \varpi_p^{f_+} & * \\ & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & \widehat{c}_0/\varpi^{(1)} \\ & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} \varpi_p^{f_+} & *+\widehat{c}_0 \\ & 1 \end{smallmatrix}\right).$$

On the other hand, $\varpi_p^{f_+}\widehat{c}/\varpi^{(1)} = \varpi^{(0)}\widehat{c}$, whence $\theta^\iota(\varpi_p^{f_+}\widehat{c}/\varpi^{(1)}) = \theta^\iota(\varpi^{(0)})\theta^\iota(\widehat{c})$. We finally remark that $\theta^\iota(\widehat{c}) = \theta^\iota(c+b)$ by construction of $\widehat{c}$. Putting these observations into (7.6.2) and (7.6.3), we have completed the proof. $\square$

In the statement of the next theorem, we write U_{ϖ_p} for the Hecke operator defined by the double coset of $\left(\begin{smallmatrix} \varpi_p & \\ & 1 \end{smallmatrix}\right)$. We could have also called it $U_{\mathbf{p}}$ but we fear it looks too close to U_p. In any case, the point is that $U_{\varpi_p}^{f_+} = \prod_{v|p} U_v^{f_{v,+}}$.

Theorem 7.6.4. *Let $\mathfrak{m} \subset \mathbf{p}$, $\Psi \in H^d_c(Y_1(\mathfrak{m}), \mathscr{D}_\lambda \otimes_{k_\lambda} L)$, and let m be an integer which is critical with respect to λ. Then,*

$$\begin{aligned} &\langle U_{\varpi_p}^{f_+}\Psi, \mathscr{Q}_\lambda(\mathbf{N}_p^m\theta^\iota)\rangle \\ &\qquad = \varpi_p^{-f_+\frac{w-\kappa}{2}} \left\langle \left(\prod_{v|p, f_v=0} \theta^\iota(\varpi_v)^{-1}(U_v - V_v^+) \prod_{v|p, f_v>0} (V_v^+)^{f_v}\right) \mathrm{tw}^{\mathrm{cl}}_{\theta^\iota} I_\lambda(\Psi), \mathrm{cl}_p(m)\right\rangle \end{aligned} \tag{7.6.4}$$

Before giving the proof, we want to clarify two points about the statement of the theorem.

Remark 7.6.5. On the right-hand side, the element $I_\lambda(\Psi)$ is meant to be an $\mathscr{L}_\lambda(L)$-valued cohomology class, not an $\mathscr{L}_\lambda^\sharp(L)$-valued one. (Thus the same goes for its twist by θ^ι.) The only difference is the Hecke action at p, and if you want an $\mathscr{L}_\lambda^\sharp(L)$-valued class, which is arguably more a more natural choice, then of course you remove the ϖ_p-factor from the front of the formula. See (7.6.9) below.

But for the sake of comparing to classical L-values, if we make the switch in the previous paragraph then we also have to remember to view $\mathrm{cl}_p(m)$ as a $(\mathscr{L}_\lambda^\vee)^\sharp$-valued homology class and take this into account during computations. (Compare with Corollary 7.6.7 below).

Remark 7.6.6. In the proof below we are going to work at the level of adelic cochains (as indicated by the previous lemmas). Since we elide the actual cohomology in the arguments, and thus omit making precise the levels, let us clarify further the two sides of the formula (7.6.4).

We hope that the left-hand side of (7.6.4) is clear: we are taking the class $U_{\varpi_p}^{f_+}\Psi$ in $H_c^d(Y_1(\mathfrak{m}), \mathscr{D}_\lambda \otimes_{k_\lambda} L)$ and pairing it with the class $\mathscr{Q}_\lambda(\mathbf{N}_p^m\theta^\iota) \in H_d^{\mathrm{BM}}(Y_1(\mathfrak{m}), \mathscr{A}_\lambda \otimes_{k_\lambda} L)$.

Let's unwind the right-hand side of (7.6.4). First, $\mathrm{tw}_{\theta^\iota} I_\lambda(\Psi)$ is a class in $H_c^d(Y_1(\mathfrak{m}\mathfrak{f}^2), \mathscr{L}_\lambda(L))$. If we write W for the operator acting on this class in (7.6.4) (and the proof below), then $W\,\mathrm{tw}_{\theta^\iota}(I_\lambda(\Psi))$ defines a class in $H_c^d(Y_1^0(\mathfrak{m}\mathfrak{f}^2;\mathfrak{f}^2); \mathscr{L}_\lambda(L))$ by the discussion preceding Lemma 7.6.2. And since $\begin{pmatrix} \widehat{\mathcal{O}}_F^\times & \\ & 1 \end{pmatrix} \subset K_1^0(\mathfrak{m}\mathfrak{f}^2;\mathfrak{f}^2)$, we can make sense of the evaluation class $\mathrm{cl}_p(m) \in H_d^{\mathrm{BM}}(Y_1^0(\mathfrak{m}\mathfrak{f}^2;\mathfrak{f}^2), \mathscr{L}_\lambda^\vee(L))$ which was carefully and universally defined in Definition 7.5.5. We then pair these classes, and this is what we mean by the right-hand side of (7.6.4).

Proof of Theorem 7.6.4. For the purposes of the proof, write

$$W := \prod_{v \mid p, f_v = 0} \theta^\iota(\varpi_v)^{-1}(U_v - V_v^+) \prod_{v\mid p, f_v > 0} (V_v^+)^{f_v}$$

for the operator appearing on the right-hand side of (7.6.4), as in Remark 7.6.6. (It is a scaling of the operator "W" in Lemma 7.6.2.)

Recall that $\mathscr{Q}_\lambda = \mathrm{t}_* \circ \mathrm{PD} \circ Q_\lambda$. Thus, according to (2.1.8) we have

$$\langle U_{\varpi_p}^{f_+}\Psi, \mathscr{Q}_\lambda(\mathbf{N}_p^m\theta^\iota)\rangle = \langle \mathrm{t}^*(U_{\varpi_p}^{f_+}\Psi) \cup Q_\lambda(\mathbf{N}_p^m\theta^\iota), [\mathrm{C}_\infty]\rangle, \tag{7.6.5}$$

where $[\mathrm{C}_\infty]$ is the Borel–Moore fundamental class for C_∞. For the purposes of this equation, the cup-product $\mathrm{t}^*(U_{\varpi_p}^{f_+}\Psi) \cup Q_\lambda(\mathbf{N}_p^m\theta^\iota) \in H_c^d(\mathrm{C}_\infty, \mathrm{t}^*(\mathscr{D}_\lambda \otimes_L \mathscr{A}_\lambda))$ is implicitly its image in $H_c^d(\mathrm{C}_\infty, L)$ under the natural map.

Similarly, since $\mathrm{cl}_p(m) = \mathrm{t}_*(\mathrm{PD}(\delta_{m,p}))$ (Definition 7.5.5) we have

$$\varpi_p^{-f_+\cdot\frac{w-\kappa}{2}}\langle W\,\mathrm{tw}_{\theta^\iota} I_\lambda(\Psi), \mathrm{cl}_p(m)\rangle = \varpi_p^{-f_+\cdot\frac{w-\kappa}{2}}\langle \mathrm{t}^*\left(W\,\mathrm{tw}_{\theta^\iota} I_\lambda(\Psi)\right) \cup \delta_{m,p}, [\mathrm{C}_\infty]\rangle \tag{7.6.6}$$

(with the same caveat on the meaning of the cup product). Comparing (7.6.5) and (7.6.6), it is enough to show that the cup products appearing define the same elements of $H_c^d(\mathrm{C}_\infty, L)$. For that, we will explicitly compute using adelic cochains.

Fix a singular d-chain $\sigma = \sigma_\infty \otimes [x]$ on $F_{\infty,+}^\times \times \mathbf{A}_{F,f}^\times$, and a representative $\widetilde{\Psi}$ for Ψ in the adelic cochains $C_{\mathrm{ad},c}^\bullet(K_1(\mathfrak{m}), \mathscr{D}_\lambda \otimes_{k_\lambda} L)$. To cut down on parentheses, let us write $\mathrm{t}_\sigma := \mathrm{t}(\sigma)$ for the image of σ under t. Then, the definition of the cup product on the level of cochains means that we want to show

$$\underbrace{(U_{\varpi_p}^{f_+}\widetilde{\Psi})(\mathrm{t}_\sigma)}_{\in \mathscr{D}_\lambda(L)}\underbrace{\left(Q_\lambda(\mathbf{N}_p^m\theta^\iota)(x)\right)}_{\in \mathscr{A}_\lambda(L)} = \varpi_p^{-f_+\cdot\frac{w-\kappa}{2}}\underbrace{\left(W\,\mathrm{tw}_{\theta^\iota} I_\lambda(\widetilde{\Psi})\right)(\mathrm{t}_\sigma)}_{\in\mathscr{L}_\lambda(L)}\underbrace{\left(\delta_{m,p}(x)\right)}_{\in \mathscr{L}_\lambda(L)^\vee}. \tag{7.6.7}$$

(To aid the reader, we have indicated where each object lives with underbraces.)

We begin computing the left-hand side of (7.6.7). In general, if $s \in H^0(\mathrm{C}_\infty, \mathrm{t}^*\mathscr{A}_\lambda)$, then

$$(U_{\varpi_p}^{f_+}\widetilde{\Psi})(\mathrm{t}_\sigma)\,(s(x)) = \sum_{a \in \mathcal{O}_p/\varpi_p^{f_+}\mathcal{O}_p} \widetilde{\Psi}\left(\mathrm{t}_\sigma \begin{pmatrix} \varpi_p^{f_+} & a \\ & 1 \end{pmatrix}\right)\left(s(x)| \begin{pmatrix} \varpi_p^{f_+} & a \\ & 1 \end{pmatrix}\right). \tag{7.6.8}$$

Consider $s = Q_\lambda(\mathbf{N}_p^m\theta^\iota)$. By Lemma 7.6.1, the term in the sum on the right-hand side of (7.6.8) is zero if $a \notin (\mathcal{O}_p/\varpi_p^{f_+}\mathcal{O}_p)^\times$, but otherwise we have

(7.6.9)
$$\begin{aligned} Q_\lambda(\mathbf{N}_p^m\theta^\iota)(x)| \begin{pmatrix} \varpi_p^{f_+} & a \\ & 1 \end{pmatrix} &= \theta^\iota(ax) I_\lambda^\vee(\delta_{m,p}(x))| \begin{pmatrix} \varpi_p^{f_+} & a \\ & 1 \end{pmatrix} && \text{(by Lemmas 7.5.4 \& 7.6.1)} \\ &= \varpi_p^{-f_+\cdot\frac{w-\kappa}{2}}\theta^\iota(ax) I_\lambda^\vee\left(\delta_{m,p}(x)| \begin{pmatrix} \varpi_p^{f_+} & a \\ & 1 \end{pmatrix}\right) && \text{(by (5.4.2)).} \end{aligned}$$

Combining this with (7.6.8), and transposing I_λ, we see that

$$\begin{aligned} &(U_{\varpi_p}^{f_+}\widetilde{\Psi})(\mathrm{t}_\sigma)\left(Q_\lambda(\mathbf{N}_p^m\theta^\iota)(x)\right) \\ &\quad= \varpi_p^{-f_+\cdot\frac{w-\kappa}{2}} \sum_{a \in (\mathcal{O}_p/\varpi_p^{f_+}\mathcal{O}_p)^\times} \theta^\iota(ax) I_\lambda(\widetilde{\Psi})\left(\mathrm{t}_\sigma \begin{pmatrix} \varpi_p^{f_+} & a \\ & 1 \end{pmatrix}\right)\left(\delta_{m,p}(x)| \begin{pmatrix} \varpi_p^{f_+} & a \\ & 1 \end{pmatrix}\right) \\ &\quad= \varpi_p^{-f_+\cdot\frac{w-\kappa}{2}} \sum_{a \in (\mathcal{O}_p/\varpi_p^{f_+}\mathcal{O}_p)^\times} \theta^\iota(ax)\left(\begin{pmatrix} \varpi_p^{f_+} & a \\ & 1 \end{pmatrix}\cdot I_\lambda(\widetilde{\Psi})\right)(\mathrm{t}_\sigma)(\delta_{m,p}(x)). \end{aligned} \tag{7.6.10}$$

We want to see that this expression is the same as the right-hand side of (7.6.7). For that, let $\widetilde{\psi} = I_\lambda(\widetilde{\Psi})$ and then Lemma 7.6.2 and Lemma 7.6.3 combine to show that

$$\begin{aligned} W\,\mathrm{tw}_{\theta^\iota}\,\widetilde{\psi}(\mathrm{t}_\sigma) &= \Big(\prod_{v|p, f_v=0}\theta^\iota(\varpi_v)^{-1}\Big)\sum_{b\in R_0^\times}\begin{pmatrix} \varpi_p^{f_+} & b \\ & 1 \end{pmatrix}\cdot\widetilde{\psi}\left(\mathrm{t}_\sigma\begin{pmatrix} \varpi_p^{f_+} & b \\ & 1 \end{pmatrix}\right). \\ &= \theta^\iota(x)\sum_{c\in R_1^\times}\sum_{b \in R_0^\times}\theta^\iota(c+b)\begin{pmatrix} \varpi_p^{f_+} & c+b \\ & 1 \end{pmatrix}\widetilde{\psi}\left(\mathrm{t}_\sigma\begin{pmatrix} \varpi_p^{f_+} & c+b \\ & 1 \end{pmatrix}\right) \\ &= \theta^\iota(x)\sum_{a\in(\mathcal{O}_p/\varpi_p^{f_+}\mathcal{O}_p)^\times}\theta^\iota(a)\left(\begin{pmatrix} \varpi_p^{f_+} & a \\ & 1 \end{pmatrix}\cdot\widetilde{\psi}\right)(\mathrm{t}_\sigma). \end{aligned} \tag{7.6.11}$$

Multiplying (7.6.11) by $\varpi_p^{-f_+\cdot\frac{w-\kappa}{2}}$ and evaluating at $\delta_{m,p}(x)$, we see exactly (7.6.10). This completes the proof. □

Our interest is in eigenclasses, so we separate out the following corollary of Theorem 7.6.4.

Corollary 7.6.7. *Suppose that $\Psi \in H_c^d(Y_1(\mathfrak{m}), \mathscr{D}_\lambda \otimes_{k_\lambda} L)$ is an eigenvector for each operator U_v, with eigenvalue $\alpha_v^\sharp$. Set $\alpha_v = \varpi_v^{\frac{w-\kappa}{2}}\alpha_v^\sharp$. Then, for all integers m critical with respect to λ,*

$$\mathscr{P}_\lambda(\Psi)(\mathbf{N}_p^m\theta^\iota) = \prod_{f_v>0}(\alpha_v^{-1}q_v^m)^{f_v}\cdot\prod_{f_v=0}(1-\theta^\iota(\varpi_v)^{-1}\alpha_v^{-1}q_v^m)\cdot\langle\mathrm{tw}_{\theta^\iota}^{\mathrm{cl}}(I_\lambda(\Psi)), \mathrm{cl}_p(m)\rangle.$$

Proof. To summarize our assumptions: we are assuming that $U_{\varpi_p}^{f_+}\Psi = \prod_{v|p}(\alpha_v^\sharp)^{f_{v,+}}\Psi$ and hence $U_v \operatorname{tw}_{\theta^\iota}^{\mathrm{cl}} I_\lambda(\Psi) = \theta^\iota(\varpi_v)\alpha_v \operatorname{tw}_{\theta^\iota}^{\mathrm{cl}} I_\lambda(\Psi)$ (see Remark 7.6.5). Then, by Theorem 7.6.4 we get that

$$\begin{aligned}\mathscr{P}_\lambda(\Psi)(\mathbf{N}_p^m\theta^\iota) &= \left(\prod_{v|p}(\alpha_v^\sharp)^{-f_{v,+}}\right)\langle U_{\varpi_p}^{f_+}\Psi, \mathscr{Q}_\lambda(\mathbf{N}_p^m\theta^\iota)\rangle\\ &= \left(\prod_{v|p}\alpha_v^{-f_{v,+}}\right)\left\langle \left(\prod_{v|p, f_v=0}\alpha_v - \theta^\iota(\varpi_v)^{-1}V_v^+)\prod_{v|p,f_v>0}(V_v^+)^{f_v}\right)\operatorname{tw}_{\theta^\iota}^{\mathrm{cl}} I_\lambda(\Psi), \mathrm{cl}_p(m)\right\rangle.\end{aligned}$$

But here V_v^+ is the pullback along $\left(\begin{smallmatrix}\varpi_v & \\ & 1\end{smallmatrix}\right)$ and so it is adjoint to the pushforward of the same matrix under the pairing $\langle -,-\rangle$. By Lemma 7.5.7, we can thus replace each instance of V_v^+ with q_v^m. The result follows. □

8. p-ADIC L-FUNCTIONS

8.1. **Consequences of smoothness.** We begin by proving a lemma in commutative algebra. If $(R, \mathfrak{m}_R)$ is a Noetherian local ring and M is a module over R then we write $\mathrm{pd}_R(M)$ for its projective dimension over R and $\mathrm{depth}_R(M)$ for its $\mathfrak{m}_R$-depth.

Lemma 8.1.1. *Suppose that $(R, \mathfrak{m}_R)$ and $(T, \mathfrak{m}_T)$ are Noetherian local rings with R regular and T Cohen–Macaulay and $R \to T$ is a finite injective local morphism. The following conclusions hold.*

(1) T is flat over R.
(2) If T is regular then $T/\mathfrak{m}_R T$ is a local complete intersection.

Suppose that M is a finite T-module such that $\mathrm{pd}_T(M) < \infty$.

(3) $\mathrm{pd}_R(M) = \mathrm{pd}_T(M)$.
(4) M is projective over T if and only if M is projective over R, in which case the natural map $T/\mathfrak{m}_R T \to \mathrm{End}_{R/\mathfrak{m}_R R}(M/\mathfrak{m}_R M)$ is injective.

Proof. Part (1) follows from [60, Theorem 23.1]. For (2), since R is regular and $R \to T$ is flat by (1), the ideal $\mathfrak{m}_R T$ is generated by a T-regular sequence. Thus $T/\mathfrak{m}_R T$ is a local complete intersection by [60, Theorem 21.2(iii)].

Now write $n = \dim R = \dim T$. Since R and T are both Cohen–Macaulay, $n = \mathrm{depth}_R(R) = \mathrm{depth}_T(T)$. Since R is regular, $\mathrm{pd}_R(M) < \infty$ by [69]. So, if $\mathrm{pd}_T(M) < \infty$ as well, the Auslander–Buchsbaum formula ([7, Theorem 3.7]) implies that

$$\mathrm{depth}_R(M) + \mathrm{pd}_R(M) = n = \mathrm{depth}_T(M) + \mathrm{pd}_T(M). \tag{8.1.1}$$

Since $R \to T$ is a local morphism, [41, Proposition 16.4.8] implies that $\mathrm{depth}_R(M) = \mathrm{depth}_T(M)$ and thus (8.1.1) reduces to $\mathrm{pd}_R(M) = \mathrm{pd}_T(M)$ as we claimed in (3).

For (4), the first clause immediately follows from (3). For the second clause, if M is projective over R then $M/\mathfrak{m}_R M$ is finite projective over $T/\mathfrak{m}_R T$ and so clearly $T/\mathfrak{m}_R T$ acts faithfully on $M/\mathfrak{m}_R M$. □

Remark 8.1.2. If T is regular in Lemma 8.1.1 (which will always be the case below) then the hypothesis on projective dimension before (3) is automatic by [69].

We now return to the setting and notation of Section 6.4. Let $x \in \mathscr{E}(\mathfrak{n})_{\mathrm{mid}}(\overline{\mathbf{Q}}_p)$ be a point of weight λ and $h = v_p(\psi_x(U_p))$. Choose an affinoid neighborhood $\Omega \subset \mathscr{W}(1)$ containing λ so that (Ω, h) is slope adapted. Thus, x defines a maximal ideal $\mathfrak{m}_x \subset \mathbf{T}_{\Omega,h}$. Now define $R_x := \mathscr{O}(\Omega)_{\mathfrak{m}_\lambda}$, $\mathbf{T}_x := (\mathbf{T}_{\Omega,h})_{\mathfrak{m}_x}$, and $M_x := \left(H_c^d(\mathfrak{n}, \mathscr{D}_\Omega)_{\leq h}\right)_{\mathfrak{m}_x} = H_c^d(\mathfrak{n}, \mathscr{D}_\Omega)_{\mathfrak{m}_x}$. We also write T_x for the image $\mathbf{T}(\mathfrak{n})$ in $\mathrm{End}_{k_\lambda}(M_x/\mathfrak{m}_\lambda M_x) = \mathrm{End}_{k_\lambda}\left(H_c^d(\mathfrak{n}, \mathscr{D}_\lambda)_{\mathfrak{m}_x}\right)$.

Proposition 8.1.3. *If $\mathscr{E}(\mathfrak{n})_{\mathrm{mid}}$ is smooth at x, then M_x is finite projective over $\mathbf{T}_x$ and $\mathbf{T}_x/\mathfrak{m}_\lambda\mathbf{T}_x \simeq T_x$.*

Proof. Since $\mathscr{E}(\mathfrak{n})_{\mathrm{mid}}$ is equidimensional of dimension equal to the dimension of $\mathscr{W}(1)$, the map $R_x \to \mathbf{T}_x$ is a finite injective map of local noetherian rings with R_x regular. Moreover, M_x is finite projective over R_x (Proposition 6.4.4). So, given that $\mathbf{T}_x$ is also regular, Lemma 8.1.1(4) implies that M_x is finite projective over $\mathbf{T}_x$ and $\mathbf{T}_x/\mathfrak{m}_\lambda\mathbf{T}_x \hookrightarrow \mathrm{End}_{k_\lambda}(M_x/\mathfrak{m}_\lambda M_x)$. The image is onto T_x, completing the proof. $\square$

If $\epsilon \in \{\pm 1\}^{\Sigma_F}$ then write $M_x^\epsilon = H_c^d(\mathfrak{n}, \mathscr{D}_\Omega)^\epsilon_{\mathfrak{m}_x}$. In the next proposition we write $\mathrm{soc}_T(M)$ for the socle of M as a T-module, i.e. the sum of the simple T-submodules.

Theorem 8.1.4. *Suppose that (π, α) is a p-refined cuspidal automorphic representation of cohomological weight λ and conductor $\mathfrak{n}$. If α is a decent refinement, $x = x(\pi, \alpha) \in \mathscr{E}(\mathfrak{n})_{\mathrm{mid}}(\overline{\mathbf{Q}}_p)$, and $\epsilon \in \{\pm 1\}^{\Sigma_F}$, then*

(1) $\mathrm{soc}_{T_x}(H_c^d(\mathfrak{n}, \mathscr{D}_\lambda \otimes_{k_\lambda} k_x)^\epsilon_{\mathfrak{m}_x})$ is one-dimensional over k_x.

If, further, condition 2(c) in Definition 6.6.1 holds, then

(2) The $\mathbf{T}_x$-module M_x^ϵ is free of rank one.

Proof. We will actually check the second claim first. Suppose that x is decent and satisfies condition 2(c) of Definition 6.6.1. Then, x is a smooth point on $\mathscr{E}(\mathfrak{n})_{\mathrm{mid}}$ by Theorem 6.6.3. By Proposition 8.1.3, M_x is projective over $\mathbf{T}_x$, and hence so is its direct summand M_x^ϵ and furthermore the rank is equal to the rank of $M_x^\epsilon/\mathfrak{m}_\lambda M_x^\epsilon$ over T_x. By (6.4.3), $M_x^\epsilon/\mathfrak{m}_\lambda M_x^\epsilon \simeq H_c^d(\mathfrak{n}, \mathscr{D}_\lambda)^\epsilon_{\mathfrak{m}_x}$ (as T_x-modules). Now, set $M^\epsilon = H_c^d(\mathfrak{n}, \mathscr{D}_\Omega)^\epsilon_{\leq h}$, which we regard as a coherent sheaf over $X = \mathrm{Sp}\,\mathbf{T}_{\Omega,h}$. Since M_x^ϵ is free over $(\mathbf{T}_{\Omega,h})_{\mathfrak{m}_x}$, M^ϵ is free over some connected (Zariski-)open neighborhood U of x in X. In particular, to calculate the rank of M_x^ϵ, it suffices to calculate the rank of the fiber of M^ϵ at any closed point $y \in U$; but by Proposition 6.4.6 we can assume that y is extremely non-critical classical, in which case the rank is one. So this completes the proof of (2).

Now we check point (1) is true. If x is non-critical, this is a purely automorphic calculation. Otherwise, since x is decent, point (2) applies to x. Thus reduced to showing that $\dim_{k_x} \mathrm{soc}_{T_x}(T_x) = 1$. But $T_x \simeq \mathbf{T}_x/\mathfrak{m}_\lambda\mathbf{T}_x$ by Proposition 8.1.3, so T_x is a local complete intersection ring by Lemma 8.1.1(2). In particular, T_x is Gorenstein (and of dimension zero) and our result follows from [60, Theorem 18.1]. $\square$

8.2. **p-adic L-functions.** Throughout this subsection, we fix a cuspidal automorphic representation π of weight λ and conductor $\mathfrak{n}$. We make the following choices:

(1) α is a decent p-refinement for π.
(2) For each $\epsilon \in \{\pm 1\}^{\Sigma_F}$ we choose $\Omega_\pi^\epsilon \in \mathbf{C}^\times$ as in Corollary 4.2.6.

We write E for the subfield of $\mathbf{C}$ containing $\mathbf{Q}(\pi)$, $\mathbf{Q}(\alpha)$, and the Galois closure of F. Let $L = \mathbf{Q}_p(\iota(E))$. Recall that ι induces an isomorphism $H_c^d(\mathfrak{n}, \mathscr{L}_\lambda(E)) \simeq H_c^d(\mathfrak{n}, \mathscr{L}_\lambda(L))$.

Given (1) and (2) we define $\Phi^\epsilon_{\pi,\alpha} \in H_c^d(Y_1(\mathfrak{n}), \mathscr{L}_\lambda(L))^\epsilon$ to be

$$\Phi^\epsilon_{\pi,\alpha} = \iota\left(\frac{\mathrm{pr}^\epsilon\,\mathrm{ES}(\phi_{\pi,\alpha})}{\Omega_\pi^\epsilon}\right),$$

where $\phi_{\pi,\alpha}$ is the p-refined eigenform associated to (π, α). In the notation of Section 6.3 we have $\Phi^\epsilon_{\pi,\alpha} \in H_c^d(\mathfrak{n}, \mathscr{L}_\lambda(L))^\epsilon[\mathfrak{m}_{\pi,\alpha}]$. On the other hand, since α is a decent p-refinement for π, Theorem 8.1.4 above implies that $\dim H_c^d(\mathfrak{n}, \mathscr{D}_\lambda \otimes_{k_\lambda} L)^\epsilon[\mathfrak{m}^\sharp_{\pi,\alpha}] = 1$ and there is a natural integration map

$$I_\lambda : H_c^d(\mathfrak{n}, \mathscr{D}_\lambda \otimes_{k_\lambda} L)^\epsilon[\mathfrak{m}^\sharp_{\pi,\alpha}] \to H_c^d(\mathfrak{n}, \mathscr{L}_\lambda(L))^\epsilon[\mathfrak{m}_{\pi,\alpha}]. \tag{8.2.1}$$

We note the following lemma.

Lemma 8.2.1. *$I_\lambda(H^d_c(\mathfrak{n}, \mathscr{D}_\lambda \otimes_{k_\lambda} L)^\epsilon[\mathfrak{m}^\sharp_{\pi,\alpha}]) \neq (0)$ if and only if α is non-critical.*

Proof. If α is non-critical then I_λ is an isomorphism, so one implication is clear.

Now suppose that α is not non-critical, but recall that α is decent. Thus condition 2(c) of Definition 6.6.1 holds. This implies that $H^d_c(\mathfrak{n}, \mathscr{L}_\lambda(L))^\epsilon_{\mathfrak{m}_{\pi,\alpha}} \simeq H^d_c(\mathfrak{n}, \mathscr{L}_\lambda(L))^\epsilon[\mathfrak{m}_{\pi,\alpha}]$, and part (2) of Theorem 8.1.4 implies that $M = H^d_c(\mathfrak{n}, \mathscr{D}_\lambda \otimes_{k_\lambda} L)^\epsilon_{\mathfrak{m}^\sharp_{\pi,\alpha}}$ is free of rank one over T, where T is the largest quotient of $\mathbf{T}(\mathfrak{n})$ acting faithfully on M. We note that T is a local complete intersection (by the above discussion).

Since α is not non-critical, the

$$I_\lambda : M \to H^d_c(\mathfrak{n}, \mathscr{L}_\lambda(L))^\epsilon[\mathfrak{m}_{\pi,\alpha}] \tag{8.2.2}$$

is not an isomorphism. If it is zero we are done. If it is non-zero, then the target is a simple T-module and thus (8.2.2) is the surjection of M onto its largest T-simple quotient (the co-socle). In particular, the socle $M[\mathfrak{m}^\sharp_{\pi,\alpha}] \subset M$ maps to zero under I_λ, as claimed. □

Now recall that we defined a period map

$$\mathscr{P}_\lambda : H^d_c(\mathfrak{n}, \mathscr{D}_\lambda \otimes_{k_\lambda} L) \to \mathscr{D}(\Gamma_F, L)$$

in Definition 7.2.4 and we may post-compose it with the Amice transform $\mathcal{A}$ to get elements in $\mathscr{O}(\mathscr{X}(\Gamma_F)) \otimes_{\mathbf{Q}_p} L$ (Proposition 5.1.6).

For the next definition and the results afterward, we assume that (π, α) is a decently p-refined cohomological cuspidal automorphic representation of weight λ and conductor $\mathfrak{n}$.

Definition 8.2.2. $L^\epsilon_p(\pi, \alpha) = \mathcal{A}\left(\mathscr{P}_\lambda(\Psi^\epsilon_{\pi,\alpha})\right)$ where $\Psi^\epsilon_{\pi,\alpha} \in H^d_c(\mathfrak{n}, \mathscr{D}_\lambda \otimes_{k_\lambda} L)^\epsilon[\mathfrak{m}^\sharp_{\pi,\alpha}]$ is any choice of non-zero vector that, if α is non-critical, we assume satisfies $I_\lambda(\Psi^\epsilon_{\pi,\alpha}) = \Phi^\epsilon_{\pi,\alpha}$.

Note that, by Lemma 7.3.3, if χ is a continuous character on Γ_F then it defines a locally analytic function on Γ_F and $L^\epsilon_p(\pi, \alpha)(\chi) = \mathscr{P}_\lambda(\Psi^\epsilon_{\pi,\alpha})(\chi) = \langle \Psi^\epsilon_{\pi,\alpha}, \mathscr{Q}_\lambda(\chi)\rangle$ as in Section 7.2.

With this definition, we can catalog the properties of these p-adic L-functions.

Proposition 8.2.3 (Canoncity)**.** *$L^\epsilon_p(\pi, \alpha)$ is naturally defined up to an element of $L^\times$ in general, and an element of $\iota(E^\times)$ if α is non-critical.*

Proof. Obviously there is a choice of $L^\times$-multiple in Definition 8.2.2 in general. But if α is non-critical then the ambiguity is up to the construction of $\Phi^\epsilon_{\pi,\alpha}$, which is only up to $\iota(E^\times)$ through the choice of periods Ω^ϵ_π as in Corollary 4.2.6. □

Given a sign $\epsilon \in \{\pm 1\}$ we write $\mathscr{X}(\Gamma_F)^\epsilon$ for the union of components of $\mathscr{X}(\Gamma_F)$ consisting of characters χ for which $\chi(\zeta) = \epsilon(\zeta)$ for all $\zeta \in \pi_0(F^\times_\infty)$ (see Remark 7.3.5).

Proposition 8.2.4 (Support)**.** *If $\epsilon \neq \epsilon'$, then $L^\epsilon_p(\pi, \alpha)\big|_{\mathscr{X}(\Gamma_F)^{\epsilon'}} = 0$.*

Proof. See Proposition 7.3.4. □

If $h \geq 0$ is a real number and $f \in \mathscr{O}(\mathscr{X}(\Gamma_F)) \otimes_{\mathbf{Q}_p} L$ then we say f has order of growth $\leq h$ if $f = \mathcal{A}(\mu)$ for some (unique) distribution μ that has order of growth $\leq h$ as in Definition 7.4.1.

Proposition 8.2.5 (Growth)**.** *If $h_v = v_p(\iota(\alpha_v))$ and $h = \sum_{v \mid p} e_v h_v + \sum_{\sigma \in \Sigma_F} \frac{\kappa_\sigma - w}{2}$, then $L^\epsilon_p(\pi, \alpha)$ has order of growth $\leq h$.*

Proof. Proposition 7.4.2 implies $L^\epsilon_p(\pi, \alpha)$ has order of growth $\leq h$ where $h = \sum_{v \mid p} e_v v_p(\alpha^\sharp_v)$. The translation to the claimed statement is clear. □

Before the next proposition, we recall the notation:

$$\Lambda(\pi\otimes\theta,m+1)^{\mathrm{alg}} := \frac{\mathrm{sgn}(\theta_\infty)i^{1+m+\frac{\kappa-w}{2}}\Delta_{F/\mathbf{Q}}^{m+1}\Lambda(\pi\otimes\theta,m+1)}{G(\theta)\Omega_\pi^\epsilon}.$$

Here θ is a finite order Hecke character, and ϵ is chosen so that $\theta(\zeta)\zeta^m = \epsilon(\zeta)$ for all $\zeta \in \pi_0(F_\infty^\times)$. We have $\Lambda(\pi\otimes\theta,m+1)^{\mathrm{alg}} \in E(\theta)$ (it is only off by the absolute norm of the conductor of θ from the value in Theorem 4.5.7). We also recall that if $\mathfrak{p}_v \nmid \mathfrak{n}$ then α_v is a root of a quadratic polynomial (Definition 3.4.2) and we write $\beta_v = a_v(\pi) - \alpha_v$ for the other root. To save notation, in what follows, we stress that α_v and β_v are viewed as p-adic numbers under the isomorphism $\iota : \mathbf{C} \simeq \overline{\mathbf{Q}}_p$.

Proposition 8.2.6 (Interpolation)**.** *Suppose that m is an integer that is critical with respect to λ, θ is a finite order Hecke character of conductor $\prod_{v|p}\mathfrak{p}_v^{f_v}$ and $\epsilon(\zeta) = \theta(\zeta)\zeta^m$ for all $\zeta \in \pi_0(F_\infty^\times)$. Then,*

(1) If α is critical, then $L_p^\epsilon(\pi,\alpha)(\mathbf{N}_p^m\theta^\iota) = 0$.
(2) If α is non-critical, then

$$L_p^\epsilon(\pi,\alpha)(\mathbf{N}_p^m\theta^\iota) = \prod_{f_v>0}\left(\frac{q_v^{m+1}}{\alpha_v}\right)^{f_v}\prod_{f_v=0}(1-\theta^\iota(\varpi_v)\alpha_v^{-1}q_v^m)\prod_{\substack{v|p\\ \mathfrak{p}_v\nmid\mathfrak{n}\\ f_v=0}}(1-\beta_v\theta^\iota(\varpi_v)q_v^{-(m+1)})\cdot\iota\left(\Lambda(\pi\otimes\theta,m+1)^{\mathrm{alg}}\right).$$

Proof. Choose $\Psi_{\pi,\alpha}^\epsilon$ as in Definition 8.2.2. Then, by Lemma 7.3.3 we want to compute $\mathscr{P}_\lambda(\Psi_{\pi,\alpha}^\epsilon)(\mathbf{N}_p^m\theta^\iota)$ with the notations as in Section 7.2. For each $v \mid p$, $\Psi_{\pi,\alpha}^\epsilon$ is a U_v-eigenform with eigenvalue $\alpha_v^\sharp = \varpi_v^{\frac{\kappa-w}{2}}\alpha_v$. Thus Corollary 7.6.7 implies that

$$\mathscr{P}_\lambda(\Psi_{\pi,\alpha}^\epsilon)(\mathbf{N}_p^m\theta^\iota) = \prod_{f_v>0}\left(\frac{q_v^m}{\alpha_v}\right)^{f_v}\prod_{f_v=0}(1-\theta^\iota(\varpi_v)\alpha_v^{-1}q_v^m)\cdot\langle\mathrm{tw}_{\theta^\iota}^{\mathrm{cl}}(I_\lambda(\Psi_{\pi,\alpha}^\epsilon)),\mathrm{cl}_p(m)\rangle. \tag{8.2.3}$$

If α is critical, then the right-hand side vanishes by Lemma 8.2.1. This proves (1). If α is non-critical though, we have $I_\lambda(\Psi_{\pi,\alpha}^\epsilon) = \Phi_{\pi,\alpha}^\epsilon$, by definition. Thus

$$\begin{aligned}\iota^{-1}\left(\langle\mathrm{tw}_{\theta^\iota}^{\mathrm{cl}}(I_\lambda(\Psi_{\pi,\alpha}^\epsilon)),\mathrm{cl}_p(m)\rangle\right) &= \iota^{-1}\left(\langle\mathrm{tw}_{\theta^\iota}^{\mathrm{cl}}(\Phi_{\pi,\alpha}^\epsilon),\mathrm{cl}_p(m)\rangle\right)\\ &= \frac{1}{\Omega_\pi^\epsilon}\langle\mathrm{tw}_\theta\,\mathrm{pr}^\epsilon\,\mathrm{ES}(\phi_{\pi,\alpha}),\mathrm{cl}_\infty(m)\rangle \qquad \text{(by Proposition 7.5.6)}\\ &= \frac{1}{\Omega_\pi^\epsilon}\langle\mathrm{tw}_\theta(\mathrm{ES}(\phi_{\pi,\alpha})),\mathrm{cl}_\infty(m)\rangle \qquad \text{(by Lemma 4.5.5)}\\ &= \frac{G(\theta^{-1})}{\Omega_\pi^\epsilon}\langle\mathrm{ES}(\phi_{\pi,\alpha}\otimes\theta),\mathrm{cl}_\infty(m)\rangle.\end{aligned}$$

Combining this calculation with (8.2.3), we are finished by Corollary 4.5.4. (The Gauss sum can be moved to the denominator using (4.3.3); this is where the m's in the q_v exponents of (8.2.3) becomes $m+1$'s.) $\square$

Finally, we have a many-variable version of the above constructions. It follows easily from the functorial nature of our construction of the period maps. The proof is directly inspired from [11, Remark 4.16].

Proposition 8.2.7 (Variation)**.** *Let $x = x_{\pi,\alpha}$ be a smooth classical point on $\mathscr{E}(\mathfrak{n})_{\mathrm{mid}}$. Then, for each sufficiently small good open neighborhood U of x in $\mathscr{E}(\mathfrak{n})_{\mathrm{mid}}$ there exists an element $\mathbf{L}^\epsilon \in$*

$\mathscr{O}(U)\widehat{\otimes}_{\mathbf{Q}_p}\mathscr{O}(\mathscr{X}(\Gamma_F))$ specified up to $\mathscr{O}(U)^\times$-multiple and such that for each decent point $x' \in U$ associated with a p-refined cohomological cuspidal automorphic representation (π', α') we have

$$\mathbf{L}_p^\epsilon|_{u=x'} = c_{x'} L_p^\epsilon(\pi, \alpha)$$

for some constant $c_{x'} \in k_{x'}^\times$.

Proof. Given x, every good open neighborhood U of x is regular (Theorem 6.6.3). Fix such a neighborhood, and assume that it is belongs to a slope adapted pair (Ω, h). By Proposition 6.4.4 we may assume that $\mathscr{O}(U)$ acts faithfully on the finite projective $\mathscr{O}(\Omega)$-module $\mathscr{M}_c^d(U) = e_U H_c^d(\mathfrak{n}, \mathscr{D}_\Omega)$. By Lemma 8.1.1, $\mathscr{M}_c^d(U)$ is finite projective over $\mathscr{O}(U)$. Furthermore, for each ϵ, $M = \mathscr{M}_c^d(U)^\epsilon$ is free of rank one over $\mathscr{O}(U)$ by the same argument as in Theorem 8.1.4.

On the other hand, in Section 7.2 we constructed a canonical period map

$$\mathscr{P}_\Omega : H_c^d(\mathfrak{n}, \mathscr{D}_\Omega) \to \mathscr{D}(\Gamma_F, \mathscr{O}(\Omega)).$$

We can then specialize this to

$$\mathscr{P}_\Omega|_M \in M^\vee \widehat{\otimes}_{\mathscr{O}(\Omega)} \mathscr{D}(\Gamma_F, \mathscr{O}(\Omega)) \simeq M^\vee \widehat{\otimes}_{\mathbf{Q}_p} \mathscr{D}(\Gamma_F, \mathbf{Q}_p)$$

where $M^\vee = \mathrm{Hom}_{\mathscr{O}(\Omega)}(\mathscr{O}(\Omega), M)$ is the dual $\mathscr{O}(U)$-module.

We now combine the previous two paragraphs. Since U is smooth, $\mathscr{O}(U)$ is regular. In particular, it is Gorenstein. Since $M \simeq \mathscr{O}(U)$ as an $\mathscr{O}(U)$-module we deduce that $M^\vee$ is also free of rank one over $\mathscr{O}(U)$. Choose an $\mathscr{O}(U)$-linear isomorphism $M^\vee \simeq \mathscr{O}(U)$ and then we get

$$\mathscr{P}_\Omega|_M \in M^\vee \widehat{\otimes}_{\mathbf{Q}_p} \mathscr{D}(\Gamma_F, \mathbf{Q}_p) \simeq \mathscr{O}(U) \widehat{\otimes}_{\mathbf{Q}_p} \mathscr{D}(\Gamma_F, \mathbf{Q}_p).$$

We finally define $\mathbf{L}_p^\epsilon := \mathcal{A}(\mathscr{P}_\Omega|_M)$ where $\mathcal{A}$ is the Amice transform, as usual.

From the construction, $\mathbf{L}_p^\epsilon$ was uniquely defined up to $\mathscr{O}(U)^\times$-multiple and it is an exercise to see that it specializes the construction(s) given above. □

Appendix A. A deformation calculation

The goal of this appendix is to extend the local calculation in [14, Section 3] to certain rank two semistable, non-crystalline cases as needed in Section 6.6 (specifically Proposition 6.6.5).

We fix the following notations throughout: K is a finite extension of $\mathbf{Q}_p$; $\mathcal{O}_K$ is its ring of integers; $\varpi_K \in \mathcal{O}_K$ is a uniformizing element; L is another finite extension of $\mathbf{Q}_p$ and $\#\Sigma_K = (K : \mathbf{Q}_p)$; where $\Sigma_K := \mathrm{Hom}_{\mathbf{Q}_p}(K, L)$; $\mathcal{R}_{K,L} = \mathcal{R}_K \otimes_{\mathbf{Q}_p} L$ for the Robba ring over K extended linearly to L; if $\delta : K^\times \to L^\times$ is a continuous character then we let $\mathcal{R}_{K,L}(\delta)$ be the corresponding rank one (φ, Γ_K)-module; if E is a (φ, Γ_K)-module over $\mathcal{R}_{K,L}$ then we write $H^1(E)$ for its cohomology, $H_f^1(E)$ and $H_g^1(E)$ for the usual Selmer groups (see [13, Section 1.4.1] for instance).

For this entire appendix we fix a rank two (φ, Γ_K)-module D which is triangulated

$$0 \to \mathcal{R}_{K,L}(\delta_1) \to D \to \mathcal{R}_{K,L}(\delta_2) \to 0. \tag{A.1}$$

We write $\mathfrak{t}_D = \mathrm{Ext}^1_{(\varphi,\Gamma_K)}(D, D) = H^1(\mathrm{ad}\, D)$ for the Zariski tangent space to the functor of deformations of D to complete local noetherian L-algebras with residue field L ([14, Section 2.2]). We will begin making assumptions now.

(HT-reg): D is Hodge–Tate and for $\tau \in \Sigma_K$, the τ-Hodge–Tate weights are distinct.

Following (HT-reg) we write $h_{1,\tau} < h_{2,\tau}$ for the τ-Hodge–Tate weights of D in the direction $\tau \in \Sigma_K$. Any deformation $\widetilde{D} \in \mathfrak{t}_D$ has two distinct Hodge–Sen–Tate weights $\eta_{i,\tau} = h_{i,\tau} + \varepsilon d\eta_{i,\tau} \in L[\varepsilon]$. Write $\log : \mathcal{O}_L^\times \to L$ for the logarithm defined on $1 + p\mathcal{O}_L$ as usual, extended by zero on torsion elements, and homomorphically otherwise.

If $\eta \in L[\varepsilon]^{\Sigma_K}$ is given by $\eta_\tau = h_\tau + \varepsilon d\eta_\tau$ with $h_\tau \in \mathbf{Z}$, then write z^η for the character $\mathcal{O}_L^\times \to L^\times$ given by

$$z \overset{z^\eta}{\longmapsto} \prod_{\tau \in \Sigma_K} \tau(z)^{h_\tau}(1 + \varepsilon d\eta_\tau \log(\tau(z))).$$

The Hodge–Sen–Tate weight of z^η is $-\eta$.

If $\delta^\circ : \mathcal{O}_K^\times \to L^\times$ is a continuous character, write $\delta = \mathrm{LT}_{\varpi_K}(\delta^\circ) : K^\times \to L^\times$ for the unique character so that $\delta(\varpi_K) = 1$ and $\delta|_{\mathcal{O}_K^\times} = \delta^\circ$. We now make our second assumption regarding D.

(st): D is semi-stable but non-crystalline.

Since D is semi-stable, so is each character δ_i. They are crystalline in fact and, for instance, φ^f acts on $D_{\mathrm{crys}}(\delta_1 \otimes \mathrm{LT}_{\varpi_K}(z^{h_1}))$ by $\Phi_{\varpi_K} := \delta_1(\varpi_K)\prod_{\tau \in \Sigma_K} \tau(\varpi_K)^{\mathrm{HT}_\tau(\delta_1) - h_{1,\tau}}$. By (A.4) we have

$$D_{\mathrm{crys}}(D \otimes \mathrm{LT}_{\varpi_K}(z^{h_1}))^{\varphi^f = \Phi_{\varpi_K}} \neq 0. \tag{A.2}$$

The non-crystalline portion of the assumption (st) has the following consequences.

Lemma A.0.1.

(1) The injective map $D_{\mathrm{crys}}(\delta_1) \to D_{\mathrm{crys}}(D)$ is an isomorphism.

(2) φ^f acts on $D_{\mathrm{crys}}(\delta_2)$ with an eigenvalue different from φ^f acting on $D_{\mathrm{crys}}(\delta_1)$.

Proof. To prove (a), we note that $D_{\mathrm{crys}}(D) = D_{\mathrm{st}}(D)^{N=0}$ is always a $K_0 \otimes_{\mathbf{Q}_p} L$-direct summand of $D_{\mathrm{st}}(D)$. In particular, since D is not crystalline, but it is semi-stable, we have that $D_{\mathrm{crys}}(D)$ is free of rank one over $K_0 \otimes_{\mathbf{Q}_p} L$. This makes the map $D_{\mathrm{crys}}(\delta_1) \to D_{\mathrm{crys}}(D)$ an isomorphism, for otherwise some non-zero element of $K_0 \otimes_{\mathbf{Q}_p} L$ would annihilate $D_{\mathrm{crys}}(\delta_2)$.

For (b), let ϕ_i be the eigenvalue of φ^f acting on $D_{\mathrm{crys}}(\delta_i)$. Write ε for the cyclotomic character. Since the extension (A.4) is assumed to be semi-stable but non-crystalline, a standard Galois cohomology calculation ([13, Corollary 1.4.5]) implies that φ^f acts trivially on $D_{\mathrm{crys}}(\delta_2\delta_1^{-1}\varepsilon)$. Since φ^f acts on $D_{\mathrm{crys}}(\varepsilon)$ as the scalar p^{-f}, we see $\phi_2\phi_1^{-1} = p^f \neq 1$. □

In particular, it follows from Lemma A.0.8 that $D_{\mathrm{crys}}(D \otimes \mathrm{LT}_{\varpi_K}(z^{h_1}))^{\varphi^f = \Phi_{\varpi_K}}$ is free of rank one over $K_0 \otimes_{\mathbf{Q}_p} L$ (not just that it is non-zero as in (A.5)).

Definition A.0.2. Let $\widetilde{D} \in \mathfrak{t}_D$ be an infinitesimal deformation, and write $\widetilde{\eta}_i$ for its Hodge–Sen–Tate weight deforming h_i.

(1) $\widetilde{D}$ is called refined if $D_{\mathrm{crys}}(\widetilde{D} \otimes \mathrm{LT}_{\varpi_K}(z^{\widetilde{\eta}_1}))^{\varphi^f = \tilde{\Phi}}$ is free of rank one over $K_0 \otimes_{\mathbf{Q}_p} L[\varepsilon]$ for some $\widetilde{\Phi} \equiv \Phi_{\varpi_K} \bmod \varepsilon$.

(2) $\widetilde{D}$ is called Hodge–Tate if $\widetilde{\eta}_i = h_i$ for each i.

It is straightforward that $\widetilde{D}$ is a Hodge–Tate deformation if and only the underlying rank four (φ, Γ_K)-module is Hodge–Tate in the usual sense (compare with the proof of Lemma A.7 below).

We write $\mathfrak{t}_D^{\mathrm{Ref}} \subset \mathfrak{t}_D$ for the L-linear subspace of refined deformations and $\mathfrak{t}_D^{\mathrm{HT}}$ for the subspace of Hodge–Tate deformations. Their intersection is written $\mathfrak{t}_D^{\mathrm{Ref,HT}}$. The Selmer group $H^1_f(\mathrm{ad}\, D)$, by definition, denotes those deformations $\widetilde{D} \in \mathfrak{t}_D$ such that the extension

$$0 \to D[1/t] \to \widetilde{D}[1/t] \to D[1/t] \to 0$$

is split as Γ_K-modules. In particular, if $\widetilde{D} \in H^1_f(\mathrm{ad}\, D)$, then the *a priori* left-exact sequence

$$0 \to D_{\mathrm{crys}}(D) \to D_{\mathrm{crys}}(\widetilde{D}) \to D_{\mathrm{crys}}(D) \to 0 \tag{A.3}$$

is exact.[23] On the other hand, since D is semi-stable, the Selmer group $H^1_g(\operatorname{ad} D)$ really parameterizes semi-stable deformations.

We note the following lemma for later use.

Lemma A.0.3. *Suppose that E is any (φ,Γ_K)-module over $\mathcal{R}_{K,L}$. Then, for any crystalline character $\delta : K^\times \to L^\times$, the natural map $D_{\mathrm{crys}}(E)\otimes D_{\mathrm{crys}}(\delta) \to D_{\mathrm{crys}}(E(\delta))$ is an isomorphism.*

Proof. In fact, if E' is another (φ,Γ_K)-module over $\mathcal{R}_{K,L}$ then the natural map $D_{\mathrm{crys}}(E)\otimes D_{\mathrm{crys}}(E') \to D_{\mathrm{crys}}(E\otimes E')$ is always injective. Now use that a character has a natural inverse. $\square$

Lemma A.0.4. $H^1_g(\operatorname{ad} D) = H^1_f(\operatorname{ad} D) \subset \mathfrak{t}_D^{\mathrm{Ref},\mathrm{HT}}$.

Proof. The first equality follows from [13, Corollary 1.4.5] (and the computation in Lemma A.0.8, say). This shows, in particular, that $H^1_f(\operatorname{ad} D)\subset \mathfrak{t}_D^{\mathrm{HT}}$. So, it suffices to prove $H^1_f(\operatorname{ad} D)\subset \mathfrak{t}_D^{\mathrm{Ref}}$.

Consider $\widetilde{D}\in H^1_f(\operatorname{ad} D)$. Then, it suffices to show that $D_{\mathrm{crys}}(\widetilde{D}\otimes \mathrm{LT}_{\varpi_K}(z^{\widetilde{\eta}_1}))$ is free of rank one over $K_0\otimes_{\mathbf{Q}_p} L[\varepsilon]$ (since then clearly φ^f acts by some eigenvalue on any basis). Note that in fact $\widetilde{\eta}_1 = h_1$ is constant, so write $M = D_{\mathrm{crys}}(\widetilde{D}\otimes \mathrm{LT}_{\varpi_k}(z^{h_1}))$. Since $\widetilde{D}$ is an f-extension, the sequence

$$0 \to D_{\mathrm{crys}}(D\otimes \mathrm{LT}_{\varpi_k}(z^{h_1})) \to M \to D_{\mathrm{crys}}(D\otimes \mathrm{LT}_{\varpi_k}(z^{h_1}))\to 0$$

is exact (as follows from (A.6) and Lemma A.0.10). Thus M is a $K_0\otimes_{\mathbf{Q}_p} L[\varepsilon]$-module, and $M/\varepsilon M$ is free of rank one over $K_0\otimes_{\mathbf{Q}_p} L$. If m is the lift to M of any basis vector, then the submodule $(K_0\otimes_{\mathbf{Q}_p} L[\varepsilon])\cdot m\subset M$ can be checked to be free of rank one over $K_0\otimes_{\mathbf{Q}_p} L[\varepsilon]$ (compare with the proof of "(d) implies (b)" in [14, Lemma 3.3]). Since M and $(K_0\otimes_{\mathbf{Q}_p} L[\varepsilon])\cdot m$ have the same length over $K_0\otimes_{\mathbf{Q}_p} L$, they must be equal. This completes the proof. $\square$

By Lemma A.0.11 we now have a short exact sequence of L-vector spaces

$$0\to \mathfrak{t}_D^{\mathrm{Ref},\mathrm{HT}}/H^1_f(\operatorname{ad} D) \to \mathfrak{t}_D^{\mathrm{Ref}}/H^1_f(\operatorname{ad} D) \xrightarrow{d\widetilde{\eta}} \bigoplus_{\tau\in\Sigma_K} L^{\oplus 2} \tag{A.4}$$

Recall that the critical type of the triangulation (A.4) is the element $c\in S_2^{\Sigma_K}$ (S_2 being permutations on $\{1,2\}$) so that $\mathrm{HT}_\tau(\delta_i) = h_{c_\tau(i),\tau}$.

Lemma A.0.5. *If $\widetilde{D}\in \mathfrak{t}_D^{\mathrm{Ref}}$ then $d\widetilde{\eta}_{i,\tau} = d\widetilde{\eta}_{c_\tau(i),\tau}$ for each $i=1,2$ and all $\tau\in\Sigma_K$. In particular,*

$$\dim_L \mathfrak{t}_D^{\mathrm{Ref}}/H^1_f(\operatorname{ad} D) \leq \dim_L \mathfrak{t}_D^{\mathrm{Ref},\mathrm{HT}}/H^1_f(\operatorname{ad} D) + 2(K:\mathbf{Q}_p) - \#\{\tau \mid c_\tau\neq 1\}$$

Proof. The first claim of the lemma easily implies the second claim by (A.7) and bounding the dimension of the image of $d\widetilde{\eta}$. The first claim of the lemma is contained in the proof of [15, Theorem 7.1 and Lemma 7.2], but with some unecessary hypotheses. We give a proof here for convenience.

First, write $D' = D\otimes \mathrm{LT}_{\varpi_K}(z^{h_1})$, $\delta_1' = \delta_1 \mathrm{LT}_{\varpi_K}(z^{h_1})$ and $\widetilde{D}' = D\otimes \mathrm{LT}_{\varpi_K}(z^{\widetilde{\eta}_1})$. Thus $\widetilde{D}'$ is an element of $\mathrm{Ext}^1_{(\varphi,\Gamma_K)}(D',D')$. We consider the map $\mathrm{Ext}^1_{(\varphi,\Gamma_K)}(D',D')\to \mathrm{Ext}^1_{(\varphi,\Gamma_K)}(\delta_1',D') = H^1(D'({\delta_1'}^{-1}))$, and we write $\widetilde{D}_1'$ for the image of $\widetilde{D}_1$ in that space.

To prove the lemma we claim it is enough to show that $\widetilde{D}_1'$ lands inside the subspace $H^1_f(D'({\delta_1'}^{-1}))$. Indeed, the matrix of Sen's operator on $\widetilde{D}'$ (viewed just a (φ,Γ_K)-module over $\mathcal{R}_{K,L}$ now) in the basis induced from (A.4) is given by

$$\begin{pmatrix} \mathrm{HT}_\tau(\delta_1)-h_{1,\tau} & & d\widetilde{\eta}_{c_\tau(1),\tau} - d\widetilde{\eta}_{1,\tau} & \\ & \mathrm{HT}_\tau(\delta_2)-h_{1,\tau} & & d\widetilde{\eta}_{c_\tau(2),\tau}-d\widetilde{\eta}_{1,\tau} \\ & & \mathrm{HT}_\tau(\delta_1)-h_{1,\tau} & \\ & & & \mathrm{HT}_\tau(\delta_2)-h_{1,\tau}\end{pmatrix}$$

[23]Warning: these are not equivalent conditions unless D is crystalline.

and the matrix of the Sen operator on $\widetilde{D}_1'$ is the upper 3×3-block

$$\begin{pmatrix} \mathrm{HT}_\tau(\delta_1) - h_{1,\tau} & & d\widetilde{\eta}_{c_\tau(1),\tau} - d\widetilde{\eta}_{1,\tau} \\ & \mathrm{HT}_\tau(\delta_2) - h_{1,\tau} & \\ & & \mathrm{HT}_\tau(\delta_1) - h_{1,\tau} \end{pmatrix}.$$

If $\widetilde{D}_1' \in H^1_f(D'(\delta_1'^{-1}))$ then $\widetilde{D}_1'$ is Hodge–Tate, though, and so we must have $d\widetilde{\eta}_{c_\tau(1),\tau} = d\widetilde{\eta}_{1,\tau}$. For $i=2$ one can re-do the proof with determinants (or, equivalently, duals).

It remains to prove $\widetilde{D}_1' \in H^1_f(D'(\delta_1'^{-1}))$. Explicitly, $\widetilde{D}_1'$ is explicitly constructed as

$$\widetilde{D}_1' = \ker\left(\widetilde{D}' \twoheadrightarrow D' \twoheadrightarrow \delta_2 \otimes \mathrm{LT}_{\varpi_K}(z^{h_1})\right),$$

and we also have a short exact sequence

$$0 \to D_{\mathrm{crys}}(D')^{\varphi^f = \Phi_{\varpi_K}} \to D_{\mathrm{crys}}(\widetilde{D}_1')^{(\varphi^f = \Phi_{\varpi_K})} \to D_{\mathrm{crys}}(\delta_1')^{\varphi^f=\Phi_{\varpi_K}} \tag{A.5}$$

Here $(-)^{(*)}$ means "generalized eigenspace" for $(*)$. By construction of $\widetilde{D}_1'$ and Lemma A.0.8(b), we know that $D_{\mathrm{crys}}(\widetilde{D}_1')^{(\varphi^f=\Phi_{\varpi_K})} = D_{\mathrm{crys}}(\widetilde{D})^{(\varphi^f=\Phi_{\varpi_K})}$ has dimension $2(K_0 : \mathbf{Q}_p)$ over L (since the right-hand side is free of rank one over $K_0 \otimes_{\mathbf{Q}_p} L[\varepsilon]$. Applying Lemma A.0.8(a), the two outside terms of (A.8) separately each have L-dimension $(K_0 : \mathbf{Q}_p)$. We thus deduce that (A.8) is exact on the right-hand side. And now it follows that $\widetilde{D}_1' \in H^1_f(D'({\delta_1'}^{-1}))$ (by Lemma A.0.10). $\square$

Lemma A.0.6. *$H^2(D \otimes \delta_2^{-1}) = (0)$.*

Proof. By local Tate duality it is enough to show that $\mathrm{Hom}_{(\varphi,\Gamma_K)}(D, \mathcal{R}_{K,L}(\delta_2\varepsilon)) = (0)$. Consider the inclusion

$$0 \to \mathrm{Hom}_{(\varphi,\Gamma_K)}(D, \mathcal{R}_{K,L}(\delta_2\varepsilon)) \to \mathrm{Hom}_{(\varphi,\Gamma_K)}(\mathcal{R}_{K,L}(\delta_1), \mathcal{R}_{K,L}(\delta_2\varepsilon))$$

Write $P = \ker(f)$ where $f : D \to \mathcal{R}_{K,L}(\delta_2\varepsilon)$. Assume $f \neq 0$. Then, P is a rank one (φ,Γ_K)-submodule of D. Moreover, the quotient $D/P \subset \mathcal{R}_{K,L}(\delta_2\varepsilon)$ must contain $\mathcal{R}_{K,L}(\delta_1)$ is a (φ,Γ_K)-submodule, and so $D_{\mathrm{crys}}(D/P) = D_{\mathrm{crys}}(\delta_1)$. Computing crystalline eigenvalues on D_{st} we see that $D_{\mathrm{crys}}(P) = D_{\mathrm{crys}}(\delta_2)$. Since *a priori*, $D_{\mathrm{crys}}(P) \subset D_{\mathrm{crys}}(D)$ we deduce that $D_{\mathrm{crys}}(D)$ has two distinct crystalline eigenvalues (Lemma A.0.8(b)). This is a contradiction to hypothesis (st). $\square$

Remark A.0.7. If the triangulation (A.4) is critical (i.e. has a non-trivial critical-type) then the previous lemma follows immediately from observing that $H^2(\delta_1\delta_2^{-1})) = (0)$.

We are now ready to give the crucial estimate for the left-hand term in (A.7).

Proposition A.0.8. *$\dim_L \mathfrak{t}_D^{\mathrm{Ref,HT}}/H^1_f(\mathrm{ad}\, D) \leq \#\{\tau \mid c_\tau \neq 1\}$.*

Proof. We will prove this in a series of steps.

Claim (Step 1). There is a natural diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1_f(D\otimes\delta_2^{-1}) & \longrightarrow & H^1_f(\mathrm{ad}\, D) & \longrightarrow & H^1_f(D\otimes \delta_1^{-1}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^1(D\otimes\delta_2^{-1}) & \longrightarrow & H^1(\mathrm{ad}\, D) & \longrightarrow & H^1(D\otimes \delta_1^{-1}) & & \end{array} \tag{A.6}$$

with exact rows.

To prove this claim, it is enough to show that the natural map $H^0(\mathrm{ad}\, D) \to H^0(D \otimes \delta_1^{-1})$ is surjective (the second row is then exact from the long exact sequence in cohomology, and the first row from [13, Corollary 1.4.6]). A non-zero map between rank one (φ, Γ_K)-modules is automatically injective and induces an isomorphism on $D_{\mathrm{crys}}(-)$. Thus, by Lemma A.0.8 we see that $H^0(\delta_2\delta_1^{-1}) = (0)$. Because D is non-split (since D is not crystalline) it follows as once that $H^0(D \otimes \delta_2^{-1}) = (0)$ and $H^0(D \otimes \delta_1^{-1})$ is one-dimensional. The surjectivity now follows.

Claim (Step 2). There is a natural inclusion $\mathfrak{t}_D^{\mathrm{Ref,HT}}/H^1_f(\mathrm{ad}\, D) \subset H^1_{/f}(D \otimes \delta_2^{-1})$.

Indeed, apply the snake lemma to the diagram (A.9) to deduce a second diagram

$$
\begin{tikzcd}
& \mathfrak{t}_D^{\mathrm{Ref,HT}}/H^1_f(\mathrm{ad}\, D) \arrow[d] & \\
0 \arrow[r] & H^1_{/f}(D \otimes \delta_2^{-1}) \arrow[r] & H^1_{/f}(\mathrm{ad}\, D) \arrow[r] & H^1_{/f}(D \otimes \delta_1^{-1}).
\end{tikzcd} \tag{A.7}
$$

And then the proof of Lemma A.0.12 implies that the composition from the top to the lower right is trivial. This completes the proof of Step 2.

Note now that, since $H^2(D \otimes \delta_2^{-1}) = (0)$ (Lemma A.0.13), we have

$$
\begin{aligned}
\dim_L H^1_{/f}(D \otimes \delta_2^{-1}) &= 2(K : \mathbf{Q}_p) - \#\{\tau \mid \mathrm{HT}(\delta_1) - \mathrm{HT}_\tau(\delta_2) < 0\} \\
&= (K : \mathbf{Q}_p) + \#\{\tau \mid \mathrm{HT}_\tau(\delta_1) > \mathrm{HT}_\tau(\delta_2)\} \\
&= (K : \mathbf{Q}_p) + \#\{\tau \mid c_\tau \neq 1\}.
\end{aligned} \tag{A.8}
$$

This bound is too coarse, so we must continue computing.

Claim (Step 3). The natural map $H^1_f(\delta_2 \otimes \delta_2^{-1}) \to H^2(\delta_1\delta_2^{-1})$ is surjective.

By local Tate duality and the orthogonality of the H^1_f, it is equivalent to show that $H^0(\delta_2\delta_1^{-1}\varepsilon) \to H^1_{/f}(\delta_2\delta_2^{-1}\varepsilon)$ is injective. But this map is explicitly defined by sending a non-zero morphism $\iota : \mathcal{R}_{K,L} \to \mathcal{R}_{K,L}(\delta_2\delta_1^{-1}\varepsilon)$ to the pullback D_ι fitting into a diagram

$$
\begin{tikzcd}
0 \arrow[r] & \mathcal{R}_{K,L}(\delta_2\delta_2^{-1}\varepsilon) \arrow[r] \arrow[d, equal] & D_\iota \arrow[r] \arrow[d] & \mathcal{R}_{K,L} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{R}_{K,L}(\delta_2\delta_2^{-1}\varepsilon) \arrow[r] & D^\vee(\delta_2\varepsilon) \arrow[r] & \mathcal{R}_{K,L}(\delta_2\delta_1^{-1}\varepsilon) \arrow[r] & 0.
\end{tikzcd} \tag{A.9}
$$

The vertical arrows in (A.12) are all injections by construction. Since D is semi-stable, non-crystalline, the same is true for $D^\vee(\delta_2\varepsilon)$ and thus also for D_ι. This shows that $D_\iota \notin H^1_f(\delta_2\delta_2^{-1}\varepsilon)$, completing the proof of Step 3.

In Step 2 we proved that $\mathfrak{t}_D^{\mathrm{Ref,HT}}/H^1_f(\mathrm{ad}\, D) \subset H^1_{/f}(D \otimes \delta_2^{-1})$. We now upgrade this to the following.

Claim (Step 4). $\mathfrak{t}_D^{\mathrm{Ref,HT}}/H^1_f(\mathrm{ad}\, D) \subset \ker\Big(H^1_{/f}(D \otimes \delta_2^{-1})) \to H^1_{/f}(\delta_2\delta_2^{-1})\Big)$.

The proof of this claim follows from the methods in [14]. Indeed, let $\widetilde{D} \in \mathfrak{t}_D^{\mathrm{Ref,HT}}$. After changing $\widetilde{D}$ by an element in $H^1_f(\mathrm{ad}\, D)$ we may suppose that $\widetilde{D}$ lies in the image of $H^1(D \otimes \delta_2^{-1}) \to H^1(\mathrm{ad}\, D)$. By [14, Lemma 3.6(a)] there is a constant deformation $\mathcal{R}_{K,L}(\delta_1)[\varepsilon] \hookrightarrow \widetilde{D}$ with saturated image. Write $\mathcal{R}_{K,L[\varepsilon]}(\widetilde{\delta}_2)$ for the cokernel. By [14, Lemma 3.6(b)] the image of $\widetilde{D}$ in $H^1(\delta_2\delta_2^{-1})$ is the deformation $\widetilde{\delta}_2$ of δ_2. But $\widetilde{\delta}_2$ is Hodge–Tate because $\widetilde{D}$ is, and a Hodge–Tate deformation of a crystalline character is a crystalline character, whence $\widetilde{D}$ has trivial image in $H^1_{/f}(\delta_2\delta_2^{-1})$. This completes the proof of Step 4.

We can now put together the proof of the proposition. First, in Step 3 we proved that $H^1_f(\delta_2\otimes\delta_2^{-1}) \to H^2(\delta_1\delta_2^{-1})$ is onto, so it follows from the long exact sequence in cohomology that the natural map

$$H^1_{/f}(D\otimes\delta_2^{-1}) \to H^1_{/f}(\delta_2\delta_2^{-1}) = H^1_{/f}(L)$$

is surjective. Then, $\dim_L H^1_{/f}(L) = (K:\mathbf{Q}_p)$ and by (A.11) we have $\dim H^1_{/f}(D\otimes\delta_2^{-1}) = (K:\mathbf{Q}_p) + \#\{\tau \mid c_\tau \neq 1\}$. Thus from Step 4 we deduce that

$$\dim_L \mathfrak{t}_D^{\mathrm{Ref},\mathrm{HT}}/H^1_f(\operatorname{ad} D) \leq \#\{\tau \mid c_\tau \neq 1\}$$

as promised. □

Corollary A.0.9. $\dim_L \mathfrak{t}_D^{\mathrm{Ref}}/H^1_f(\operatorname{ad} D) \leq 2(K:\mathbf{Q}_p)$.

Proof. This follows from Lemma A.0.12 and Proposition A.0.15. □

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