Remarks on nearby cycles of formal schemes

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Abstract

We prove that the nearby cycles functor associated with the generic fiber of a p-adic formal scheme has many properties in common with derived pushforward along a proper map of schemes. In particular, we show that this functor has a right adjoint, satisfies the projection formula, and commutes with Verdier duality.

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1 Introduction

Let K be a nonarchimedean field, with ring of integers $\mathcal{O}$ and residue field k, and let $\mathfrak{X}$ be a formal scheme over $\mathrm{Spf}\,\mathcal{O}$ which is locally of topologically finite type. By a classical construction of Raynaud, there is a functorially associated rigid analytic generic fiber $X = \mathfrak{X}^{\mathrm{an}}$, which we interpret as an adic space over $\mathrm{Spa}(K, \mathcal{O})$. There is a natural map of sites

$$\lambda_{\mathfrak{X}} : X_{\text{ét}} \to \mathfrak{X}_{\text{ét}}$$

corresponding to the functor sending an étale map $\mathfrak{Y} \to \mathfrak{X}$ to the associated map on rigid generic fibers. Fix a Noetherian coefficient ring Λ killed by some positive integer, so $\lambda_{\mathfrak{X}}$ induces a pushforward functor $\lambda_{\mathfrak{X}*}$ on sheaves of Λ-modules. We refer to the associated derived functor

$$R\lambda_{\mathfrak{X}*} : D(X_{\text{ét}}, \Lambda) \to D(\mathfrak{X}_{\text{ét}}, \Lambda)$$

as the *nearby cycles functor* of $\mathfrak{X}$. This functor was introduced and studied (separately) by Berkovich and Huber [Ber94, Hub96], who established many of its important properties; in particular, when $\mathfrak{X}$

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is the formal completion of a finite type $\mathcal{O}$-scheme, they proved a comparison with classical nearby cycles.

In this note we prove some new properties of $R\lambda_{\mathfrak{X}*}$. Roughly, our results show that $R\lambda_{\mathfrak{X}*}$ behaves like Rf_* for f a *proper* morphism of schemes: we prove that $R\lambda_{\mathfrak{X}*}$ admits a well-behaved right adjoint, satisfies the "projection formula", commutes with Verdier duality, and satisfies some other standard formulas. We first proved some of these results in [Han17], but the arguments presented here are (in our opinion) much more satisfying.

We now state our results precisely. In the remainder of this note, we impose the following condition on the pair (K, Λ):

(†) **The groups $\mathrm{Gal}(\overline{K}/K)$ and $\mathrm{Gal}(\overline{k}/k)$ have finite ℓ-cohomological dimension for all primes ℓ nilpotent in Λ.**

This condition is very mild, and holds for example if K is algebraically closed, or if K is a local field.

Theorem 1.1. *Let $\mathfrak{X}$ be a formal scheme locally of topologically finite type over $\mathrm{Spf}\,\mathcal{O}$, with rigid generic fiber X. Fix a coefficient ring Λ as above, and assume that the condition (†) is true. Then*

o. *The derived categories $D(X_{\text{ét}}, \Lambda)$ and $D(\mathfrak{X}_{\text{ét}}, \Lambda)$ are compactly generated.*

i. *The functor $R\lambda_{\mathfrak{X}*}$ admits a right adjoint*

$$R\lambda_{\mathfrak{X}}^! : D(\mathfrak{X}_{\text{ét}}, \Lambda) \to D(X_{\text{ét}}, \Lambda).$$

ii. (Projection Formula) *For any $A \in D(\mathfrak{X}_{\text{ét}}, \Lambda)$ and $B \in D(X_{\text{ét}}, \Lambda)$, the natural projection map*

$$A \otimes^{\mathbf{L}} R\lambda_{\mathfrak{X}*}B \to R\lambda_{\mathfrak{X}*}\left(\lambda_{\mathfrak{X}}^* A \otimes^{\mathbf{L}} B\right)$$

is an isomorphism.

iii. *For any $A \in D(X_{\text{ét}}, \Lambda)$ and $B \in D(\mathfrak{X}_{\text{ét}}, \Lambda)$, there is a natural isomorphism*

$$R\lambda_{\mathfrak{X}*}R\mathscr{H}\mathrm{om}_X(A, R\lambda_{\mathfrak{X}}^! B) \cong R\mathscr{H}\mathrm{om}_{\mathfrak{X}}(R\lambda_{\mathfrak{X}*}A, B).$$

iv. *For any $A, B \in D(\mathfrak{X}_{\text{ét}}, \Lambda)$, there is a natural isomorphism*

$$R\lambda_{\mathfrak{X}}^! R\mathscr{H}\mathrm{om}_{\mathfrak{X}}(A, B) \cong R\mathscr{H}\mathrm{om}_X(\lambda_{\mathfrak{X}}^* A, R\lambda_{\mathfrak{X}}^! B).$$

As in the case of schemes, parts iii. and iv. are formal consequences of i. and ii. For o., which is presumably well-known, one explicitly writes down generating sets of compact objects S_X and $S_{\mathfrak{X}}$; this is where the condition (†) shows up. The proof of i. is a simple application of Neeman's adjoint functor theorems [Nee96], together with the fact that $\lambda_{\mathfrak{X}}^*$ transforms objects in $S_{\mathfrak{X}}$ into objects in S_X. Note that it is essential to work with unbounded derived categories here. A related argument shows that to prove the projection formula, it is enough to treat the special case where $A \in S_{\mathfrak{X}}$. The special shape of these objects then reduces us to the following result.

Proposition 1.2. *Let $\mathfrak{j} : \mathfrak{Y} \to \mathfrak{X}$ be an étale map of affine formal schemes, and let $j : Y \to X$ be the induced étale map on rigid generic fibers. Then the natural transformation $\mathfrak{j}_! R\lambda_{\mathfrak{Y}*} \to R\lambda_{\mathfrak{X}*} j_!$ defined as the adjoint to the map*

$$R\lambda_{\mathfrak{Y}*} \to R\lambda_{\mathfrak{Y}*} j^* j_! \cong \mathfrak{j}^* R\lambda_{\mathfrak{X}*} j_!.$$

is an isomorphism.

This was observed by Huber in the special case where j is an open immersion, but the general case is somewhat trickier. Note that in the schemes setting, the projection formula is usually reduced by proper base change to the case where the base scheme is a geometric point, in which case it is essentially trivial. Since the analogue of proper base change fails for the nearby cycles functor, it is perhaps surprising that the projection formula still holds.

We next give some applications to Verdier duality.

Theorem 1.3. *Let $\mathfrak{X}$ and $X = \mathfrak{X}^{\mathrm{an}}$ be as above. Suppose that K is algebraically closed, and that Λ is a Gorenstein ring of dimension zero with $\mathrm{char}(k)$ invertible in Λ. Let $\omega_{\mathfrak{X}} \in D(\mathfrak{X}_{\text{ét}}, \Lambda)$ be a dualizing complex for $\mathfrak{X}$. Then*

i. *The complex $\omega_X = R\lambda_{\mathfrak{X}}^! \omega_{\mathfrak{X}}$ is the dualizing complex of X, cf. Definition 2.9.*

ii. *Writing $\mathbf{D}_\bullet(-) = R\mathscr{H}\mathrm{om}_\bullet(-, \omega_\bullet)$ for the natural Verdier duality functors on $D(\bullet_{\text{ét}}, \Lambda)$ for either $\bullet \in \{\mathfrak{X}, X\}$, there is a natural isomorphism of functors $\mathbf{D}_{\mathfrak{X}} R\lambda_{\mathfrak{X}*} \cong R\lambda_{\mathfrak{X}*} \mathbf{D}_X$.*

iii. *There is a natural isomorphism of functors $\mathbf{D}_X \lambda_{\mathfrak{X}}^* \cong R\lambda_{\mathfrak{X}}^! \mathbf{D}_{\mathfrak{X}}$. In particular, for any $A \in D_c^b(\mathfrak{X}_{\text{ét}}, \Lambda)$, $R\lambda_{\mathfrak{X}}^! A$ can be computed as*

$$R\lambda_{\mathfrak{X}}^! A \cong \mathbf{D}_X \lambda_{\mathfrak{X}}^* \mathbf{D}_{\mathfrak{X}} A.$$

Note that $\mathfrak{X}$ always admits an (essentially unique) dualizing complex, by e.g. the work of Liu-Zheng [LZ17] (cf. also Laszlo-Olsson [LO08]). Part ii. and the first claim in part iii. of this result are immediate upon combining part i. with the isomorphisms of Theorem 1.1.iii-iv., by setting $B = \omega_{\mathfrak{X}}$ in the latter results. The second claim in part iii. follows by combining the first claim with the biduality isomorphism $A \cong \mathbf{D}_{\mathfrak{X}} \mathbf{D}_{\mathfrak{X}} A$, which holds for any $A \in D_c^b(\mathfrak{X}_{\text{ét}}, \Lambda)$ by [Del77, Théorèmes de finitude, Th. 4.3].

Theorem 1.3.ii was first observed by Scholze, who gave a proof (unpublished) relying on the theory of ∞-categories. In [Han17] we gave a proof of this result which used less heavy machinery, but our argument was still somewhat tortuous, since at the time we weren't aware of the projection formula. The argument given here seems to be the "correct" one.

As in [Han17], one deduces from Theorem 1.3.ii the following important corollary, which again was first observed by Scholze.

Corollary 1.4. *Let X be a separated quasicompact rigid analytic space over a complete algebraically closed nonarchimedean field K, so X has a dualizing complex ω_X and a Verdier duality functor $\mathbf{D}_X$ as above. Let $A \in D^b(X_{\text{ét}}, \Lambda)$ be an object such that for any affine formal scheme $\mathfrak{U} = \mathrm{Spf}\, R$ with an étale map $f : U = \mathfrak{U}^{\mathrm{an}} \to X$, the complex $R\lambda_{\mathfrak{U}*} f^* A \in D^b(\mathfrak{U}_{\text{ét}}, \Lambda)$ has constructible cohomology sheaves. Then A is* reflexive*: the natural biduality map $A \to \mathbf{D}_X \mathbf{D}_X A$ is an isomorphism.*

Remarks on terminology and conventions

Let $\mathcal{O}$ be the ring of integers in a nonarchimedean field as above; by a "formal scheme" over $\mathrm{Spf}\, \mathcal{O}$, we shall always mean a formal scheme which is locally of topologically finite type. If $\mathfrak{X} = (|\mathfrak{X}|, \mathcal{O}_{\mathfrak{X}})$ is such a formal scheme, then the ringed space $\mathfrak{X}_s = (|\mathfrak{X}|, \mathcal{O}_{\mathfrak{X}} \otimes_{\mathcal{O}} k)$ is a scheme locally of finite type over the residue field k, which we call the special fiber of $\mathfrak{X}$. We will repeatedly use without comment the fact that the functor $\mathfrak{Y} \mapsto \mathfrak{Y}_s$ induces an equivalence from the étale site of $\mathfrak{X}$ to the étale site of $\mathfrak{X}_s$.

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2.6.

2 Proofs

2.1 Compact generation and the right adjoint

In this section we prove Theorem 1.1.o.-i. We use Neeman's adjoint functor theorems in the following form.

Proposition 2.1. *Let $\mathcal{S}$ and $\mathcal{T}$ be compactly generated triangulated categories, and let $F : \mathcal{S} \to \mathcal{T}$ be a triangulated functor respecting coproducts, with right adjoint $G : \mathcal{T} \to \mathcal{S}$. Suppose moreover that $\mathcal{S}$ has a generating set of compact objects $S \subset \mathrm{Ob}(\mathcal{S})$ such that $F(s)$ is compact for every $s \in S$. Then G respects coproducts, and admits a right adjoint $H : \mathcal{S} \to \mathcal{T}$.*

Proof. This is immediate from combining Theorems 4.1 and 5.1 of [Nee96]. $\square$

Proposition 2.2. *Let $\mathfrak{X}$ be a formal scheme with rigid generic fiber X. Let $\mathfrak{j} : \mathfrak{Y} \to \mathfrak{X}$ be an étale map of formal schemes, and let $j : Y \to X$ be the associated étale map on rigid generic fibers. Then there is a natural isomorphism of functors on abelian étale sheaves $\lambda_{\mathfrak{X}}^* \mathfrak{j}_! \cong j_! \lambda_{\mathfrak{Y}}^*$.*

Proof. This is left adjoint to the obvious equivalence $\mathfrak{j}^* \lambda_{\mathfrak{X}*} \cong \lambda_{\mathfrak{Y}*} j^*$, which is a special case of "base change for a slice". $\square$

Proposition 2.3. *Let K, $\mathcal{O}$, k and Λ be as in the introduction.*

i. *Suppose that $\mathrm{Gal}(k^s/k)$ has finite ℓ-cohomological dimension for every prime ℓ nilpotent in Λ. Let $\mathfrak{X}$ be a formal scheme over $\mathrm{Spf}\,\mathcal{O}$, and let $S_{\mathfrak{X}} \subset \mathrm{Ob}\,(D(\mathfrak{X}_{\text{ét}}, \Lambda))$ be the set of objects of the form $\mathfrak{j}_! \Lambda[n]$, where n is any integer and $\mathfrak{j} : \mathfrak{Y} \to \mathfrak{X}$ is any étale map of formal schemes with $\mathfrak{Y}$ affine. Then $S_{\mathfrak{X}}$ is a generating set of compact objects for $D(\mathfrak{X}_{\text{ét}}, \Lambda)$.*

ii. *Likewise, suppose that $\mathrm{Gal}(K^s/K)$ has finite ℓ-cohomological dimension for every prime ℓ nilpotent in Λ. Then if X is any rigid space over K, the set $S_X \subset \mathrm{Ob}\,(D(X_{\text{ét}}, \Lambda))$ of objects of the form $j_! \Lambda[n]$, where again n is any integer and $j : Y \to X$ is any étale map of rigid spaces with Y affinoid, is a generating set of compact objects for $D(X_{\text{ét}}, \Lambda)$.*

Proof. We treat the case of rigid spaces; by passing to special fibers, the case of formal schemes reduces immediately to the case of schemes locally of finite type over the residue field k, which is only easier.

Let $A \in D(X_{\text{ét}}, \Lambda)$. The stalk of any cohomology sheaf $\mathscr{H}^n(A)$ at any geometric point $\overline{x} \to X$ can be realized as a colimit

$$\varinjlim \mathrm{Hom}_{D(X_{\text{ét}},\Lambda)}(B_i, A)$$

with all $B_i \in S_X$, by taking $B_i = j_{i!}\Lambda[-n]$ for $j_i : Y_i \to X$ ranging over a cofinal set of affinoid étale neighborhoods of $\overline{x} \to X$. Thus S_X is a generating set. For compacity, note that for a given étale map $j : Y \to X$ with Y affinoid we have

$$\mathrm{Hom}_{D(X_{\text{ét}},\Lambda)}(j_! \Lambda[n], A) = \mathrm{Hom}_{D(Y_{\text{ét}},\Lambda)}(\Lambda, j^* A[-n]) = H^{-n}(R\Gamma(Y, j^* A)).$$

Since $H^n(Y, \mathscr{F})$ vanishes for all $\mathscr{F} \in \mathrm{Sh}(Y_{\text{ét}}, \Lambda)$ and all $n > 2\mathrm{dim} X + \sup_{\ell \notin \Lambda^\times} \mathrm{coh.dim}_\ell(\mathrm{Gal}(K^s/K))$ by combining [Hub96, Corollary 2.8.3] and [Hub96, Corollary 1.8.7], there is some large integer N independent of A such that the natural map

$$H^{-n}(R\Gamma(Y, j^* A)) \to H^{-n}(R\Gamma(Y, j^* \tau^{\geq -N} A)) = H^{-n}(R\Gamma(Y, \tau^{\geq -N} j^* A))$$

is an isomorphism. Moreover $\tau^{\geq -N}$ and j^* commute with direct sums by inspection (in fact, they are both left adjoints). It thus suffices to prove that the functor $R\Gamma(Y_{\text{ét}}, -) : D^{\geq -N}(Y_{\text{ét}}, \Lambda) \to D(\text{Mod}_\Lambda)$ commutes with direct sums. This reduces to the fact that $H^i(Y_{\text{ét}}, -)$ commutes with direct sums, which is a special case of [Hub96, Lemma 2.3.13]. □

Proposition 2.4. *Maintain the notation and assumptions of the previous proposition, and suppose that $X = \mathfrak{X}^{\text{an}}$ is the rigid generic fiber of $\mathfrak{X}$. Then $\lambda_{\mathfrak{X}}^* B \in S_X$ for any $B \in S_{\mathfrak{X}}$. Thus $R\lambda_{\mathfrak{X}*}$ commutes with coproducts, and therefore has a right adjoint $R\lambda_{\mathfrak{X}}^!$.*

Proof. Immediate upon combining Propositions 2.1-2.3. □

2.2 The projection formula and its consequences

In this section we conclude the proof of Theorem 1.1. The essential point is to prove the projection formula, which we restate for the reader's convenience:

Theorem 2.5. *Let $\mathfrak{X}$ be a formal scheme with rigid generic fiber X. Then for any $A \in D(\mathfrak{X}_{\text{ét}}, \Lambda)$ and $B \in D(X_{\text{ét}}, \Lambda)$, the natural projection map*

$$A \otimes^{\mathbf{L}} R\lambda_{\mathfrak{X}*} B \to R\lambda_{\mathfrak{X}*} \left(\lambda_{\mathfrak{X}}^* A \otimes^{\mathbf{L}} B \right)$$

adjoint to the composition

$$\lambda_{\mathfrak{X}}^* \left(A \otimes^{\mathbf{L}} R\lambda_{\mathfrak{X}*} B \right) \cong \lambda_{\mathfrak{X}}^* A \otimes^{\mathbf{L}} \lambda_{\mathfrak{X}}^* R\lambda_{\mathfrak{X}*} B \to \lambda_{\mathfrak{X}}^* A \otimes^{\mathbf{L}} B$$

is an isomorphism.

Proof. Quite generally, let $F, F' : \mathcal{S} \to \mathcal{T}$ be triangulated functors between triangulated categories, and let $\phi : F \to F'$ be a natural transformation compatible with shifts. Suppose moreover that F and F' commute with coproducts, that $\mathcal{S}$ is compactly generated, and that $\phi(s) : F(s) \to F'(s)$ is an isomorphism for all s in a generating set of compact objects $S \subset \text{Ob}(\mathcal{S})$. Then ϕ is a natural isomorphism. Indeed, the assumptions guarantee that ϕ restricts to a natural isomorphism on the smallest full subcategory $\mathcal{S}'$ of $\mathcal{S}$ which contains all objects of S and is closed under coproducts and triangles; on the other hand, $\mathcal{S}' = \mathcal{S}$ by [Nee96, Theorem 2.1.2].

We now apply this observation to the case where $\mathcal{S} = \mathcal{T} = D(\mathfrak{X}_{\text{ét}}, \Lambda)$, $B \in D(X_{\text{ét}}, \Lambda)$ is fixed, $F(-) = - \otimes^{\mathbf{L}} R\lambda_{\mathfrak{X}*} B$, $F'(-) = R\lambda_{\mathfrak{X}*} \left(\lambda_{\mathfrak{X}}^* - \otimes^{\mathbf{L}} B \right)$, and ϕ is the projection map defined in the theorem. The only nontrivial point is that F' commutes with coproducts, which follows from Proposition 2.4. Since moreover everything commutes with shifts, it thus suffices to prove that the projection map is an isomorphism in the special case where $A = \mathfrak{j}_! \Lambda$, for $\mathfrak{j} : \mathfrak{Y} \to \mathfrak{X}$ any étale map from an affine formal scheme. Since everything is local on $\mathfrak{X}$, we can assume that $\mathfrak{X}$ is affine as well.

Fix such an A, so the projection map in question is adjoint to the map

$$\lambda_{\mathfrak{X}}^* \left(\mathfrak{j}_! \Lambda \otimes^{\mathbf{L}} R\lambda_{\mathfrak{X}*} B \right) \cong \lambda_{\mathfrak{X}}^* \mathfrak{j}_! \Lambda \otimes^{\mathbf{L}} \lambda_{\mathfrak{X}}^* R\lambda_{\mathfrak{X}*} B \to \lambda_{\mathfrak{X}}^* \mathfrak{j}_! \Lambda \otimes^{\mathbf{L}} B.$$

We can rewrite the source of this as $\lambda_{\mathfrak{X}}^* \mathfrak{j}_! \mathfrak{j}^* R\lambda_{\mathfrak{X}*} B \cong \lambda_{\mathfrak{X}}^* \mathfrak{j}_! R\lambda_{\mathfrak{Y}*} j^* B$ and the target as $j_! \Lambda \otimes^{\mathbf{L}} B \cong j_! j^* B$, so we get a map

$$\lambda_{\mathfrak{X}}^* \mathfrak{j}_! R\lambda_{\mathfrak{Y}*} j^* B \to j_! j^* B$$

which one checks coincides with the natural map

$$\lambda_{\mathfrak{X}}^* \mathfrak{j}_! R\lambda_{\mathfrak{Y}*} j^* B \cong j_! \lambda_{\mathfrak{Y}}^* R\lambda_{\mathfrak{Y}*} j^* B \to j_! j^* B,$$

and which on taking right adjoints presents the projection map as a map

$$\mathfrak{j}_! R\lambda_{\mathfrak{Y}*} j^* B \to R\lambda_{\mathfrak{X}*} j_! j^* B.$$

Thus it suffices to prove that the transformation τ defined in the subsequent claim is an isomorphism.

Claim. *The natural transformation $\tau : \mathfrak{j}_! R\lambda_{\mathfrak{Y}*} \to R\lambda_{\mathfrak{X}*} j_!$ defined as the right adjoint to the composition*

$$\gamma : \lambda^*_{\mathfrak{X}} \mathfrak{j}_! R\lambda_{\mathfrak{Y}*} \cong j_! \lambda^*_{\mathfrak{Y}} R\lambda_{\mathfrak{Y}*} \to j_!$$

coincides with the natural transformation defined as the left adjoint to the composition

$$\delta : R\lambda_{\mathfrak{Y}*} \to R\lambda_{\mathfrak{Y}*} j^* j_! \cong \mathfrak{j}^* R\lambda_{\mathfrak{X}*} j_!.$$

Proof. This follows from Construction 2.7 in [Zhe15]. Precisely, in the notation of loc. cit., take $\mathcal{Y}$ and $\mathcal{X}$ to be the étale topoi of X and $\mathfrak{X}$, respectively; take $f = \lambda_{\mathfrak{X}}$; and take $U = h_{\mathfrak{Y}}$. Then $\mathcal{U}$ and $\mathcal{V}$ are the étale topoi of $\mathfrak{Y}$ and Y, respectively, and $g = \lambda_{\mathfrak{Y}}$, $j = \mathfrak{j}$, and $j' = j$. Then in the notation of loc. cit, the base change map B_D is just the isomorphism $\mathfrak{j}^* R\lambda_{\mathfrak{X}*} \cong R\lambda_{\mathfrak{Y}*} j^*$, the isomorphism A_D is its left adjoint $\lambda^*_{\mathfrak{X}} \mathfrak{j}_! \cong j_! \lambda^*_{\mathfrak{Y}}$, and G_D is the natural transformation τ. □

We need to show that τ is an equivalence. To do this, choose a factorization $\mathfrak{j} = \mathfrak{p} \circ \mathfrak{h}$, where $\mathfrak{h} : \mathfrak{Y} \to \mathfrak{Y}'$ is an open immersion and $\mathfrak{p} : \mathfrak{Y}' \to \mathfrak{X}$ is a finite map. Let $Y \xrightarrow{h} Y' \xrightarrow{p} X$ be the induced maps on rigid generic fibers, so again h is an open immersion and p is finite. We will use the identifications $\mathfrak{p}_* = R\mathfrak{p}_*$ and $p_* = Rp_*$ without further comment. By a standard argument, a choice of such a factorization gives rise to natural isomorphisms $\mathfrak{j}_! = \mathfrak{p}_* \mathfrak{h}_!$ and $j_! = p_* h_!$, so on taking right adjoints we also get natural isomorphisms $\mathfrak{j}^* = \mathfrak{h}^* R\mathfrak{p}^!$ and $j^* = h^* Rp^!$. As usual, there is a base change map $\lambda^*_{\mathfrak{X}} \mathfrak{p}_* \to p_* \lambda^*_{\mathfrak{Y}'}$, which on taking right adjoints induces a natural transformation $\alpha : R\lambda_{\mathfrak{Y}'*} Rp^! \to R\mathfrak{p}^! R\lambda_{\mathfrak{X}*}$. This gives rise to a sequence of natural transformations

$$R\lambda_{\mathfrak{Y}*} j^* = R\lambda_{\mathfrak{Y}*} h^* Rp^! \cong \mathfrak{h}^* R\lambda_{\mathfrak{Y}'*} Rp^! \to \mathfrak{h}^* R\mathfrak{p}^! R\lambda_{\mathfrak{X}*} = \mathfrak{j}^* R\lambda_{\mathfrak{X}*},$$

and one checks directly that the composite of these maps coincides with the (inverse of the) base change isomorphism $\mathfrak{j}^* R\lambda_{\mathfrak{X}*} \cong R\lambda_{\mathfrak{Y}*} j^*$. Note also that composing $\alpha \circ p_*$ with the unit $\mathrm{id} \to Rp^! p_*$ gives rise to a natural transformation

$$\beta : R\lambda_{\mathfrak{Y}'*} \to R\mathfrak{p}^! R\lambda_{\mathfrak{X}*} p_*.$$

The idea now is to show that the composition τ' of the sequence of natural isomorphisms

$$\mathfrak{j}_! R\lambda_{\mathfrak{Y}*} = \mathfrak{p}_* \mathfrak{h}_! R\lambda_{\mathfrak{Y}*} \overset{(1)}{\cong} \mathfrak{p}_* R\lambda_{\mathfrak{Y}'*} h_! \overset{(2)}{\cong} R\lambda_{\mathfrak{X}*} p_* h_! = R\lambda_{\mathfrak{X}*} j_!$$

coincides with the natural transformation τ defined above, where the isomorphism (1) is a special case of [Hub96, Corollary 3.5.11.ii], and the isomorphism (2) is trivial. For this, observe that passing to right adjoints for the pair $(\mathfrak{p}_*, R\mathfrak{p}^!)$, the middle three terms here give rise to a sequence of natural transformations

$$\mathfrak{h}_! R\lambda_{\mathfrak{Y}*} \cong R\lambda_{\mathfrak{Y}'*} h_! \to R\mathfrak{p}^! R\lambda_{\mathfrak{X}*} p_* h_!.$$

By Lemma 2.6, the second transformation here is induced by the transformation β defined above. Next, passing to right adjoints for the pair $(\mathfrak{h}_!, \mathfrak{h}^*)$, this transforms further to give a sequence of natural transformations

$$R\lambda_{\mathfrak{Y}*} \to \mathfrak{h}^* R\lambda_{\mathfrak{Y}'*} h_! \to \mathfrak{h}^* R\mathfrak{p}^! R\lambda_{\mathfrak{X}*} p_* h_! = \mathfrak{j}^* R\lambda_{\mathfrak{X}*} j_!.$$

By the remarks in the previous paragraph, one checks that the composite of these maps coincides with the transformation δ appearing in the definition of τ. Therefore $\tau' = \tau$, so τ is a natural isomorphism. $\square$

In the course of the previous proof, we made use of the following lemma.

Lemma 2.6. *Let*

$$\begin{tikzcd} \mathcal{A} \arrow[r, "f'"] \arrow[d, "g'"'] & \mathcal{B} \arrow[d, "g"] \arrow[dl, Rightarrow] \\ \mathcal{C} \arrow[r, "f"] & \mathcal{D} \end{tikzcd}$$

be a 2-commutative diagram of ringed sites, so all four pushforward maps admit total right derived functors. Suppose moreover that Rg_* and Rg'_* admit right adjoints $Rg^!$ and $Rg'^!$, so the base change map $Lf^*Rg_* \to Rg'_*Lf'^*$ induces a map $\beta : Rf'_*Rg'^! \to Rg^!Rf_*$ by passing to right adjoints. Then the composite map

$$Rf'_* \to Rf'_*Rg'^!Rg'_* \xrightarrow{\beta} Rg^!Rf_*Rg'_*$$

is right adjoint to the canonical equivalence

$$Rg_*Rf'_* \cong Rf_*Rg'_*.$$

The following proof is due to Weizhe Zheng.

Proof. We argue more generally. Let

$$\begin{tikzcd} \mathcal{A} \arrow[r, "f'_*"] \arrow[d, "g'_*"'] & \mathcal{B} \arrow[d, "g_*"] \arrow[dl, Rightarrow] \\ \mathcal{C} \arrow[r, "f_*"] & \mathcal{D} \end{tikzcd}$$

be any 2-commutative diagram of categories. Assume that g_* and g'_* admit right adjoints $g^!$ and $g'^!$, and that f_* and f'_* admit left adjoints f^* and f'^*. Let

$$\alpha : f^*g_* \to g'_*g'^*f^*g_* \simeq g'_*f'^*g^*g_* \to g'_*f'^*$$

be the associated "base change map". There are then two natural ways to define a transformation $f'_*g'^! \to g^!f_*$. On the one hand, one can form the right adjoint β of α. On the other hand, one can form the "dual base change map"

$$\beta' : f'_*g'^! \to g^!g_*f'_*g'^! \simeq g^!f_*g'_*g'^! \to g^!f_*.$$

By [Zhe15, Construction 9.6], we have $\beta = \beta'$. Moreover, by the first square of (1.3) in [Ayo07, Proposition 1.1.9], β' is the unique transformation such that the composite

$$f'_* \to f'_*g'^!g'_* \xrightarrow{\beta'} g^!f_*g'_*$$

equals the composite

$$f'_* \to g^!g_*f'_* \simeq g^!f_*g'_*.$$

$\square$

Granted the projection formula, the proofs of Theorem 1.1.iii-iv. are standard. For the convenience of the reader, we indicate the proof of Theorem 1.1.iii, part iv. being similar.

For any $A \in D(X_{\text{ét}}, \Lambda)$ and $B, C \in D(\mathfrak{X}_{\text{ét}}, \Lambda)$ we have a sequence of natural isomorphisms

$$\begin{aligned} \operatorname{Hom}_{D(\mathfrak{X}_{\text{ét}},\Lambda)}(C, R\lambda_{\mathfrak{X}*} R\mathscr{H}\mathrm{om}_X(A, R\lambda_{\mathfrak{X}}^! B)) &\cong \operatorname{Hom}_{D(X_{\text{ét}},\Lambda)}(\lambda_{\mathfrak{X}}^* C, R\mathscr{H}\mathrm{om}_X(A, R\lambda_{\mathfrak{X}}^! B)) \\ &\cong \operatorname{Hom}_{D(X_{\text{ét}},\Lambda)}(\lambda_{\mathfrak{X}}^* C \otimes^{\mathbf{L}} A, R\lambda_{\mathfrak{X}}^! B) \\ &\cong \operatorname{Hom}_{D(\mathfrak{X}_{\text{ét}},\Lambda)}(R\lambda_{\mathfrak{X}*}(\lambda_{\mathfrak{X}}^* C \otimes^{\mathbf{L}} A), B) \\ &\cong \operatorname{Hom}_{D(\mathfrak{X}_{\text{ét}},\Lambda)}(C \otimes^{\mathbf{L}} R\lambda_{\mathfrak{X}*} A, B) \\ &\cong \operatorname{Hom}_{D(\mathfrak{X}_{\text{ét}},\Lambda)}(C, R\mathscr{H}\mathrm{om}_{\mathfrak{X}}(R\lambda_{\mathfrak{X}*} A, B)) \end{aligned}$$

by (respectively) adjointness of $(\lambda_{\mathfrak{X}}^*, R\lambda_{\mathfrak{X}*})$, tensor-hom adjunction, adjointness of $(R\lambda_{\mathfrak{X}*}, R\lambda_{\mathfrak{X}}^!)$, the projection formula, and tensor-hom adjunction. Since C is arbitrary, the result now follows from the Yoneda lemma.

2.3 Duality

In this section we prove Theorem 1.3.i. For affinoid $\mathfrak{X}$, the result is easy. The general case is more subtle, since it's not a priori clear what "the" dualizing complex of X should be. To resolve this, we make use of the following result, which relies on the ∞-categorical techniques developed in [LZ17]. In the following proposition, we freely employ some notation from [LZ17].

Proposition 2.7. *Let Rig_K denote the category of rigid analytic spaces over a fixed nonarchimedean field K. Fix a coefficient ring Λ killed by some positive integer. Then for every morphism $f : X \to Y$ in Rig_K, there is a functor $Rf_! : D(X_{\text{ét}}, \Lambda) \to D(Y_{\text{ét}}, \Lambda)$, and the assignment $f \rightsquigarrow Rf_!$ satisfies all expected properties:*

i. *When f is separated, taut and finite-dimensional, $Rf_!$ coincides with the functor defined by Huber.*

ii. *For any morphisms $f : X \to Y$ and $g : Y \to Z$, there is a natural isomorphism $Rg_! \circ Rf_! \simeq R(g \circ f)_!$ satisfying the usual triple composability condition.*

iii. *The functor $Rf_!$ satisfies proper base change and the projection formula, and commutes with all colimits. In particular, it admits a right adjoint $Rf^!$.*

iv. *The functor $Rf_!$ is the shadow of its "∞-categorical enhancement"*

$$\mathscr{R} : \mathrm{N}(\mathrm{Rig}_K) \to \mathcal{C}\mathrm{at}_\infty,$$

which is a functor $\mathscr{R}$ sending any vertex X to the ∞-derived category $\mathcal{D}(X_{\text{ét}}, \Lambda)$ and sending any edge $f : X \to Y$ to a functor $\mathscr{R}f_!$ which induces the functor $Rf_!$ after passing to homotopy categories. Moreover, $\mathscr{R}$ is the restriction of an "enhanced operation map"

$${}_{\mathrm{Rig}_K}\mathrm{EO}_!^* : \delta_{2,\{2\}}^* \mathrm{N}(\mathrm{Rig}_K)^{\mathrm{cart}} \to \mathcal{P}r_{\mathrm{st}}^{\mathrm{L}}$$

analogous to the map (3.9) in [LZ17].

v. *For any morphism $f : X \to Y$, the functor $\mathscr{R}f_!$ satisfies étale co-descent: for any étale cover $U_0 \to X$ in Rig_K with associated Cech nerve $j_\bullet : U_\bullet \to X$, $U_n = \underbrace{U \times_X U \times_X \cdots \times_X U}_{n+1}$, the natural transformation*

$$\varinjlim_p \mathscr{R}(f \circ j_p)_! j_p^* \to \mathscr{R}f_!$$

is an equivalence.

This proposition will be proved elsewhere, by applying the machinery of [LZ17], as a special case of a more general result proving the existence and good behavior of $Rf_!$ for certain stacky maps of "small v-stacks" in the sense of [Sch17]. Note that part v. here is much more powerful than its classical shadow, which is an awkward statement involving spectral sequences. In particular, the argument for the next Proposition relies on v. and seems impossible to implement in the classical language.

Proposition 2.8. *Let $\mathcal{O}$ be the ring of integers in an algebraically closed nonarchimedean field K, so there are canonical equivalences $D((\mathrm{Spf}\,\mathcal{O})_{\text{ét}}, \Lambda) \cong D(\Lambda) \cong D((\mathrm{Spa}\,K)_{\text{ét}}, \Lambda)$. Let $\mathfrak{X}$ be a formal scheme over $\mathrm{Spf}\,\mathcal{O}$ with rigid generic fiber X, and let $\mathfrak{f} : \mathfrak{X} \to \mathrm{Spf}\,\mathcal{O}$ and $f : X \to \mathrm{Spa}\,K$ be the structure maps.*

Then there is a natural equivalence of functors

$$R\mathfrak{f}_! R\lambda_{\mathfrak{X}*} \simeq Rf_! : D(X_{\text{ét}}, \Lambda) \to D(\Lambda).$$

Proof. When $\mathfrak{X}$ is affine this is easy and well-known, cf. [Hub98, Lemma 2.13]. Slightly more generally, the case where $\mathfrak{X}$ is open in a coproduct of affine formal schemes can also be handled directly, using the affine case together with [Hub96, Corollary 3.5.11.ii].

For the general case, let $\mathfrak{X} = \cup_i \mathfrak{X}_i$ be an affine open covering of $\mathfrak{X}$, so $\mathfrak{j}_0 : \mathfrak{U}_0 = \coprod \mathfrak{X}_i \to \mathfrak{X}$ is a surjective étale map. Let $j_0 : U_0 \to X$ denote the generic fiber of this map. Let $\mathfrak{j}_\bullet : \mathfrak{U}_\bullet \to \mathfrak{X}$ (resp. $j_\bullet : U_\bullet \to X$) be the Cech nerve of $\mathfrak{j}_0$ (resp. j_0). Then we have a sequence of natural equivalences

$$\begin{aligned} \mathscr{R}f_! &\simeq \varinjlim_p \mathscr{R}(f \circ j_p)_! j_p^* \\ &\simeq \varinjlim_p \mathscr{R}(\mathfrak{f} \circ \mathfrak{j}_p)_! \mathscr{R}\lambda_{\mathfrak{U}_p *} j_p^* \\ &\simeq \varinjlim_p \mathscr{R}(\mathfrak{f} \circ \mathfrak{j}_p)_! \mathfrak{j}_p^* \mathscr{R}\lambda_{\mathfrak{X}*} \\ &\simeq \mathscr{R}\mathfrak{f}_! \mathscr{R}\lambda_{\mathfrak{X}*}, \end{aligned}$$

where $\mathscr{R}\lambda_{\mathfrak{X}*}$ denotes the obvious ∞-categorical enhancement of $R\lambda_{\mathfrak{X}*}$. Here the first and fourth lines follow from the fact that $\mathscr{R}f_!$ and $\mathscr{R}\mathfrak{f}_!$ satisfy étale co-descent, the second line follows from (the ∞-categorical lift of) the special cases discussed in the previous paragraph, and the third line is trivial. Passing to homotopy categories gives the result. □

Definition 2.9. Let K be an algebraically closed nonarchimedean field, and let $f : X \to \mathrm{Spa}\,K$ be an object of Rig_K. Fix a coefficient ring Λ as in Theorem 1.3. The *dualizing complex of X* is the object $\omega_X = Rf^!\Lambda \in D(X_{\text{ét}}, \Lambda)$.

Proof of Theorem 1.3.i. Passing to right adjoints in Proposition 2.8 gives an equivalence $Rf^! \cong R\lambda_{\mathfrak{X}}^! R\mathfrak{f}^!$, so

$$\omega_X = Rf^!\Lambda \simeq R\lambda_{\mathfrak{X}}^! R\mathfrak{f}^!\Lambda \simeq R\lambda_{\mathfrak{X}}^! \omega_{\mathfrak{X}}$$

as desired. □

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