ON THE SUPERCUSPIDAL COHOMOLOGY OF BASIC LOCAL SHIMURA VARIETIES

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Abstract. We prove that the supercuspidal cohomology of many basic local Shimura varieties is concentrated in the middle degree, under a mild (and probably necessary) condition. The proof uses a mixture of local and global techniques, and relies crucially on the recent work of Fargues-Scholze. As a byproduct of our methods, we prove the cuspidal case of Fargues's geometrization conjecture for general linear groups.

Contents

1. Introduction

1.1. Background and main result. Fix a reductive group $G/\mathbf{Q}_p$ and a conjugacy class of minuscule cocharacters $\mu : \mathbf{G}_{m,\overline{\mathbf{Q}_p}} \to G_{\overline{\mathbf{Q}_p}}$. Let b be an element of the Kottwitz set $B(G,\mu)$, and let J_b the associated algebraic group. Given such a datum (G,μ,b), Scholze constructed a tower of local Shimura varieties

$$\{\mathrm{Sh}(G,\mu,b)_K\}_{K\subset G(\mathbf{Q}_p)},$$

confirming conjectures of Rapoport-Viehmann [RV14]. This is a tower of smooth partially proper rigid spaces of dimension $d = \langle 2\rho, \mu\rangle$ over $\mathbf{C}_p$, generalizing the Rapoport-Zink spaces constructed in [RZ96]. The compactly supported ℓ-adic étale cohomology groups of this tower carry natural smooth commuting actions of $G(\mathbf{Q}_p)$ and $J_b(\mathbf{Q}_p)$, and also an action of the Weil group W_F where $F/\mathbf{Q}_p$ is the field of definition of μ.

It is a fundamental problem in the Langlands program to understand how these cohomology groups decompose representation-theoretically under the $G(\mathbf{Q}_p)\times J_b(\mathbf{Q}_p)\times W_F$-action. There are precise conjectures of Kottwitz and Harris-Viehmann which completely describe the alternating sum of the cohomologies in terms of putative local Langlands and local Jacquet-Langlands correspondences [RV14, §7-8]. However, even conjecturally, the individual cohomology groups are poorly understood.

In this paper, we make some progress towards understanding the individual cohomology groups of local Shimura varieties in arguably the most interesting case, namely when b is *basic*. To state our main result, we introduce some notation. Fix a prime $\ell \neq p$ and a finite extension $E/\mathbf{Q}_\ell$ with ring of integers $\mathcal{O}$. For any open compact subgroup $K \subset G(\mathbf{Q}_p)$, the ℓ-adic cohomology $R\Gamma_c(\mathrm{Sh}(G,\mu,b)_K, E)$ (which requires some care to define correctly, cf. Definition 2.15) is a bounded complex of smooth $J_b(\mathbf{Q}_p)$-representations. Let ρ be any admissible smooth $J_b(\mathbf{Q}_p)$-representation on an E-vector space. Then we define

$$R\Gamma_c(G,\mu,b)[\rho] = \operatorname*{colim}_{K\to\{1\}} R\Gamma_c(\mathrm{Sh}(G,\mu,b)_K, E) \otimes^{\mathbf{L}}_{\mathcal{H}(J_b(\mathbf{Q}_p))} \rho$$

and

$$H^i_c(G,\mu,b)[\rho] = H^i\left(R\Gamma_c(G,\mu,b)[\rho]\right).$$

Here $\mathcal{H}(J_b(\mathbf{Q}_p)) = C_c^\infty(J_b(\mathbf{Q}_p), E)$ is the usual smooth Hecke algebra. By the finiteness results in [FS20], $H^i_c(G,\mu,b)[\rho]$ is an admissible smooth $G(\mathbf{Q}_p)$-representation, which moreover has finite length if ρ has finite length, and $H^i_c(G,\mu,b)[\rho] = 0$ unless $0 \le i \le 2d$. Note that there is a natural edge map

$$\alpha : H^*_c(\mathrm{Sh}(G,\mu,b), E) \otimes_{\mathcal{H}(J_b(\mathbf{Q}_p))} \rho \to H^*_c(G,\mu,b)[\rho],$$

which is often (but not always) an isomorphism. In particular, one can check that α is an isomorphism if ρ is supercuspidal, so in this case $H^*_c(G,\mu,b)[\rho]$ coincides with the naive ρ-part of $H^*_c(\mathrm{Sh}(G,\mu,b),E)$.

It seems to be a folklore expectation that the most interesting part of $R\Gamma_c(\mathrm{Sh}(G,\mu,b)_K, E)$ should be concentrated in the middle degree, in analogy with Arthur and Kottwitz's conjectures on the cohomology of global Shimura varieties [Art89, Kot90]. Our main result confirms a precise form of this conjecture.

Theorem 1.1. *Let (G,μ,b) be a basic local Shimura datum, and let ρ be a supercuspidal representation of $J_b(\mathbf{Q}_p)$. Suppose the following conditions hold.*

1. *The spaces $\mathrm{Sh}(G,\mu,b)_K$ occur in the basic uniformization at p of a global Shimura variety, in the sense of Definition 3.2.*
2. *The Weil parameter $\varphi_\rho : W_{\mathbf{Q}_p} \to {}^LG(\overline{\mathbf{Q}_\ell})$ associated with ρ by Fargues-Scholze [FS20] is supercuspidal.*

Then $H^i_c(G,\mu,b)[\rho] = 0$ for all $i \neq d = \dim \mathrm{Sh}(G,\mu,b)_K$.

Recall that given a reductive group $G/\mathbf{Q}_p$ with quasisplit inner form G^*, a Weil parameter

$$\varphi : W_{\mathbf{Q}_p} \to {}^LG(\overline{\mathbf{Q}_\ell}) = \widehat{G}(\overline{\mathbf{Q}_\ell}) \rtimes W_{\mathbf{Q}_p} \cong \widehat{G^*}(\overline{\mathbf{Q}_\ell}) \rtimes W_{\mathbf{Q}_p}$$

is supercuspidal if it is semisimple and does not factor through a conjugate of ${}^LP(\overline{\mathbf{Q}_\ell}) = \widehat{P}(\overline{\mathbf{Q}_\ell}) \rtimes W_{\mathbf{Q}_p}$ for any proper parabolic subgroup $P \subset G^*$. Such parameters are called discrete in [RV14] and [KW19], but this is arguably misleading.

Let us comment on the conditions in this theorem. Condition 1. is absolutely essential for our proof, although we believe the theorem holds without this assumption. Nevertheless, it is plausible that condition 1. holds for every basic local Shimura datum. With current technology, condition 1. can be verified for many (but not all) basic local Shimura data, including most cases of classical interest, cf. §3.1.

Condition 2. may appear strange at first sight, but in reality it is natural. In fact, this condition is already necessary in the case of the Lubin-Tate tower. Recall that in this situation, $G = \mathrm{GL}_{n/\mathbf{Q}_p}$ and $J_b(\mathbf{Q}_p) = D^\times$ where $D/\mathbf{Q}_p$ is the division algebra of invariant $1/n$. Then $J_b(\mathbf{Q}_p)$ is compact modulo center, and any irreducible ρ is supercuspidal (in a somewhat vacuous sense). However, for any such ρ whose Jacquet-Langlands transfer to GL_n is not supercuspidal, Boyer proved that the groups $H^i(G,\mu,b)[\rho]$ are *not* concentrated in the middle degree [Boy09]. This shows that *some* condition on ρ beyond mere supercuspidality is necessary for the conclusion of Theorem 1.1 to hold. However, the Fargues-Scholze construction agrees with the known local Langlands correspondence in this case, and these bad ρ's are exactly the $J_b(\mathbf{Q}_p)$-representations which violate condition 2.

For a more subtle example, consider the basic local Shimura datum (G,μ,b) where G is the unique nonsplit inner form of $\mathrm{GSp}_4/\mathbf{Q}_p$ and μ is the Siegel cocharacter. Note that in this case $J_b(\mathbf{Q}_p) \simeq \mathrm{GSp}_4(\mathbf{Q}_p)$ is a split group. Then for certain supercuspidal ρ, Ito-Mieda have shown that the cohomology $H^i_c(G,\mu,b)[\rho]$ is not concentrated in the middle degree. Notably, the ρ's examined by Ito-Mieda lie in a non-tempered A-packet, and condition 2. is again expected to fail for these particular choices of ρ.[1] More generally, if one believes

[1] More precisely, the Langlands parameter associated with ρ by the GSp_4 local Langlands correspondence of Gan-Takeda is not supercuspidal. Therefore, condition 2. will fail if Fargues-Scholze parameters for $\mathrm{GSp}_4/\mathbf{Q}_p$ coincide with Gan-Takeda parameters.

that the Fargues-Scholze construction realizes the "true" local Langlands correspondence, then condition 2. morally amounts to the requirement that for any pure inner form H of G and any L-packet Π for H containing some generalized Jacquet-Langlands transfer of ρ, the packet Π is supercuspidal.

Here is a rough sketch of the proof. Condition 1. implies, more or less by definition, that we can choose a global Shimura datum $(\mathbf{G}, X)$ such that the basic Newton strata in the associated Shimura varieties are uniformized by the local Shimura varieties $\mathrm{Sh}(G, \mu, b)_K$. Moreover, for such a datum, $\mathbf{G}$ admits a unique inner form such that $\mathbf{G}'_{\mathbf{Q}_p} \simeq J_b$, $\mathbf{G}'_{\mathbf{A}^{f,p}} \simeq \mathbf{G}_{\mathbf{A}^{f,p}}$, and $\mathbf{G}'(\mathbf{R})$ is compact modulo center. By a standard argument with the simple trace formula, we can choose a compact open subgroup $K^p \subset \mathbf{G}'(\mathbf{A}^{f,p})$ and an algebraic $\mathbf{G}$-representation $\mathcal{L}_\xi$ of some regular highest weight such that possibly after replacing ρ by an unramified twist, ρ occurs as a direct summand of a suitable space $\mathcal{A}(\mathbf{G}'(\mathbf{Q})\backslash\mathbf{G}'(\mathbf{A}^f)/K^p, \mathcal{L}_{\xi,E})$ of algebraic automorphic forms.

From the geometry of the uniformization isomorphism, we easily obtain a $G(\mathbf{Q}_p)$-equivariant map

$$R\Gamma_c(\mathrm{Sh}(G, \mu, b), E) \otimes^{\mathbf{L}}_{\mathcal{H}(J_b(\mathbf{Q}_p))} \mathcal{A}(\mathbf{G}'(\mathbf{Q})\backslash\mathbf{G}'(\mathbf{A}^f)/K^p, \mathcal{L}_{\xi,E}) \xrightarrow{\Theta} R\Gamma(\mathrm{Sh}(\mathbf{G}, X)_{K^p}, \mathcal{L}_{\xi,E}).$$

The idea that uniformization of the basic locus induces a map like this is certainly not new, and goes back in various guises to work of Carayol, Harris-Taylor, Fargues and Mantovan [Car90, Far04, HT01, Man04]. The key new idea in our proof is to show that for any supercuspidal ρ occurring as a direct summand of $\mathcal{A}(\mathbf{G}'(\mathbf{Q})\backslash\mathbf{G}'(\mathbf{A}^f)/K^p, \mathcal{L}_{\xi,E})$ which moreover satisfies condition 2. in Theorem 1.1, the induced map

$$R\Gamma_c(G, \mu, b)[\rho] \xrightarrow{\Theta_\rho} R\Gamma(\mathrm{Sh}(\mathbf{G}, X)_{K^p}, \mathcal{L}_{\xi,E})$$

realizes the left-hand side as a $G(\mathbf{Q}_p)$-stable *direct summand* of the right-hand side. If the global Shimura varieties are compact, Theorem 1.1 now follows immediately, since $R\Gamma(\mathrm{Sh}(\mathbf{G}, X)_{K^p}, \mathcal{L}_{\xi,E})$ is concentrated in the middle degree by Matsushima's theorem together with standard vanishing properties of $(\mathfrak{g}, K_\infty)$-cohomology in regular weights. In the general case, we use an extra duality argument (Theorem 2.23) and the more general vanishing theorems proved in [LS04].

Perhaps surprisingly, our argument that Θ_ρ is a split inclusion doesn't require any knowledge whatsoever about the globalization of ρ to an automorphic form on $\mathbf{G}'$. We also don't need any information about non-basic Newton strata in the global Shimura variety, not even their existence. Instead, we make crucial use of condition 2., the geometry of the Hodge-Tate period map, and the relationship between representation theory and sheaves on the stack Bun_G of G-bundles on the Fargues-Fontaine curve. The essential new ingredient is the following notion (which we state in a slightly incorrect way in this introduction, cf. Definition 2.21 for the true definition).

Definition 1.2. Let $b \in B(G)_{\mathrm{bas}}$ be any basic element, and let $j : B\underline{J_b(\mathbf{Q}_p)} \cong \mathrm{Bun}_G^b \to \mathrm{Bun}_G$ be the associated open immersion. An admissible smooth $J_b(\mathbf{Q}_p)$-representation ρ is *inert* if the associated ℓ-adic sheaf $\mathcal{F}_\rho$ on $B\underline{J_b(\mathbf{Q}_p)}$ satisfies $j_!\mathcal{F}_\rho \xrightarrow{\sim} Rj_*\mathcal{F}_\rho$.

By some straightforward arguments, the subcategory of $\mathrm{Rep}_E(J_b(\mathbf{Q}_p))$ spanned by inert representations is stable under twisting, extensions, finite direct sums, direct summands, and contragredients. However, the following result lies significantly deeper.

Theorem 1.3 (Fargues-Scholze)**.** *Let $\rho \in \mathrm{Irr}_E(J_b(\mathbf{Q}_p))$ be any irreducible smooth representation. If the associated Weil parameter φ_ρ is supercuspidal, then ρ is supercuspidal and inert.*

This theorem ultimately follows from the compatibility between the Fargues-Scholze construction of Langlands parameters and parabolic induction. Theorem 1.3 is a powerful result, and is the key technical input into our argument. In fact, our argument shows that Theorem 1.1 holds with condition 2. replaced by the apparently more general condition that ρ is supercuspidal and inert. However, Scholze has conjectured that the converse of Theorem 1.3 is true, so this extra generality is probably illusory.

It is instructive to consider the statement and proof of Theorem 1.1 in the most classical and well-understood situations, namely for the Lubin-Tate and Drinfeld local Shimura data $(\mathrm{GL}_n, (z, 1, ..., 1), b)$ and $(D^\times, (z^{-1}, 1, ..., 1), b^{-1})$. Recall that for these data we have $J_b = D^\times$ and GL_n, respectively. Then condition

1. holds (up to a harmless $\mathbf{G}_m$ factor) with $(\mathbf{G}, X)$ a Harris-Taylor Shimura datum. In the Drinfeld case, condition 2. always holds, and in the Lubin-Tate case it reduces (as discussed above) to the assumption that ρ is the Jacquet-Langlands transfer of a supercuspidal representation of GL_n. In the Drinfeld situation, Harris proved Theorem 1.1 by a global argument [Har97]. Faltings then proved the Lubin-Tate case of Theorem 1.1, also by a global argument, and Mieda later gave a purely local proof [Fal02, Mie10]. However, these situations are extremely special: in the Drinfeld case, the map Θ is an *isomorphism*, while in the Lubin-Tate case a soft argument ("Boyer's trick") shows that it becomes an isomorphism after passing to supercuspidal parts for the $G(\mathbf{Q}_p)$-action.

1.2. **Corollaries.** Our main theorem has a number of consequences.

First of all, combining Theorem 1.1 with [KW19, Theorem 1.0.2][2] leads to the following result.

Theorem 1.4. *Fix any (G, μ, b) and ρ satisfying the conditions of Theorem 1.1, with $d = \dim \mathrm{Sh}(G, \mu, b)_K$ as before. Suppose moreover that the refined local Langlands correspondence holds for G and its extended pure inner forms, and that $\rho \in \Pi_\phi(J_b)$ lies in a supercuspidal L-packet. Then as $G(\mathbf{Q}_p)$-representations, we have*

$$H^i_c(G, \mu, b)[\rho] \simeq \begin{cases} \oplus_{\pi \in \Pi_\phi(G)} \dim \mathrm{Hom}_{S_\phi}(\delta_{\pi,\rho}, r_\mu) \cdot \pi & \text{if } i = d \\ 0 & \text{if } i \neq d. \end{cases}$$

Here r_μ is the natural representation of $\widehat{G} \rtimes W_F$ extending the highest weight representation r_μ of $\widehat{G}$, and the remaining notation follows [KW19] and [Kal16].

In other words, conditionally and ignoring the W_F-action, we confirm a strong form of the Kottwitz conjecture, without passing to the Grothendieck group. Note that here we commit the usual venal sin of referring to **the** local Langlands correspondence, despite the fact that a uniquely determined local Langlands correspondence with all expected properties is not yet known for general groups, nor is there even a known list of conditions which would uniquely determine such a correspondence (but see [BMY20] for some recent progress towards this). Of course, it is completely natural to expect that any such correspondence will be compatible with the construction of Fargues-Scholze. In particular, it is natural to expect (in the notation of the previous theorem) that $\phi = \varphi_\rho$, but amusingly we don't need to impose any compatibility between ϕ and φ_ρ.

Next we specialize our results to the case of local Shimura data of unramified EL type. These data can be described very concretely. Precisely, we take $G = \mathrm{Res}_{L/\mathbf{Q}_p}\mathrm{GL}_n$ for some finite unramified extension $L/\mathbf{Q}_p$. Writing $\Sigma = \mathrm{Hom}(L, \overline{\mathbf{Q}_p})$, we then choose any

$$\mu = \prod_{\sigma \in \Sigma} \mu_\sigma : \mathbf{G}_{m,\overline{\mathbf{Q}_p}} \to G_{\overline{\mathbf{Q}_p}} \cong \prod_{\sigma \in \Sigma} \mathrm{GL}_{n,\overline{\mathbf{Q}_p}}$$

where $\mu_\sigma(z) = (\underbrace{z, ..., z}_{d_\sigma}, \underbrace{1, ..., 1}_{n-d_\sigma})$ for some arbitrary integers $0 \leq d_\sigma \leq n$. When Σ contains an element τ such that

$$d_\sigma \in \begin{cases} \{1, n-1\} & \text{if } \sigma = \tau \\ \{0, n\} & \text{if } \sigma \neq \tau, \end{cases}$$

the tower $\mathrm{Sh}(G, \mu, b)_K$ recovers the classical Lubin-Tate tower for $\mathrm{GL}_{n/L}$. However, for any other choice of μ, the geometry of the spaces $\mathrm{Sh}(G, \mu, b)_K$ is still very mysterious. Note that in general, $\dim \mathrm{Sh}(G, \mu, b)_K = \sum_\sigma d_\sigma(n - d_\sigma)$.

Now, for any inner form of $\mathrm{Res}_{L/\mathbf{Q}_p}\mathrm{GL}_n$, the Fargues-Scholze construction provably recovers the correct local Langlands correspondence [HT01]. In particular, $\rho \in \mathrm{Irr}_E(J_b(\mathbf{Q}_p))$ satisfies condition 2. in Theorem 1.1 if and only if the Jacquet-Langlands transfer of ρ to $\mathrm{GL}_n(L)$ is supercuspidal. Combining these observations and Theorem 1.1 with previous works of Fargues and Shin on Kottwitz's conjecture in the unramified EL case, cf. [Far04, Shi12b], we deduce the following result.

[2] In the currently posted version of [KW19], Theorem 1.0.2 is only proved up to an "error term" consisting of a non-elliptic virtual representation. However, the author recently explained to Kaletha-Weinstein how to remove this error term under the assumption that φ_ρ is supercuspidal. This will appear in a future revision of [KW19].

Theorem 1.5. *Let (G,μ,b) be a basic local Shimura datum of unramified EL type, with $d = \dim \mathrm{Sh}(G,\mu,b)_K$. Let $\pi \in \mathrm{Irr}_E(G(\mathbf{Q}_p))$ be any supercuspidal representation, with Jacquet-Langlands transfer $\mathrm{JL}(\pi) \in \mathrm{Irr}_E(J_b(\mathbf{Q}_p))$. Then as $G(\mathbf{Q}_p) \times W_F$-representations, we have*

$$H^i_c(G,\mu,b)[\mathrm{JL}(\pi)] \simeq \begin{cases} \pi \boxtimes (r_\mu \circ \varphi_\pi|_{W_F} \otimes |\cdot|^{-d/2}) & \text{if } i = d \\ 0 & \text{if } i \neq d. \end{cases}$$

Here $\varphi_\pi : W_{\mathbf{Q}_p} \to {}^L G(\overline{\mathbf{Q}_\ell}) = \widehat{G}(\overline{\mathbf{Q}_\ell}) \rtimes W_{\mathbf{Q}_p}$ is the L-parameter of π, and r_μ is the natural representation of $\widehat{G} \rtimes W_F$ extending the highest weight representation r_μ of $\widehat{G}$.

This confirms a strong form of the Kottwitz conjecture for unramified EL data. Moreover, we obtain a complete description of the supercuspidal part of the cohomology of $\mathrm{Sh}(G,\mu,b)$ as a $G(\mathbf{Q}_p) \times J_b(\mathbf{Q}_p) \times W_F$-representation. To the best of the author's knowledge, prior to this paper, the Lubin-Tate and Drinfeld towers were (up to trivialities) the only basic local Shimura towers for which a complete description of the supercuspidal cohomology was known.

We also get the following unconditional cases of Theorem 1.4.

Theorem 1.6. *Let (G,μ,b) be any basic local Shimura datum such that G is an inner form of $\mathrm{Res}_{L/\mathbf{Q}_p}\mathrm{GL}_n$ for some finite extension $L/\mathbf{Q}_p$. Let $\rho \in \mathrm{Irr}_E(J_b(\mathbf{Q}_p))$ be any irreducible representation whose Jacquet-Langlands transfer to $\mathrm{GL}_n(L)$ is supercuspidal. Then*

$$H^i_c(G,\mu,b)[\rho] \simeq \begin{cases} \dim r_\mu \cdot \mathrm{JL}(\rho) & \text{if } i = d \\ 0 & \text{if } i \neq d. \end{cases}$$

Here $\mathrm{JL}(\rho)$ is the generalized Jacquet-Langlands transfer of ρ to $G(\mathbf{Q}_p)$ [DKV84, Rog83].

This requires some cases of basic uniformization for Shimura varieties with bad reduction at p, cf. Theorem 3.5.

To explain our next theorem, let $D_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbf{Q}_\ell})$ be the triangulated category constructed in [FS20]. If $\varphi : W_{\mathbf{Q}_p} \to {}^L G(\overline{\mathbf{Q}_\ell})$ is any Langlands parameter, the machinery in [FS20] defines a full triangulated subcategory $\mathscr{C}_\varphi \subset D_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbf{Q}_\ell})$, which is roughly the subcategory of "φ-isotypic objects" for the action of the algebra of excursion operators. In general, $\mathscr{C}_\varphi$ is stable under the Hecke action of $\mathrm{Rep}(\widehat{G})$ on $D_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbf{Q}_\ell})$. If φ is supercuspidal, one can also show that any object of $\mathscr{C}_\varphi$ is !-extended from the open substack $\mathrm{Bun}_G^{\mathrm{ss}}$; in other words, $\mathscr{C}_\varphi$ is naturally contained in the full subcategory

$$D_{\mathrm{lis}}(\mathrm{Bun}_G^{\mathrm{ss}}, \overline{\mathbf{Q}_\ell}) = \prod_{b \in B(G)_{\mathrm{bas}}} D(J_b(\mathbf{Q}_p), \overline{\mathbf{Q}_\ell}),$$

where $D(G, \overline{\mathbf{Q}_\ell})$ denotes the derived category of smooth $\overline{\mathbf{Q}_\ell}$-representations of G. In particular, when φ is supercuspidal, the category $\mathscr{C}_\varphi$ carries a natural t-structure, by restriction from the standard t-structures on the categories $D(J_b(\mathbf{Q}_p), \overline{\mathbf{Q}_\ell})$. Moreover, the heart of this t-structure is a full subcategory of

$$\prod_{b \in B(G)_{\mathrm{bas}}} \mathrm{Rep}_{\overline{\mathbf{Q}_\ell}}(J_b(\mathbf{Q}_p)),$$

so this heart is a very concrete category of representation-theoretic nature.

In his 2017 IHES lectures, Scholze proposed the following.

Conjecture 1.7 (Scholze)**.** *If $\varphi : W_{\mathbf{Q}_p} \to {}^L G(\overline{\mathbf{Q}_\ell})$ is any supercuspidal L-parameter, the Hecke action of $\mathrm{Rep}(\widehat{G})$ on $\mathscr{C}_\varphi$ is t-exact for the natural t-structure. In particular, it restricts to a natural Hecke action on $\mathscr{C}_\varphi^\heartsuit$.*

Our next theorem confirms Scholze's t-exactness conjecture and Fargues's geometrization conjecture [Far16] for general linear groups.

Theorem 1.8. *Let G be a restriction of scalars of GL_n, and let $\varphi : W_{\mathbf{Q}_p} \to {}^LG(\overline{\mathbf{Q}_\ell})$ be any supercuspidal L-parameter. Then Conjecture 1.7 is true. Moreover, $\mathscr{C}_\varphi^\heartsuit$ admits a canonical generator $\mathscr{F}_\varphi$.*

The sheaf $\mathscr{F}_\varphi$ is an irreducible Hecke eigensheaf with eigenvalue φ, and its stalk at $B\underline{G_d(\mathbf{Q}_p)}$ is isomorphic to $\mathcal{F}_{\pi_{\varphi,d}}$ for all $d \in \mathbf{Z} \cong B(G)_{\mathrm{bas}}$.

Here we change notation slightly, and write $G_d = J_{b_d}$ for the inner form associated with the isoclinic isocrystal of slope d/n. Then $\pi_{\varphi,d}$ is the (unique) irreducible representation of $G_d(\mathbf{Q}_p)$ with L-parameter φ, and $\mathcal{F}_{\pi_{\varphi,d}}$ is the associated sheaf on $B\underline{G_d(\mathbf{Q}_p)}$.

The proof of Theorem 1.8 combines Theorem 1.6 with some ongoing work of Anschütz and Le Bras [ALB20]. More precisely, in the situation of Theorem 1.8, their work exhibits a canonical Hecke eigensheaf $\mathscr{G}_\varphi \in \mathscr{C}_\varphi$ with eigenvalue φ, whose stalk at $B\underline{G_1(\mathbf{Q}_p)}$ can be identified with $\mathcal{F}_{\pi_{\varphi,1}}$. However, the stalks of their sheaf at the other points in $\mathrm{Bun}_G^{\mathrm{ss}}$ are somewhat mysterious. We proceed from the opposite direction, writing down the obvious candidate sheaf $\mathscr{F}_\varphi$ by hand. Forgetting the Weil group action, Theorem 1.6 easily implies that $T_{\mathrm{Std}}\mathscr{F}_\varphi \simeq T_{\mathrm{Std}^\vee}\mathscr{F}_\varphi \simeq \mathscr{F}_\varphi^{\oplus n}$. Moreover, the stalk of $\mathscr{F}_\varphi$ at $B\underline{G_1(\mathbf{Q}_p)}$ identifies with $\mathcal{F}_{\pi_{\varphi,1}}$ by construction, so in particular

$$\mathscr{G}_\varphi|B\underline{G_1(\mathbf{Q}_p)} \simeq \mathscr{F}_\varphi|B\underline{G_1(\mathbf{Q}_p)}.$$

Taken together, this turns out to be just enough information to conclude that $\mathscr{F}_\varphi \simeq \mathscr{G}_\varphi$. Since $\mathscr{G}_\varphi$ is a Hecke eigensheaf, this implies the theorem.

Finally, unwinding the meaning of the Hecke eigensheaf property in Theorem 1.8, we get a considerable generalization of Theorem 1.6.

Theorem 1.9. *Let (G, μ, b) be any basic local shtuka datum such that G is an inner form of $\mathrm{Res}_{L/\mathbf{Q}_p}\mathrm{GL}_n$ for some finite extension $L/\mathbf{Q}_p$. Let $\rho \in \mathrm{Irr}_E(J_b(\mathbf{Q}_p))$ be any irreducible representation whose Jacquet-Langlands transfer to $\mathrm{GL}_n(L)$ is supercuspidal. Then as $G(\mathbf{Q}_p) \times W_F$-representations, we have*

$$IH_c^i(G,\mu,b)[\rho] \simeq \begin{cases} \mathrm{JL}(\rho) \boxtimes (r_\mu \circ \varphi_\rho|_{W_F} \otimes |\cdot|^{-d/2}) & \text{if } i = d = \dim \mathrm{Sht}(G,\mu,b)_K\ , \\ 0 & \text{if } i \neq d. \end{cases}$$

Here $\mathrm{JL}(\rho)$ is the generalized Jacquet-Langlands transfer of ρ to $G(\mathbf{Q}_p)$.

1.3. Perspectives and conjectures.

The global condition in Theorem 1.1 should be irrelevant, but we have not been able to remove it. The following conjecture, formulated in conversations with Scholze, would allow us to circumvent this assumption.

Conjecture 1.10. *For any local Shimura datum (G, μ, b) and any open compact subgroup $K \subset G(\mathbf{Q}_p)$, the local Shimura variety $\mathrm{Sh}(G, \mu, b)_K$ is a Stein space.*

In the setting of Theorem 1.1, the Stein property implies that $R\Gamma_c(G, \mu, b)[\rho]$ is concentrated in degrees $[d, 2d]$ by the Artin vanishing results in [Han20], and then a duality argument (which requires condition 2., cf. Theorem 2.23) finishes the proof. Conjecture 1.10 is known for the Lubin-Tate and Drinfeld cases, for any μ-ordinary local Shimura datum, and for some other cases which can be related to these cases, e.g. in some HN-reducible situations.[3] However, it is entirely open for general local Shimura varieties of unramified EL type. Already the case of $G = \mathrm{GL}_{5/\mathbf{Q}_p}$, $\mu = \mathrm{diag}(z, z, 1, 1, 1)$, and b basic seems hard.

The following less elegant conjecture would actually have much wider implications.

Conjecture 1.11. *Let $\mathbf{B} = \mathrm{Spa}\, C\left\langle T_1^{1/p^\infty}, \dots, T_n^{1/p^\infty}\right\rangle$ be an affinoid perfectoid ball over some complete algebraically closed field $C/\mathbf{F}_p$, and let $Z \subset \mathbf{B}$ be a Zariski-closed affinoid perfectoid subset. Then $H^i_{\text{ét}}(Z, \mathbf{F}_\ell) = 0$ for all $i > \mathrm{Krull.dim}\,|Z|$ and all $\ell \neq p$.*

Among other things, this would imply the truth of Conjecture 1.7 for all classical groups. This conjecture is widely open, even if Z is cut out by a single equation in a two-dimensional ball.

[3] In particular, the methods in this paper give a new and purely local proof of Theorem 1.1 in the Lubin-Tate and Drinfeld cases, quite distinct from the arguments in [Har97] and [Mie10].

Conventions. Throughout this paper, we fix a prime p. We write E for a finite extension of $\mathbf{Q}_\ell$ for some prime $\ell \neq p$, and $\mathcal{O}$ for the ring of integers in E.

Beginning in §2.3, we need to use the étale cohomology of diamonds and v-stacks rather heavily. Here we freely use the formalism developed in [Sch17], in particular the adic formalism defined in [Sch17, §26]. In the notation of loc. cit, we will only apply the adic formalism to the coefficient ring $\Lambda = \mathcal{O}$. We note that the adic formalism of [Sch17, §26] is a "full" six functor formalism, cf. the final paragraph of [Sch17, §26]: in particular, all the usual excision triangles associated with an open-closed decomposition still hold.

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(more to be added)

2. Background

2.1. **Isogeny direct summands.** The following notion plays a key technical role in this paper.

Definition 2.1. Let X and Y be objects in an additive category $\mathcal{A}$. We say X is an *isogeny direct summand* of Y if there are maps $i : X \to Y$ and $s : Y \to X$ and a nonzero integer N such that $s \circ i = N \in \mathrm{End}_{\mathcal{A}}(X)$. We say a map $i : X \to Y$ is an *isogeny inclusion* if there exists a map s such that i and s exhibit X as an isogeny direct summand of Y.

In practice, our additive categories will be $\mathbf{Z}_\ell$-linear, so only $N|\ell^\infty$ will be relevant. This notion has several favorable properties.

Proposition 2.2. *1. If X is an isogeny direct summand of Y and Y is an isogeny direct summand of Z, then X is an isogeny direct summand of Z.*

2. If X is an isogeny direct summand of Y and $F : \mathcal{A} \to \mathcal{B}$ is any additive functor of additive categories, then $F(X)$ is an isogeny direct summand of $F(Y)$. If $i : X \to Y$ is an isogeny inclusion, then $F(i)$ is an isogeny inclusion.

3. Let $X \overset{f}{\to} Y \overset{g}{\to} Z \overset{h}{\to} X[1]$ be a distinguished triangle in a triangulated category $\mathcal{T}$. Suppose that $Nh = 0$ for some nonzero integer N. Then X is an isogeny direct summand of Y and f is an isogeny inclusion. More precisely, there exists a map $\varphi : Y \to X \oplus Z$ such that $\mathrm{Cone}(\varphi)$ is killed by N, and we can choose φ so that $\mathrm{pr}_Z \circ \varphi = g$ and $\mathrm{pr}_X \circ \varphi \circ f = N \in \mathrm{End}_{\mathcal{T}}(X)$.

Conversely, if f is an isogeny inclusion, then $Nh = 0$ for some nonzero integer N.

Proof. 1. and 2. are clear from the definition. For the first part of 3., the axioms of a triangulated category imply that the diagram

$$\begin{array}{ccccccc} Z[-1] & \xrightarrow{-h[-1]} & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle N} & & & & \\ Z[-1] & \xrightarrow{-Nh[-1]} & X & & & & \end{array}$$

can be extended to a diagram

$$\begin{CD}
Z[-1] @>{-h[-1]}>> X @>{f}>> Y @>{g}>> {} \\
@V{\mathrm{id}}VV @V{N}VV @V{\varphi}VV \\
Z[-1] @>{-Nh[-1]}>> X @>>> C @>>> {} \\
@VVV @VVV @VVV \\
0 @>>> \mathrm{Cone}(X \overset{\cdot N}{\to} X) @>>> \mathrm{Cone}(X \overset{\cdot N}{\to} X) @>>> {} \\
@VVV @VVV @VVV \\
{} @. {} @. {}
\end{CD}$$

where all the rows and columns are distinguished triangles and all squares commute. Since $Nh = 0$ by assumption, we can choose an isomorphism $C \simeq X \oplus Z$ such that the maps $X \to C \to Z$ are the obvious inclusion and projection, cf. [Sta20, Tag 05QT]. The remaining verifications are now easy and are left to the reader.

For the last half of 3., choose some $s : Y \to X$ with $s \circ f = N$. The axioms of a triangulated category imply that the diagram

$$\begin{CD}
X @>{f}>> Y @>{g}>> Z @>{h}>> X[1] \\
@V{\mathrm{id}}VV @V{s}VV \\
X @>{\cdot N}>> X
\end{CD}$$

extends to a commutative diagram

$$\begin{CD}
X @>{f}>> Y @>{g}>> Z @>{h}>> X[1] \\
@V{\mathrm{id}}VV @V{s}VV @V{t}VV @V{\mathrm{id}}VV \\
X @>{\cdot N}>> X @>>> D @>{h'}>> X[1]
\end{CD}$$

where the rows are exact triangles. Then D is killed by N, so $Nh' = 0$ and then also $Nh = Nh' \circ t = 0$, as desired. $\square$

2.2. **Some representation theory.** In this subsection, we enhance all derived categories to derived ∞-categories.

Let G be a locally pro-p group admitting a countable basis of open compact pro-p subgroups, and fix a prime $\ell \neq p$. Let $E/\mathbf{Q}_\ell$ be some finite extension with ring of integers $\mathcal{O}$ and uniformizer $\varpi \in \mathcal{O}$, so we get the usual Grothendieck abelian categories $\mathrm{Rep}_E(G)$ (resp. $\mathrm{Rep}_{\mathcal{O}}(G)$, resp. $\mathrm{Rep}_{\mathcal{O}/\varpi^n}(G)$) of smooth G-representations on E-vector spaces (resp. on $\mathcal{O}$-modules, resp. on $\mathcal{O}/\varpi^n$-modules). Note that these have natural symmetric monoidal structures. As usual, there are abelian Serre subcategories $\mathrm{Rep}_A(G)_{\mathrm{adm}}$ for $A \in \{E, \mathcal{O}, \mathcal{O}/\varpi^n\}$ consisting of smooth $A[G]$-modules M such that for all open compact subgroups $K \subset G$, M^K is a finitely generated A-module. If $\omega : Z_G \to A^\times$ is a fixed smooth character, we write $\mathrm{Rep}_A(G)_\omega$ for the full subcategory of objects with central character ω.

We also need to work with ϖ-adically complete representations. Let $\mathrm{Rep}^{\mathrm{comp}}_{\mathcal{O}}(G)$ denote the category of ϖ-adically complete $\mathcal{O}[G]$-modules such that $M/\varpi^n M$ is a smooth $\mathcal{O}/\varpi^n[G]$-module for all $n \geq 1$, and let $\mathrm{Rep}^{\mathrm{comp}}_{\mathcal{O}}(G)_{\mathrm{adm}}$ denote the full subcategory for which the $M/\varpi^n M$'s are admissible smooth.

Proposition 2.3. *The functor* $\mathrm{Rep}_{\mathcal{O}}(G) \to \mathrm{Rep}^{\mathrm{comp}}_{\mathcal{O}}(G)$ *of ϖ-adic completion induces an exact equivalence of categories*

$$\mathrm{Rep}_{\mathcal{O}}(G)_{\mathrm{adm}} \overset{\sim}{\to} \mathrm{Rep}^{\mathrm{comp}}_{\mathcal{O}}(G)_{\mathrm{adm}},$$

with essential inverse given by the functor $M \mapsto \mathrm{colim}_{H \subset G \text{ open}} M^H$ *of smooth vectors. In particular,* $\mathrm{Rep}_{\mathcal{O}}^{\mathrm{comp}}(G)_{\mathrm{adm}}$ *is an abelian category.*

Proof. Easy and left to the reader. The key point is that for any $M \in \mathrm{Rep}_{\mathcal{O}}^{\mathrm{comp}}(G)_{\mathrm{adm}}$ and any open compact pro-p subgroup $H \subset G$, the natural map $M^H \to R\lim_n R\Gamma_{\mathrm{cts}}(H, M/\varpi^n M)$ is an isomorphism. For this, observe that $H^i_{\mathrm{cts}}(H, M/\varpi^n M) = 0$ for all $i > 0$ since p is invertible in $\mathcal{O}$, and then $R^1 \lim_n (M/\varpi^n M)^H = 0$ since $(M/\varpi^n M)^H$ is a finite abelian group by admissibility. □

We will also need some information about derived categories of representations. For $A \in \{E, \mathcal{O}, \mathcal{O}/\varpi^n\}$, let $D(G, A)$ denote the derived category of $\mathrm{Rep}_A(G)$, and let $D(G, A)_{\mathrm{adm}}$ denote the full subcategory spanned by objects with admissible cohomology modules. On the other hand, fixing a complete algebraically closed field $C/\mathbf{F}_p$ and applying [Sch17, Definitions 14.13 and 26.1] to the classifying stack $B\underline{G} = [\mathrm{Spd}\, C/\underline{G}]$, we get competely natural symmetric monoidal triangulated categories $D_{\text{ét}}(B\underline{G}, \mathcal{O}/\varpi^n)$ and $D_{\text{ét}}(B\underline{G}, \mathcal{O})$, which (morally) consist of complexes of étale sheaves of $\mathcal{O}/\varpi^n$-modules (resp. derived ϖ-complete complexes of étale sheaves of $\mathcal{O}$-modules) on $B\underline{G}$. By Proposition 2.4 below and [Sch17, Proposition 26.2], these categories are independent of the choice of C.

It will be extremely important for us to have some comparison between the geometrically natural categories $D_{\text{ét}}(B\underline{G}, \mathcal{O}/\varpi^n)$ and the more down-to-earth categories $D(G, \mathcal{O}/\varpi^n)$ for $n \leq \infty$. When $n < \infty$ this comparison is as clean as possible.

Proposition 2.4. *There is a natural equivalence of presentably symmetric monoidal stable ∞-categories*

$$D(G, \mathcal{O}/\varpi^n) \cong D_{\text{ét}}(B\underline{G}, \mathcal{O}/\varpi^n)$$
$$A \mapsto \mathcal{F}_A$$

functorial in G and the coefficient ring.

Proof. This is proved in [FS20, Theorem V.1.1]. □

To discuss the case of $\mathcal{O}$-coefficients, we need a "completed" version $\widehat{D}(G, \mathcal{O})$ of $D(G, \mathcal{O})$. It seems somewhat subtle to define a well-behaved category $\widehat{D}(G, \mathcal{O})$ as the actual derived category of some abelian category. Instead, we make the following definition.

Definition 2.5. For any G and $\mathcal{O}$ as above, we set $\widehat{D}(G, \mathcal{O}) = \lim_n D(G, \mathcal{O}/\varpi^n)$, where the limit is computed in the ∞-categorical sense.

Morally (but not literally) this is the subcategory of $D(\mathrm{Mod}_{\mathcal{O}[G]})$ spanned by derived ϖ-complete complexes whose derived mod-ϖ^n reductions are complexes of smooth $\mathcal{O}[G]$-modules.

Proposition 2.6. *1. There is a natural equivalence $\widehat{D}(G, \mathcal{O}) \cong D_{\text{ét}}(B\underline{G}, \mathcal{O})$ functorially in G and $\mathcal{O}$, via a functor denoted $A \mapsto \mathcal{F}_A$.*

2. There is a natural exact symmetric monoidal functor

$$\gamma : D(G, \mathcal{O}) \to \widehat{D}(G, \mathcal{O}).$$
$$A \mapsto (A \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}/\varpi^n)_{n \geq 1}.$$

The functor γ commutes with all colimits and therefore admits a right adjoint $\delta : \widehat{D}(G, \mathcal{O}) \to D(G, \mathcal{O})$.

3. For any $A \in D(G, \mathcal{O})_{\mathrm{adm}}$, the unit of the adjunction $A \to \delta\gamma A$ is an isomorphism. In particular, the functor $D(G, \mathcal{O})_{\mathrm{adm}} \to \widehat{D}(G, \mathcal{O})$ is fully faithful.

4. The composite functor

$$\mathcal{F}_{-} \circ \gamma : \mathrm{Rep}_{\mathcal{O}}(G) \to D_{\text{ét}}(B\underline{G}, \mathcal{O})$$

factors uniquely over the completion functor $\mathrm{Rep}_{\mathcal{O}}(G) \to \mathrm{Rep}_{\mathcal{O}}^{\mathrm{comp}}(G)$ via a natural functor

$$\mathcal{F}_{-} : \mathrm{Rep}_{\mathcal{O}}^{\mathrm{comp}}(G) \to D_{\text{ét}}(B\underline{G}, \mathcal{O}).$$

The reader should remember that intuitively, γ is completion and δ is decompletion.

Proof. 1. is immediate from the equivalences

$$D_{\text{ét}}(B\underline{G},\mathcal{O}) \cong \lim_n D_{\text{ét}}(B\underline{G},\mathcal{O}/\varpi^n) \cong \lim_n D(G,\mathcal{O}/\varpi^n)$$

where the first isomorphism follows from [Sch17, Proposition 26.2] and the second isomorphism is immediate from the previous proposition.

For 2., the fact that γ is symmetric monoidal follows as in [Sch17, Remark 26.3]. It is clear from the definition that γ commutes with all direct sums, or equivalently [Lur16, Proposition 1.4.4.1(2)] with all colimits, so the existence of the right adjoint follows from Lurie's ∞-categorical adjoint functor theorem [Lur09, Corollary 5.5.2.9(1)].

The desired functor

$$\mathrm{Rep}^{\mathrm{comp}}_{\mathcal{O}}(G) \to D_{\text{ét}}(B\underline{G},\mathcal{O}) \cong \lim_n D_{\text{ét}}(B\underline{G},\mathcal{O}/\varpi^n)$$

in 4. sends M to $(\mathcal{F}_{M\otimes_{\mathcal{O}}\mathcal{O}/\varpi^n})_{n\geq 1}$. The claimed commutativity is then clear.

For 3., we first note that δ has an explicit description. Precisely, for any open subgroup $H \subset G$ and any $n \leq \infty$, there is a natural functor

$$\begin{aligned} R\Gamma(H,-) : D(G,\mathcal{O}/\varpi^n) &\to D(\mathcal{O}/\varpi^n) \\ A &\mapsto R\mathrm{Hom}(\mathrm{ind}_H^G \mathcal{O}/\varpi^n, A) \end{aligned}$$

of (derived) H-invariants, as well as a completed version

$$\begin{aligned} R\widehat{\Gamma}(H,-) : \widehat{D}(G,\mathcal{O}) &\to D_{\varpi-\mathrm{comp}}(\mathcal{O}) \cong \lim_n D(\mathcal{O}/\varpi^n\mathcal{O}) \to D(\mathcal{O}) \\ (A_n)_{n\geq 1} &\mapsto R\lim_n R\mathrm{Hom}_{D(G,\mathcal{O}/\varpi^n)}(\mathrm{ind}_H^G \mathcal{O}/\varpi^n, A_n). \end{aligned}$$

When H is compact, $R\Gamma(H,-)$ actually coincides with the usual functor of continuous group cohomology on discrete H-modues, hence the notation. Note that $\mathrm{colim}_H R\widehat{\Gamma}(H,B)$ is naturally a complex of smooth $\mathcal{O}[G]$-modules, with G acting by "transport of structure". Then for any $B \in \widehat{D}(G,\mathcal{O})$, $\delta(B)$ is naturally identified with $\mathrm{colim}_H R\widehat{\Gamma}(H,B)$. For this, observe that $\delta B \simeq \mathrm{colim}_H R\Gamma(H,\delta B)$ (this holds with δB replaced by any object of $D(G,\mathcal{O})$). Then observe that $R\Gamma(H,\delta B) = R\widehat{\Gamma}(H,B)$ for any B; this follows by a direct computation from the formula for γ.

Granted this, we compute that

$$\begin{aligned} \delta(\gamma(A)) &\simeq \mathrm{colim}_H R\widehat{\Gamma}(H,\gamma(A)) \\ &\simeq \mathrm{colim}_H \lim_n R\Gamma(H, A\otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}/\varpi^n) \\ &\simeq \mathrm{colim}_H \lim_n R\Gamma(H,A)\otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}/\varpi^n \\ &\simeq \mathrm{colim}_H R\Gamma(H,A) \\ &\simeq A \end{aligned}$$

for any $A \in D(G,\mathcal{O})_{\mathrm{adm}}$. Here the key isomorphism $R\Gamma(H,A) \simeq \lim_n R\Gamma(H,A)\otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}/\varpi^n$ follows from the fact that the cohomology groups of $R\Gamma(H,A)$ are all finitely generated $\mathcal{O}$-modules, which follows from the fact that all $H^i(A)$'s are admissible by assumption. □

Corollary 2.7. *For any $M \in D(G,\mathcal{O})$ and $N \in D(G,\mathcal{O})_{\mathrm{adm}}$, the natural map $R\mathrm{Hom}_{D(G,\mathcal{O})}(M,N) \to R\mathrm{Hom}_{\widehat{D}(G,\mathcal{O})}(\gamma M,\gamma N)$ is an isomorphism.*

Proof. By part 3. of the previous proposition, we have $R\mathrm{Hom}_{D(G,\mathcal{O})}(M,N) \cong R\mathrm{Hom}_{D(G,\mathcal{O})}(M,\delta\gamma N)$, so the result follows by adjunction. □

Lemma 2.8. *Let M be an admissible smooth ℓ-torsion-free $\mathcal{O}[G]$-module. Let $\pi \in \mathrm{Rep}_E(G)$ be a finite-length admissible smooth representation of G. Suppose moreover that π is a direct summand of $M[\frac{1}{\ell}]$. Then π admits a G-stable $\mathcal{O}$-lattice π°, and any such lattice is an isogeny direct summand of M.*

In what follows, when we say "$\mathcal{O}$-lattice", we mean "G-stable $\mathcal{O}$-lattice".

Proof. Since π is of finite length, it is finitely generated, so some results of Vignéras [Vig04] imply that any $\mathcal{O}$-lattice in π is finitely generated as an $\mathcal{O}[G]$-module, and any two $\mathcal{O}$-lattices in π are commensurable. Moreover, an $\mathcal{O}[G]$-submodule of π is an $\mathcal{O}$-lattice if and only if it is a free $\mathcal{O}$-module and it contains an E-basis of π. These results easily imply that if *some* $\mathcal{O}$-lattice $\pi^\circ \subset \pi$ is an isogeny direct summand of M, then *all* $\mathcal{O}$-lattices are isogeny direct summands of M. Thus, we only need to exhibit a single such π°.

By assumption, we can choose morphisms $i : \pi \to M[\frac{1}{\ell}]$ and $s : M[\frac{1}{\ell}] \to \pi$ with $s \circ i = \mathrm{id}$. We claim that $\pi^\circ = s(M)$ is an $\mathcal{O}$-lattice in π. Clearly π° contains an E-basis of π, so it suffices to check that π° is a free $\mathcal{O}$-module. Choose a countable cofinal set of open pro-p subgroups

$$\cdots \lhd K_n \lhd \cdots \lhd K_3 \lhd K_2 \lhd K_1 \subset G.$$

Then each $s(M^{K_n}) = s(M)^{K_n}$ is a finite free $\mathcal{O}$-module, and the inclusions $s(M)^{K_n} \to s(M)^{K_{n+1}}$ are split as $\mathcal{O}$-modules by the obvious averaging maps, using that $[K_{n+1} : K_n] \in \mathcal{O}^\times$. Therefore $\pi^\circ = \mathrm{colim}_n s(M)^{K_n}$ is a free $\mathcal{O}$-module.

Since π° is a finitely generated $\mathcal{O}[G]$-submodule, we can choose some large n such that $\ell^n i(\pi^\circ) \subseteq M$. Setting $i' = \ell^n i$, we now have a pair of maps $i' : \pi^\circ \to M$ and $s : M \to \pi^\circ$ with $s \circ i = \ell^n$, so these maps exhibit π° as an isogeny direct summand of M. □

As a consequence, we can globalize lattices in square-integrable representations. For simplicity we only discuss the supercuspidal case.

Proposition 2.9. *Let $\mathbf{H}/\mathbf{Q}$ be a connected reductive group such that $\mathbf{H}(\mathbf{R})$ is compact modulo center. Fix an isomorphism $\iota : \overline{E} \to \mathbf{C}$, and let $\pi \in \mathrm{Rep}_E(\mathbf{H}(\mathbf{Q}_p))$ be any irreducible supercuspidal representation. Then some unramified twist of π admits a $\mathbf{H}(\mathbf{Q}_p)$-stable $\mathcal{O}$-lattice, and this lattice occurs as an isogeny direct summand in a space of algebraic automorphic forms for $\mathbf{H}$.*

More precisely, we can choose the data of

- *An open compact subgroup $K^p \subset \mathbf{H}(\mathbf{A}^{f,p})$,*
- *An irreducible algebraic $\mathbf{H}$-representation $\mathcal{L}_\xi$ of some regular highest weight ξ,*
- *A K^p-stable $\mathcal{O}$-lattice $\mathcal{L}_{\xi,\mathcal{O}}$ in the E-linear realization $\mathcal{L}_{\xi,E}$ of $\mathcal{L}_\xi$ (where K^p acts via the composition $K^p \to \mathbf{H}(\mathbf{A}^f) \to \mathbf{H}(\mathbf{Q}_\ell)$), and*
- *A $\mathbf{H}(\mathbf{Q}_p)$-stable $\mathcal{O}$-lattice τ° in an unramified twist $\tau = \pi \otimes \eta$ of π,*

such that τ° is an isogeny direct summand of

$$\mathcal{A}_{\mathbf{H}}(K^p, \mathcal{L}_{\xi,\mathcal{O}}) \stackrel{\text{def}}{=} \operatorname*{colim}_{K_p \to \{1\}} \mathcal{A}(\mathbf{H}(\mathbf{Q})\backslash \mathbf{H}(\mathbf{A}^f)/K^p K_p, \mathcal{L}_{\xi,\mathcal{O}})$$

in the category of admissible smooth $\mathcal{O}[\mathbf{H}(\mathbf{Q}_p)]$-modules.

Proof. By construction, $\mathcal{A}_{\mathbf{H}}(K^p, \mathcal{L}_{\xi,\mathcal{O}}) \otimes_{\mathcal{O},\iota} \mathbf{C}$ is a space of algebraic automorphic forms for $\mathbf{H}$ with infinite level at p. Since $\iota\pi$ is essentially square-integrable, a standard argument with the simple trace formula shows that we may choose ξ and K^p such that some unramified twist $\iota\pi \otimes \eta'$ occurs as a direct summand of $\mathcal{A}_{\mathbf{H}}(K^p, \mathcal{L}_{\xi,\mathcal{O}}) \otimes_{\mathcal{O},\iota} \mathbf{C}$ (cf. [Shi12a] for the most general results along these lines). Replacing E by a sufficiently large finite extension, we can assume that $\eta = \iota^{-1}\eta'$ is valued in E, so then $\tau = \pi \otimes \eta$ occurs as a direct summand of $\mathcal{A}_{\mathbf{H}}(K^p, \mathcal{L}_{\xi,\mathcal{O}})[\frac{1}{\ell}]$. Since $\mathcal{A}_{\mathbf{H}}(K^p, \mathcal{L}_{\xi,\mathcal{O}})$ is an admissible smooth ℓ-torsion-free $\mathcal{O}[\mathbf{H}(\mathbf{Q}_p)]$-module, we now conclude by Lemma 2.8. □

2.3. **Some p-adic geometry.** In this subsection, we give some recollections on p-adic geometry. Fix a complete algebraically closed extension $C/\mathbf{Q}_p$. All rigid spaces will live over $\mathrm{Spa}\, C$, and all diamonds, small v-stacks, etc. will live over $\mathrm{Spd}\, C$. If G is a locally profinite group, we write $B\underline{G} = [\mathrm{Spd}\, C/\underline{G}]$.

Fix a reductive group $G/\mathbf{Q}_p$, and let Bun_G be the stack of G-bundles on the Fargues-Fontaine curve. In our convention, this is a category fibered in groupoids over $\mathrm{Perf}_{/\mathrm{Spd}\, C} \cong \mathrm{Perfd}_{/\mathrm{Spa}\, C}$. Recall that Bun_G is a small v-stack, and by a theorem of Fargues [Far15] there is a natural continuous bijection

$$|\mathrm{Bun}_G| \xrightarrow{\sim} B(G)$$
$$x_b \leftarrow\!\shortmid b$$

where $B(G)$ is topologized by the partial order topology on Newton points. Moreover, for any basic b there is a natural open immersion $j : B\underline{J_b(\mathbf{Q}_p)} \to \mathrm{Bun}_G$.

Now fix a geometric conjugacy class of minuscule cocharacters $\mu : \mathbf{G}_{m,\overline{\mathbf{Q}_p}} \to G_{\overline{\mathbf{Q}_p}}$, and let $\mathscr{F}\ell_{G,\mu}$ be the associated rigid analytic flag variety. By [CS17, §3.4-3.5], we can also identify $\mathscr{F}\ell_{G,\mu}$ with the functor $\mathrm{Perfd}_{/\mathrm{Spa}\,C} \to \mathrm{Sets}$ sending any S to the set of isomorphism classes of μ-bounded meromorphic modifications

$$(\mathcal{E}, i : \mathcal{E}_{\mathbf{1}}|_{\mathcal{X}_{S^\flat} \smallsetminus S} \xrightarrow{\sim} \mathcal{E}|_{\mathcal{X}_{S^\flat} \smallsetminus S})$$

of G-bundles on the relative Fargues-Fontaine curve $\mathcal{X}_{S^\flat}$. In particular, there is a canonical map $\tilde{q} : \mathscr{F}\ell_{G,\mu} \to \mathrm{Bun}_G$ sending the data of such a modification to the class of the bundle $\mathcal{E}$.

Proposition 2.10. *1. The map $\tilde{q}$ factors as $\mathscr{F}\ell_{G,\mu} \to [\mathscr{F}\ell_{G,\mu}/\underline{G(\mathbf{Q}_p)}] \xrightarrow{q} \mathrm{Bun}_G$, and the map q is ℓ-cohomologically smooth for all $\ell \neq p$.*

2. For any open compact subgroup $K \subset G(\mathbf{Q}_p)$, the map

$$q_K : [\mathscr{F}\ell_{G,\mu}/\underline{K}] \to [\mathscr{F}\ell_{G,\mu}/\underline{G(\mathbf{Q}_p)}] \xrightarrow{q} \mathrm{Bun}_G$$

is ℓ-cohomologically smooth for all $\ell \neq p$.

Proof. For 1., recall that the $G(\mathbf{Q}_p)$-action on $\mathscr{F}\ell_{G,\mu}$ changes a modification $(\mathcal{E}, i)$ by precomposing i with the action of $G(\mathbf{Q}_p) \cong \mathrm{Aut}(\mathcal{E}_{\mathbf{1}})$. Since the map $\tilde{q}$ discards the data of i, the claimed factorization is clear. For the smoothness claim, one checks that for any perfectoid space S/C with a map $S \to \mathrm{Bun}_G$, the fiber product

$$[\mathscr{F}\ell_{G,\mu}/\underline{G(\mathbf{Q}_p)}] \times_{\mathrm{Bun}_G} S$$

is étale-locally on S isomorphic to an open subspace of $\mathscr{F}\ell_{G,\mu^{-1}} \times_{\mathrm{Spd}\,C} S$, which is then ℓ-cohomologically smooth over S. Part 2. is now clear, since $[\mathscr{F}\ell_{G,\mu}/\underline{K}] \to [\mathscr{F}\ell_{G,\mu}/\underline{G(\mathbf{Q}_p)}]$ is étale, hence ℓ-cohomologically smooth, and compositions of cohomologically smooth maps are cohomologically smooth. □

Recall from [CS17] that $\mathscr{F}\ell_{G,\mu}$ admits a $G(\mathbf{Q}_p)$-invariant Newton stratification into locally spatial locally closed subdiamonds $\mathscr{F}\ell^b_{G,\mu} \subset \mathscr{F}\ell_{G,\mu}$ indexed by $b \in B(G,\mu)$. By definition, these are the subdiamonds corresponding to the subspaces $|\tilde{q}|^{-1}(x_b) \subset |\mathscr{F}\ell_{G,\mu}|$ for any individual $b \in B(G) = |\mathrm{Bun}_G|$.

The following observation settles a question left open in [CS17], although we won't need it in this paper.

Proposition 2.11. *The stratification $|\mathscr{F}\ell_{G,\mu}| = \coprod_{b \in B(G,\mu)} |\mathscr{F}\ell^b_{G,\mu}|$ is a true stratification, in the sense that the closure of any stratum is a union of strata. Moreover, if $G = \mathrm{GL}_n$ then $\overline{|\mathscr{F}\ell^b_{G,\mu}|} = \coprod_{b' \geq b} |\mathscr{F}\ell^{b'}_{G,\mu}|$.*

Proof. As above, the map $\tilde{q}$ factors as a $G(\mathbf{Q}_p)$-torsor followed by a cohomologically smooth map, so $|\tilde{q}|$ is an open map. The first claim now follows from the fact that $\overline{f^{-1}(S)} = f^{-1}(\overline{S})$ for any open continuous map of topological spaces, together with the fact that the decomposition $|\mathrm{Bun}_G| = \coprod_{b \in B(G)} \{x_b\}$ is obviously a true stratification. The second claim then follows from [Han17]. □

We now turn to local Shimura varieties. Choose some element $b \in B(G,\mu)$, so (G,μ,b) is a local Shimura datum.

Definition 2.12. The local Shimura variety $\mathrm{Sh}(G,\mu,b)_\infty$ is the functor on perfectoid spaces over $\mathrm{Spa}\,C$ sending any S to the set of μ-bounded meromorphic modifications

$$\mathcal{E}_{\mathbf{1}}|_{\mathcal{X}_{S^\flat} \smallsetminus S} \xrightarrow{\sim} \mathcal{E}_b|_{\mathcal{X}_{S^\flat} \smallsetminus S}.$$

For $K \subset G(\mathbf{Q}_p)$ an open compact subgroup, $\mathrm{Sh}(G,\mu,b)_K$ is the unique rigid analytic space such that $\mathrm{Sh}(G,\mu,b)^\lozenge_K = \mathrm{Sh}(G,\mu,b)_\infty/\underline{K}$.

It turns out that for any given representation $\rho^\circ \in \mathrm{Rep}_{\mathcal{O}}(J_b(\mathbf{Q}_p))_{\mathrm{adm}}$, there are several natural ways to define "the ρ°-part of the cohomology of the tower $\{\mathrm{Sh}(G,\mu,b)_K\}_K$". Fortunately these recipes all agree, up to the essentially harmless difference between complete admissible representations and smooth admissible representations. Before explaining this, we need to define the cohomology of the individual spaces $\mathrm{Sh}(G,\mu,b)_K$. Here there are already two natural candidate definitions.

Definition 2.13. Let X be any compactifiable locally spatial diamond of finite dim.trg over C, with structure morphism $f : X \to \operatorname{Spd} C$. Then for any $n \leq \infty$, we define $R\Gamma_c(X, \mathcal{O}/\varpi^n) = Rf_!(\mathcal{O}/\varpi^n)$, where $Rf_!$ is defined as in [Sch17, §22 and §26]. The assignment $X \mapsto R\Gamma_c(X, \mathcal{O})$ is covariant for open immersions, so we can also define

$$R\Gamma_{c,\mathrm{sm}}(X, \mathcal{O}) = \mathrm{colim}_{U \subset X} R\Gamma_c(U, \mathcal{O})$$

where the colimit runs over all quasicompact open subspaces $U \subset X$.[4]

Note that we could obviously also define $R\Gamma_{c,\mathrm{sm}}(X, \mathcal{O}/\varpi^n)$ for finite n, but this coincides with $R\Gamma_c(X, \mathcal{O}/\varpi^n)$.

Proposition 2.14. *1. In the notation of the previous definition, there is a natural map*

$$R\Gamma_{c,\mathrm{sm}}(X, \mathcal{O}) \to R\Gamma_c(X, \mathcal{O})$$

identifying the target with the derived ℓ-completion of the source.

2. If X is ℓ-cohomologically smooth and equipped with a $\underline{G}$-action for some locally pro-p group G, then $R\Gamma_{c,\mathrm{sm}}(X, \mathcal{O})$ is naturally a complex of smooth $\mathcal{O}[G]$-modules, and $R\Gamma_c(X, \mathcal{O}) = \gamma R\Gamma_{c,\mathrm{sm}}(X, \mathcal{O})$ where γ is the completion functor defined in Proposition 2.6.3.

Note that in the setup of part 2., one gets a map $R\Gamma_{c,\mathrm{sm}}(X, \mathcal{O}) \to \delta R\Gamma_c(X, \mathcal{O})$ by adjunction. It is plausible that this map is often an isomorphism, but we didn't try to check this.

Proof. 1. is immediate form the observation that

$$R\Gamma_{c,\mathrm{sm}}(X, \mathcal{O}) \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}/\varpi^n \to R\Gamma_c(X, \mathcal{O}) \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}/\varpi^n \cong R\Gamma_c(X, \mathcal{O}/\varpi^n)$$

is an isomorphism. For the first half of 2., note that any qc open $U \subset X$ is stable under an open pro-p subgroup $K \subset G$. Then $R\Gamma_c(U, \mathcal{O}) \in D_{\text{ét}}(B\underline{K}, \mathcal{O})$ is a compact object, and in particular it is a bounded complex of finitely generated smooth $\mathcal{O}[K]$-modules. Passing to the colimit, we deduce that $R\Gamma_{c,\mathrm{sm}}(X, \mathcal{O})$ is naturally a complex of smooth $\mathcal{O}[G]$-modules. The second half of 2. is immediate from the definition of γ. □

Definition 2.15. For (G, μ, b) a local Shimura datum and $K \subset G(\mathbf{Q}_p)$ an open compact subgroup, we define

$$R\Gamma_c(\mathrm{Sh}(G, \mu, b)_K, E) := R\Gamma_{c,\mathrm{sm}}(\mathrm{Sh}(G, \mu, b)_K, \mathcal{O}) \otimes_{\mathcal{O}} E$$

and

$$R\Gamma_c(\mathrm{Sh}(G, \mu, b), E) = \operatorname*{colim}_{K \to \{1\}} R\Gamma_c(\mathrm{Sh}(G, \mu, b)_K, E).$$

By the previous proposition, $R\Gamma_c(\mathrm{Sh}(G, \mu, b)_K, E)$ is a smooth $J_b(\mathbf{Q}_p)$-representation, and $R\Gamma_c(\mathrm{Sh}(G, \mu, b), E)$ is a smooth $G(\mathbf{Q}_p) \times J_b(\mathbf{Q}_p)$-representation.

We now restrict our attention to the case where $b \in B(G, \mu)$ is the unique basic element.

Proposition 2.16. *For any open compact subgroup $K \subset G(\mathbf{Q}_p)$ there is a natural Cartesian diagram*

$$\begin{array}{ccc} \mathrm{Sh}(G, \mu, b)_K & \longrightarrow & [\mathscr{F}\ell^b_{G,\mu}/\underline{K}] \\ \downarrow & & \downarrow {\scriptstyle q^b_K} \\ \operatorname{Spd} C & \longrightarrow & B\underline{J_b(\mathbf{Q}_p)} \end{array}$$

compatibly with varying K. Here the right-hand vertical map is the pullback of $q_K : [\mathscr{F}\ell_{G,\mu}/\underline{K}] \to \mathrm{Bun}_G$ along the open immersion $B\underline{J_b(\mathbf{Q}_p)} \subset \mathrm{Bun}_G$.

Proof. Clear from the definitions. □

Now, fix an admissible representation $\rho^\circ \in \mathrm{Rep}_{\mathcal{O}}(J_b(\mathbf{Q}_p))_{\mathrm{adm}}$.

[4] The colimit here is computed in the derived ∞-category $D(\mathcal{O})$.

Proposition 2.17. *Notation as above, consider the following complexes of $\mathcal{O}[G(\mathbf{Q}_p)]$-modules.*

1. $M_1 = \mathrm{colim}_K R\Gamma_{c,\mathrm{sm}}(\mathrm{Sh}(G,\mu,b)_K, \mathcal{O}) \otimes^{\mathbf{L}}_{\mathcal{H}_{\mathcal{O}}(J_b(\mathbf{Q}_p))} \rho^\circ$.
2. $M_2 = R\Gamma_c(\mathscr{F}\ell^b_{G,\mu}, \tilde{q}^{b*}\mathcal{F}_{\widehat{\rho^\circ}})$, *where* $\tilde{q}^b : \mathscr{F}\ell^b_{G,\mu} \to B\underline{J_b(\mathbf{Q}_p)}$ *is the obvious pullback of* $\tilde{q}$.
3. $M_3 = \mathrm{colim}_K R\Gamma([\mathscr{F}\ell_{G,\mu}/K], \tilde{j}_! q^{b*}_K \mathcal{F}_{\widehat{\rho^\circ}})$, *where* $\tilde{j} : [\mathscr{F}\ell^b_{G,\mu}/K] \to [\mathscr{F}\ell_{G,\mu}/K]$ *is the natural open immersion.*

Then M_1 and M_3 are bounded complexes of smooth $\mathcal{O}[G(\mathbf{Q}_p)]$-modules with admissible cohomology, and M_2 is a bounded derived ℓ-complete complex with complete admissible cohomology. Moreover, there are natural maps $M_1 \to M_2 \leftarrow M_3$ which become isomorphisms after derived ℓ-completion.

Applying Proposition 2.6.3 twice, we deduce that there is a natural quasi-isomorphism $M_1 \cong M_3$ in $D(G(\mathbf{Q}_p), \mathcal{O})$. Note that $M_1[\frac{1}{\ell}]$ coincides with $R\Gamma_c(G,\mu,b)[\rho]$ as defined in the introduction. On the other hand, $M_3[\frac{1}{\ell}]$ is the object which will emerge naturally in our proof of Theorem 1.1.

Proof. We first construct the maps. To build the map $M_1 \to M_2$, note that there is an obvious map

$$\mathrm{colim}_K R\Gamma_{c,\mathrm{sm}}(\mathrm{Sh}(G,\mu,b)_K, \mathcal{O}) \otimes^{\mathbf{L}}_{\mathcal{H}_{\mathcal{O}}(J_b(\mathbf{Q}_p))} \rho^\circ \to R\Gamma_c(\mathrm{Sh}(G,\mu,b)_\infty, \mathcal{O}) \widehat{\otimes}^{\mathbf{L}}_{\mathcal{H}_{\mathcal{O}}(J_b(\mathbf{Q}_p))} \widehat{\rho^\circ} \overset{\mathrm{def}}{=} N.$$

Indeed, the right-hand side is the derived ℓ-completion of the left-hand side. On the other hand, the local Hodge-Tate period map $\mathrm{Sh}(G,\mu,b)_\infty \to \mathscr{F}\ell^b_{G,\mu}$ is a $J_b(\mathbf{Q}_p)$-torsor, so by Lemma 2.18 below there is a natural isomorphism $N \cong M_2$.

To build the map $M_3 \to M_2$, note that by general nonsense, the natural map

$$\mathrm{colim}_K R\Gamma([\mathscr{F}\ell_{G,\mu}/K], \tilde{j}_! q^{b*}_K \mathcal{F}_{\widehat{\rho^\circ}}) \to R\Gamma(\mathscr{F}\ell_{G,\mu}, \tilde{j}_! \tilde{q}^{b*} \mathcal{F}_{\widehat{\rho^\circ}})$$

identifies the target with the derived ℓ-completion of the source.[5] On the other hand,

$$R\Gamma_c(\mathscr{F}\ell^b_{G,\mu}, q^{b*}\mathcal{F}_{\widehat{\rho^\circ}}) \cong R\Gamma(\mathscr{F}\ell_{G,\mu}, \tilde{j}_! q^{b*}\mathcal{F}_{\widehat{\rho^\circ}})$$

since $\mathscr{F}\ell_{G,\mu}$ is proper.

We've already shown that the maps become isomorphisms after derived ℓ-completion. Thus, by two applications of Proposition 2.6.3, it remains to see that M_3 is admissible, i.e. that for any given K

$$R\Gamma([\mathscr{F}\ell_{G,\mu}/K], \tilde{j}_! q^{b*}_K \mathcal{F}_{\widehat{\rho^\circ}})$$

is a perfect complex of $\mathcal{O}$-modules. This follows from the admissibility of ρ° and the fundamental finiteness theorems proved in [FS20]. □

In the previous proof, we used the following lemma.

Lemma 2.18. *Let X/C be a compactifiable locally spatial diamond of finite dim.trg, and let $\tilde{X} \to X$ be a G-torsor for some locally pro-p-group G with $p \neq \ell$, corresponding to a map $q : X \to B\underline{G}$. Let $A \in \widehat{D}(G, \mathcal{O})$ be any object, with $\mathcal{F}_A \in D_{\text{ét}}(B\underline{G}, \mathcal{O})$ the associated sheaf. Then there is a natural isomorphism*

$$R\Gamma_c(X, q^*\mathcal{F}_A) \cong R\Gamma_c(\tilde{X}, \mathcal{O}) \widehat{\otimes}^{\mathbf{L}}_{\mathcal{H}_{\mathcal{O}}(G)} A.$$

Proof. This follows from an easy ℓ-complete variant of the analysis in [Man04, §5] and [HK19, Appendix B]. □

Finally, we need to discuss inert representations.

Definition 2.19. Let $b \in B(G)_{\mathrm{bas}}$ be any basic element, with $j : B\underline{J_b(\mathbf{Q}_p)} \to \mathrm{Bun}_G$ the associated open immersion. A smooth admissible representation $\pi^\circ \in \mathrm{Rep}_{\mathcal{O}}(J_b(\mathbf{Q}_p))_{\mathrm{adm}}$ is *inert* if for any quasicompact open substack $W \subset \mathrm{Bun}_G$,

$$\mathrm{Cone}(j_! \mathcal{F}_{\widehat{\pi^\circ}} \to Rj_* \mathcal{F}_{\widehat{\pi^\circ}})|_W \in D_{\text{ét}}(W, \mathcal{O})$$

is killed by ℓ^n for some $n \gg 0$.

[5] Let X_i be a cofiltered system of qcqs small v-stacks with 0-truncated qcqs transition maps indexed by a category I with an initial object 0. Let $X = \lim X_i$ be the limiting small v-stack. Let $\mathcal{F}_0 \in D_{\text{ét}}(X_0, \mathcal{O})$ be some object, with pullbacks $\mathcal{F}_i \in D_{\text{ét}}(X_i, \mathcal{O})$ and $\mathcal{F} \in D_{\text{ét}}(X, \mathcal{O})$. Then there is a natural map $\mathrm{colim}_i R\Gamma(X_i, \mathcal{F}_i) \to R\Gamma(X, \mathcal{F})$ identifying the target with the derived ℓ-completion of the source.

Proposition 2.20. *Suppose that $\pi \in \mathrm{Rep}_E(J_b(\mathbf{Q}_p))_{\mathrm{adm}}$ is finitely generated and admissible. If some $\mathcal{O}$-lattice $\pi^\circ \subset \pi$ is inert, then every $\mathcal{O}$-lattice in π is inert.*

Proof. This follows easily from the commensurability of any two $\mathcal{O}$-lattices in π. □

Definition 2.21. Let $\pi \in \mathrm{Irr}_E(J_b(\mathbf{Q}_p))$ be any irreducible representation. We say π is inert if for every unramified character η such that $\pi \otimes \eta$ contains an $\mathcal{O}$-lattice, every $\mathcal{O}$-lattice in $\pi \otimes \eta$ (or equivalently, some $\mathcal{O}$-lattice in $\pi \otimes \eta$) is inert.

We will need the following crucial result.

Theorem 2.22 (Fargues-Scholze)**.** *Suppose that $\rho \in \mathrm{Irr}_E(J_b(\mathbf{Q}_p))$ is a irreducible representation admitting an $\mathcal{O}$-lattice, and that the Weil parameter*

$$\varphi_\rho : W_{\mathbf{Q}_p} \to {}^L J_b(\overline{\mathbf{Q}_\ell}) \cong {}^L G(\overline{\mathbf{Q}_\ell})$$

associated with ρ by [FS20, Theorem ???] *is supercuspidal. Then ρ is supercuspidal and inert.*

Proof. [...] □

When ρ is inert, $R\Gamma(G, \mu, b)[\rho]$ satisfies a clean duality principle.

Theorem 2.23. *Fix a basic local Shimura datum (G, μ, b), and let ρ be an inert representation of $J_b(\mathbf{Q}_p)$. Writing $(-)^*$ for the contragredient, there is a natural isomorphism*

$$R\Gamma_c(G, \mu, b)[\rho]^* \cong (R\Gamma_c(G, \mu, b)[\rho^*])\,[2d](d)$$

of W_F-equivariant objects in $D(G(\mathbf{Q}_p), E)$. In particular, there is a natural W_F-equivariant isomorphism

$$H^i_c(G, \mu, b)[\rho]^* \cong H^{2d-i}_c(G, \mu, b)[\rho^*](d)$$

of smooth $G(\mathbf{Q}_p)$-representations.

Proof. We can assume that ρ contains an $\mathcal{O}$-lattice ρ°. By Proposition 2.17, it's enough to produce a natural map

$$R\Gamma([\mathscr{F}\ell_{G,\mu}/\underline{K}], \tilde{j}_! q_K^{b*} \mathcal{F}_{\widehat{\rho^{\circ *}}})[2d](d) \to R\mathrm{Hom}(R\Gamma([\mathscr{F}\ell_{G,\mu}/\underline{K}], \tilde{j}_! q_K^{b*} \mathcal{F}_{\widehat{\rho^\circ}}), \mathcal{O})$$

functorially in open pro-p subgroups $K \subset G(\mathbf{Q}_p)$ whose cone is killed by ℓ^n for some $n \gg 0$ independent of K. For this, look at the diagram

$$\begin{array}{ccc} & [\mathscr{F}\ell_{G,\mu}/\underline{K}] & \\ {}^{p_K}\swarrow & & \searrow^{q_K} \\ B\underline{K} & & \mathrm{Bun}_G \end{array}$$

where q_K is smooth and p_K is proper and smooth. An exercise with the six functors shows that

$$R\mathrm{Hom}(R\Gamma(B\underline{K}, Rp_{K*} q_K^* \mathcal{G}), \mathcal{O}) \cong R\Gamma(B\underline{K}, (Rp_{K*} q_K^* \mathbf{D}\mathcal{G})[2d](d))$$

for any $\mathcal{G} \in D_{\text{ét}}(\mathrm{Bun}_G, \mathcal{O})$, where $\mathbf{D}\mathcal{G} = R\mathscr{H}\mathrm{om}(\mathcal{G}, \mathcal{O})$ denotes the Verdier dual. Applying this with $\mathcal{G} = j_! \mathcal{F}_{\widehat{\rho^\circ}}$, in which case $\mathbf{D}\mathcal{G} = Rj_* \mathcal{F}_{\widehat{\rho^{\circ *}}}$ we deduce that

$$R\mathrm{Hom}(R\Gamma([\mathscr{F}\ell_{G,\mu}/\underline{K}], \tilde{j}_! q_K^{b*} \mathcal{F}_{\widehat{\rho^\circ}}), \mathcal{O}) \cong R\Gamma([\mathscr{F}\ell_{G,\mu}/\underline{K}], q_K^* Rj_* \mathcal{F}_{\widehat{\rho^{\circ *}}})[2d](d).$$

But inert representations are preserved under Verdier duality, and the image of q_K is quasicompact and independent of K, so the cone of the natural map

$$\tilde{j}_! q_K^{b*} \mathcal{F}_{\widehat{\rho^{\circ *}}} \cong q_K^* j_! \mathcal{F}_{\widehat{\rho^{\circ *}}} \to q_K^* Rj_* \mathcal{F}_{\widehat{\rho^{\circ *}}}$$

is killed by some ℓ^n with n independent of K. Applying $R\Gamma([\mathscr{F}\ell_{G,\mu}/K], -)$ gives the desired map. □

Corollary 2.24. *If $\rho \in \mathrm{Rep}_E(J_b(\mathbf{Q}_p))$ is an inert supercuspidal representation, there is a natural isomorphism*

$$R\Gamma_c(G,\mu,b)[\rho] \cong \operatorname*{colim}_K R\mathrm{Hom}_{D(J_b(\mathbf{Q}_p),E)}\left(R\Gamma_c(\mathrm{Sh}(G,\mu,b)_K,E)[2d](d),\rho\right).$$

In particular, when ρ is inert, $R\Gamma_c(G,\mu,b)[\rho]$ as defined in this paper coincides with $R\Gamma_c(G,\mu,b)[\rho]$ as defined in [KW19].

Proof. It is clear from the definitions that

$$R\Gamma_c(G,\mu,b)[\rho]^* \cong \mathrm{colim}_K R\mathrm{Hom}_{D(J_b(\mathbf{Q}_p),E)}(R\Gamma_c(\mathrm{Sh}(G,\mu,b)_K,E),\rho^*)$$

for any ρ. When ρ is inert, the left-hand side is identified with $(R\Gamma_c(G,\mu,b)[\rho^*])\,[2d](d)$ by the previous theorem, so the result follows by moving the shifts and twists. □

3. Proofs

3.1. **Basic uniformization.** Let $(\mathbf{G},X)$ be a Shimura datum, with conjugacy class of (inverse) Hodge cocharacters $\mu : \mathbf{G}_{m,\mathbf{C}} \to \mathbf{G}_{\mathbf{C}}$. Fix a prime p, and set $G = \mathbf{G}\otimes_{\mathbf{Q}}\mathbf{Q}_p$. Via our fixed isomorphisms $\mathbf{C} \simeq \overline{\mathbf{Q}_p}$, we can and do regard μ as a conjugacy class of cocharacters $\mu : \mathbf{G}_{m,\overline{\mathbf{Q}_p}} \to G_{\overline{\mathbf{Q}_p}}$. In particular, we may speak of the Kottwitz set $B(G,\mu)$.

Conjecture 3.1. *Let $(\mathbf{G},X)$ be a Shimura datum, and let p be any prime. Then, with notation as above, $\mathbf{G}$ admits a unique (up to isomorphism) $\mathbf{Q}$-inner form $\mathbf{G}'$ such that:*

i. $\mathbf{G}'_{\mathbf{A}^{\{\infty p\}}} \simeq \mathbf{G}_{\mathbf{A}^{\{\infty p\}}}$ *as algebraic groups over $\mathbf{A}^{\{p\infty\}}$,*

ii. $\mathbf{G}'_{\mathbf{Q}_p} \simeq J_b$, *where $J_b/\mathbf{Q}_p$ is the inner form of G associated with the unique basic element $b \in B(G,\mu)$, and*

iii. $\mathbf{G}'(\mathbf{R})$ *is compact modulo center.*

When $(\mathbf{G},X)$ is a Hodge type Shimura datum with good reduction at p, this conjecture was proved by Xiao-Zhu [XZ17, Corollary 7.2.15].

Now, for any open compact subgroup $K \subset \mathbf{G}(\mathbf{A}^f)$, let $\mathcal{S}(\mathbf{G},X)_K$ be the associated rigid analytic Shimura variety over $\mathbf{C}_p$ with level K. If $K^p \subset \mathbf{G}(\mathbf{A}^{f,p})$ is an open compact subgroup, set

$$\mathcal{S}(\mathbf{G},X)_{K^p} = \varprojlim_{K_p\to\{1\}} \mathcal{S}(\mathbf{G},X)_{K^pK_p}.$$

If $(\mathbf{G},X)$ is of pre-abelian type, this is representable by a perfectoid space; in general it is just a diamond. By the results of [Han16], there is a canonical $\mathbf{G}(\mathbf{Q}_p)$-equivariant Hodge-Tate period map

$$\pi_{\mathrm{HT}} : \mathcal{S}(\mathbf{G},X)_{K^p} \to \mathscr{F}\ell_{G,\mu}$$

of diamonds over $\mathrm{Spd}\,\mathbf{C}_p$, compatible with varying K^p, which recovers the construction of [CS17] for Hodge type Shimura varieties. By $\mathbf{G}(\mathbf{Q}_p)$-equivariance, π_{HT} descends to a map

$$\pi_{\mathrm{HT},K_p} : \mathcal{S}(\mathbf{G},X)_{K^pK_p} \to [\mathscr{F}\ell_{G,\mu}/\underline{K_p}]$$

for any open compact subgroup $K_p \subset \mathbf{G}(\mathbf{Q}_p)$.

Next, let $b \in B(G,\mu)$ be the unique basic element, and let $\mathscr{F}\ell^b_{G,\mu}$ be the basic Newton stratum in the flag variety as defined in §2.2. This is an open $G(\mathbf{Q}_p)$-stable subspace of $\mathscr{F}\ell_{G,\mu}$. By pullback along π_{HT}, this defines an open subspace $\mathcal{S}(\mathbf{G},X)^b_{K^p} \subset \mathcal{S}(\mathbf{G},X)_{K^p}$, which by $G(\mathbf{Q}_p)$-invariance descends to an open subspace $\mathcal{S}(\mathbf{G},X)^b_K \subset \mathcal{S}(\mathbf{G},X)_K$ for any $K \subset \mathbf{G}(\mathbf{A}^f)$.

Definition 3.2. Notation as above, we say a global Shimura datum $(\mathbf{G},X)$ satisfies *basic uniformization at p* if Conjecture 3.1 holds for $(\mathbf{G},X)$, and if there is a $\mathbf{G}(\mathbf{A}^f)$-equivariant isomorphism

$$\varprojlim_{K^p} \mathcal{S}(\mathbf{G},X)^b_{K^p} \overset{\psi}{\simeq} (\underline{\mathbf{G}'(\mathbf{Q})\backslash\mathbf{G}'(\mathbf{A}^f)} \times_{\mathrm{Spd}\,\mathbf{C}_p} \mathrm{Sh}(G,\mu,b)_\infty)/\underline{J_b(\mathbf{Q}_p)} \qquad (\dagger)$$

of diamonds over $\mathrm{Spd}\,\mathbf{C}_p$ such that under the identification $\mathscr{F}\ell^b_{G,\mu} \cong \mathrm{Sh}(G,\mu,b)_\infty/\underline{J_b(\mathbf{Q}_p)}$, the Hodge-Tate period map

$$\pi_{\mathrm{HT}} : \varprojlim_{K^p} \mathcal{S}(\mathbf{G},X)^b_{K^p} \to \mathscr{F}\ell^b_{G,\mu}$$

identifies with the projection

$$(\underline{\mathbf{G}'(\mathbf{Q})\backslash\mathbf{G}'(\mathbf{A}^f)} \times_{\mathrm{Spd}\,\mathbf{C}_p} \mathrm{Sh}(G,\mu,b)_\infty)/\underline{J_b(\mathbf{Q}_p)} \to \mathrm{Sh}(G,\mu,b)_\infty/\underline{J_b(\mathbf{Q}_p)}$$

onto the second factor. Moreover, we require that $\mathbf{G}(\mathbf{A}^f) \simeq \mathbf{G}'(\mathbf{A}^{f,p}) \times G(\mathbf{Q}_p)$ acts on the right-hand side of (†) via the natural actions $\mathbf{G}'(\mathbf{A}^{f,p}) \circlearrowright \mathbf{G}'(\mathbf{Q})\backslash\mathbf{G}'(\mathbf{A}^f)$ and $G(\mathbf{Q}_p) \circlearrowright \mathrm{Sh}(G,\mu,b)_\infty$.

Let us make some remarks on this definition.

- This definition implicitly depends on the particular choices of isomorphisms $\mathbf{C} \simeq \overline{\mathbf{Q}_p}$, $\mathbf{G}'_{\mathbf{A}^{\{\infty p\}}} \simeq \mathbf{G}_{\mathbf{A}^{\{\infty p\}}}$, and $\mathbf{G}'_{\mathbf{Q}_p} \simeq J_b$, as well as on a choice of an actual element $b \in G(\breve{\mathbf{Q}}_p)$ representing the basic class in $B(G,\mu)$. In all cases where a Shimura datum is known to satisfy basic uniformization at p, the existence of an isomorphism ψ with the required properties is not sensitive to these choices (more precisely, the actual proofs that ψ exists are equivariant with respect to these choices).
- The quotient $\mathbf{G}'(\mathbf{Q})\backslash\mathbf{G}'(\mathbf{A}^f)$ is naturally a profinite set, via the isomorphism
$$\mathbf{G}'(\mathbf{Q})\backslash\mathbf{G}'(\mathbf{A}^f) \cong \lim_{K\subset\mathbf{G}'(\mathbf{A}^f)} \mathbf{G}'(\mathbf{Q})\backslash\mathbf{G}'(\mathbf{A}^f)/K$$
and the finiteness of the individual quotients $\mathbf{G}'(\mathbf{Q})\backslash\mathbf{G}'(\mathbf{A}^f)/K$.
- The name "uniformization" is indeed appropriate: for any open compact subgroup $K^pK_p \subset \mathbf{G}(\mathbf{A}^f)$, passing to the quotients by K^pK_p on both sides of (†) induces an isomorphism
$$\mathcal{S}(\mathbf{G},X)^b_{K^pK_p} \simeq \coprod \mathrm{Sh}(G,\mu,b)_{K_p}/\Gamma_i$$
for some finite set of discrete subgroups $\Gamma_i \subset J_b(\mathbf{Q}_p)$.

Of course, we are not proposing this definition in a vacuum.

Theorem 3.3. *If $(\mathbf{G},X)$ is a Shimura datum of abelian type and $p > 2$ is a prime such that $\mathbf{G}_{\mathbf{Q}_p}$ is unramified, then $(\mathbf{G},X)$ satisfies basic uniformization at p.*

Proof. In the Hodge type setting, Conjecture 3.1 was proved by Xiao-Zhu [XZ17, Corollary 7.2.15], and the basic uniformization isomorphism then follows from work of Kim and Howard-Pappas [HP17, Kim18]. These results were then extended to the abelian type setting by Shen [She19]. □

One can also treat some cases of bad reduction; this already goes back to Rapoport-Zink's book.

Theorem 3.4. *Let $(\mathbf{G},X)$ is a PEL Shimura datum arising from a rational PEL datum $(F,B,*,V,\langle,\rangle,h)$ in Kottwitz's sense. Let p be a prime such that B contains a $\mathbf{Z}_{(p)}$-order $\mathcal{O}_B$ with $\mathcal{O}_B \otimes \mathbf{Z}_p$ maximal and $*$-stable in $B \otimes \mathbf{Q}_p$. Then $(\mathbf{G},X)$ satisfies basic local uniformization at p.*

Proof. This follows from the analysis in [RZ96, Chapter 6]. Note that Rapoport-Zink in fact require the existence of a $\mathbf{Z}$-order in B with the specified properties, but this more restrictive condition is irrelevant and is actually not used anywhere in their arguments. □

Theorem 3.5. *Let (G,μ,b) be a basic local Shimura datum of the following type: assume that $G = \mathrm{Res}_{L/\mathbf{Q}_p}H$ where $H \simeq \mathrm{GL}_m(A)$ is an inner form of $\mathrm{GL}_{n/L}$ for some finite extension $L/\mathbf{Q}_p$, and that*

$$\mu : \mathbf{G}_{m,\overline{\mathbf{Q}_p}} \to G_{\overline{\mathbf{Q}_p}} = \prod_{\mathrm{Hom}(L,\overline{\mathbf{Q}_p})} \mathrm{GL}_{n,\overline{\mathbf{Q}_p}}$$

is as in the discussion preceding Theorem 1.5. Then we can find an isomorphism $\iota : \mathbf{C} \to \overline{\mathbf{Q}_p}$ and a PEL Shimura datum $(\mathbf{G},X)$ satisfying the conditions of Theorem 3.4 such that:

1. $\mathbf{G}_{\mathbf{Q}_p} \simeq G \times \mathbf{G}_m$, *and*

2. $\iota\mu_h \times \mathrm{id} = \mu$ *where* μ_h *is the (inverse) Hodge cocharacter associated with the given Shimura datum. In particular, the local Shimura varieties* $\mathrm{Sh}(G, \mu, b)_K$ *occur (up to a harmless* $\mathbf{G}_m$*-factor) in the basic uniformization at* p *of the Shimura varieties associated with the datum* $(\mathbf{G}, X)$*.*

Proof. We loosely follow the analysis in [HT01, pp. 51-57]. Choose a totally real field $F^+/\mathbf{Q}$ such that $F^+ \otimes \mathbf{Q}_p = L$ and an imaginary quadratic field E in which p is split. Let v be the unique place of F^+ above p. Set $F = F^+E$, and let w and w^c be the places of F above p, so $L \simeq F^+_v \simeq F_w$. Let B/F be a central division algebra of degree n^2 such that $B \otimes_{E,c} E \simeq B^{\mathrm{opp}}$, B_x splits for any place of F which is not split over F^+, and $B \otimes_F F_w \simeq M_m(A)$. Choose a positive involution $*$ of the second kind on B, and set $V = B$ regarded as a $B \otimes_F B^{\mathrm{op}}$-module in the natural way.

Any alternating $*$-Hermitian pairing $V \times V \to \mathbf{Q}$ is of the form $(x, y)_\beta = \mathrm{tr}_{B/\mathbf{Q}}(x\beta y^*)$ for some $\beta \in B^{*=-1}$. Any such pairing induces a perfect duality between $V \otimes_F F_w$ and $V \otimes_F F_{w^c}$. Fix a maximal order $\mathcal{O} \subset B_w$, and set

$$\Lambda_w = \mathcal{O} \subset B_w = V \otimes_F F_w = M_m(A)$$

Let $\Lambda_{w^c} \subset V \otimes_F F_{w^c}$ be the $\mathbf{Z}_p$-dual of Λ_w under the pairing $(,)_\beta : V \otimes \mathbf{Q}_p \times V \otimes \mathbf{Q}_p \to \mathbf{Q}_p$. Then $\Lambda = \Lambda_w \oplus \Lambda_{w^c} \subset V \otimes \mathbf{Q}_p$ is a $\mathbf{Z}_p$-lattice and the induced pairing $(,)_\beta : \Lambda \times \Lambda \to \mathbf{Z}_p$ is perfect. Now there is a unique maximal $*$-stable $\mathbf{Z}_{(p)}$-order $\mathcal{O}_B \subset B$ such that $\mathcal{O}_{B,w} = \mathcal{O}$, which by definition is the set of elements in B carrying Λ into itself under the natural B-action on $V \otimes \mathbf{Q}_p$.

For any β as in the discussion above, let $\mathbf{G}_\beta$ be the associated unitary similitude group over $\mathbf{Q}$ as defined in [HT01], and let $\mathbf{G}_{\beta,1}$ be the kernel of the similitude character. Note that the structure maps of these groups factor naturally over $\mathrm{Spec}\, F^+$. Then $\mathbf{G}_{\beta,\mathbf{Q}_p} \simeq G \times \mathbf{G}_m$ for any such β, so it remains to check that for any tuple

$$\{(p_\tau, q_\tau) \in (\mathbf{Z}^2_{\geq 0})^{\mathrm{Hom}(F^+,\mathbf{R})}, p_\tau + q_\tau = n\},$$

the element β can be chosen such that $\mathbf{G}_{\beta,1} \times_{\mathrm{Spec}\, F^+,\tau} \mathrm{Spec}\, \mathbf{R} \simeq U(p_\tau, q_\tau)$ for all $\tau \in \mathrm{Hom}(F^+, \mathbf{R})$. Since we don't impose any conditions on $\mathbf{G}_{\beta,\mathbf{Q}_q}$ at any finite prime $q \neq p$, the existence of such a β follows from a much easier variant of the analysis in [HT01, pp. 52-55]. □

Remark 3.6. The $\mathbf{G}_m$ factor appearing in the previous theorem is indeed harmless. To explain this, let (G, μ, b) be any local Shimura datum. Consider a product local Shimura datum $(G \times T, \mu \times \mu_T, b \times b_T)$ where (T, μ_T, b_T) is any toral local Shimura datum. Let T_b be the inner form of T associated with b_T. Let ρ be any irreducible smooth $J_b(\mathbf{Q}_p)$-representation, and let χ be any smooth character of $T_b(\mathbf{Q}_p)$. Then a Künneth formula argument shows that

$$R\Gamma_c(G \times T, \mu \times \mu_T, b \times b_T)[\rho \boxtimes \chi] \cong R\Gamma_c(G, \mu, b)[\rho] \otimes R\Gamma_c(T, \mu_T, b_T)[\chi].$$

Moreover $R\Gamma_c(T, \mu_T, b_T)[\chi]$ is nonzero and concentrated in degree zero. Together with the compatibility of the construction $\rho \rightsquigarrow \varphi_\rho$ with products, this shows that Theorem 1.1 holds for the datum (G, μ, b) and all ρ satisfying condition 2. iff it holds for the datum $(G \times T, \mu \times \mu_T, b \times b_T)$ and all $\rho \boxtimes \chi$ satisfying condition 2. In particular, when proving Theorem 1.1, it is permissible to replace the datum (G, μ, b) by any product with a toral local Shimura datum.

It would be interesting to prove basic uniformization at p under the hypotheses of [KP18, Theorem 0.1]. This is probably within reach.

Corollary 3.7. *Let* $(\mathbf{G}, X)$ *be a Shimura datum satisfying basic uniformization at* p *as in Definition 3.2, and let* $\mathbf{G}'$ *be the associated inner form of* $\mathbf{G}$*. Fix some open compact subgroup* $K^pK_p \subset \mathbf{G}(\mathbf{A}^f)$*. Let* $\mathcal{L}_{\xi,\mathcal{O}}$ *be a* K^p*-stable* $\mathcal{O}$*-lattice in the* E*-linear realization of some irreducible algebraic representation of* $\mathbf{G}$ *of highest weight* ξ*, so (with the obvious abuse of notation) there is an associated* ℓ*-adic étale sheaf* $\mathcal{L}_{\xi,\mathcal{O}}$ *on* $\mathcal{S}(\mathbf{G}, X)_{K^pK_p}$*.*

Then there is a natural isomorphism

$$R\pi_{\mathrm{HT},K_p*}\mathcal{L}_{\xi,\mathcal{O}}|_{[\mathscr{F}\ell^b_{G,\mu}/\underline{K_p}]} \cong q^{b*}_{K_p}\mathcal{F}_\Pi$$

in $D_{\text{ét}}([\mathscr{F}\ell^b_{G,\mu}/\underline{K_p}],\mathcal{O})$ *compatibly with varying* K_p*. Here*

$$q^b_{K_p} : [\mathscr{F}\ell^b_{G,\mu}/\underline{K_p}] \to B\underline{J_b(\mathbf{Q}_p)}$$

is the map defined in Proposition 2.16, and $\Pi = \mathcal{A}_{\mathbf{G}'}(K^p,\mathcal{L}_{\xi,\mathcal{O}})^\wedge$ *is the* ℓ*-adic completion of the space of algebraic automorphic forms defined in Proposition 2.9.*

Proof. Quotienting the basic uniformization isomorphism by K^pK_p, we get a Cartesian diagram

$$\begin{array}{ccc} \mathcal{S}(\mathbf{G},X)^b_{K^pK_p} & \xrightarrow{\pi^b_{\mathrm{HT},K_p}} & [\mathscr{F}\ell^b_{G,\mu}/\underline{K_p}] \\ \downarrow g & & \downarrow q^b_{K_p} \\ [\underline{\mathbf{G}'(\mathbf{Q})\backslash\mathbf{G}'(\mathbf{A}^f)/K^p}/\underline{J_b(\mathbf{Q}_p)}] & \xrightarrow{f} & B\underline{J_b(\mathbf{Q}_p)} \end{array}$$

where the horizontal maps are proper and the vertical maps are ℓ-cohomologically smooth. Moreover, the definition of basic uniformization implies that the restriction of $\mathcal{L}_{\xi,\mathcal{O}}$ to $\mathcal{S}(\mathbf{G},X)^b_{K^pK_p}$ is the pullback along g of the correct local system (which we also denote by $\mathcal{L}_{\xi,\mathcal{O}}$) on the v-stack $[\underline{\mathbf{G}'(\mathbf{Q})\backslash\mathbf{G}'(\mathbf{A}^f)/K^p}/\underline{J_b(\mathbf{Q}_p)}]$. In particular,

$$R\pi_{\mathrm{HT},K_p*}\mathcal{L}_{\xi,\mathcal{O}}|_{[\mathscr{F}\ell^b_{G,\mu}/\underline{K_p}]} \cong R\pi^b_{\mathrm{HT},K_p*}g^*\mathcal{L}_{\xi,\mathcal{O}}$$

with the obvious abuse of notation. Moreover,

$$Rf_*\mathcal{L}_{\xi,\mathcal{O}} \in D_{\text{ét}}(B\underline{J_b(\mathbf{Q}_p)},\mathcal{O}) \cong \widehat{D}(J_b(\mathbf{Q}_p),\mathcal{O})$$

is admissible and concentrated in degree zero, and is given concretely as (the sheaf $\mathcal{F}_\Pi$ associated with) the claimed representation $\Pi = \mathcal{A}_{\mathbf{G}'}(K^p,\mathcal{L}_{\xi,\mathcal{O}})^\wedge$. Since the vertical maps are smooth, the result now follows from the smooth base change isomorphism $q^{b*}_{K_p}Rf_* \cong R\pi^b_{\mathrm{HT},K_p*}g^*$. ☐

3.2. **Proof of Theorem 1.1.** In this section, we state and prove the key technical lemma, and conclude the proof of Theorem 1.1.

Lemma 3.8. *Let* X *be a quasicompact small v-stack with an* ℓ*-cohomologically smooth map* $f : X \to \mathrm{Bun}_G$*. Let* $j : B\underline{J_b(\mathbf{Q}_p)} \to \mathrm{Bun}_G$ *be the inclusion of a basic stratum which meets the image of* f*, so we get a cartesian diagram*

$$\begin{array}{ccc} U & \xrightarrow{\tilde{j}} & X \\ \downarrow f^b & & \downarrow f \\ B\underline{J_b(\mathbf{Q}_p)} & \xrightarrow{j} & \mathrm{Bun}_G \end{array}$$

where $\tilde{j}$ *is an open immersion.*

Let $\mathcal{F}$ *be a complex in* $D_{\text{ét}}(X,\mathcal{O})$*. Assume that* $\mathcal{F}|_U = f^{b*}\mathcal{G}_b$ *for some complex* $\mathcal{G}_b \in D_{\text{ét}}(B\underline{J_b(\mathbf{Q}_p)},\mathcal{O})$*. Let* ρ *be an inert supercuspidal representation of* $J_b(\mathbf{Q}_p)$ *with a* $J_b(\mathbf{Q}_p)$*-stable* $\mathcal{O}$*-lattice* ρ°*, and let* $\mathcal{F}_{\widehat{\rho^\circ}} \in D_{\text{ét}}(B\underline{J_b(\mathbf{Q}_p)},\mathcal{O})$ *be the associated* ℓ*-adic sheaf.*

If $\mathcal{F}_{\widehat{\rho^\circ}}$ *occurs as an isogeny direct summand of* $\mathcal{G}_b$*, then* $f^*j_!\mathcal{F}_{\widehat{\rho^\circ}} \cong \tilde{j}_!f^{b*}\mathcal{F}_{\widehat{\rho^\circ}}$ *occurs as an isogeny direct summand of* $\mathcal{F}$ *in* $D_{\text{ét}}(X,\mathcal{O})$*.*

The essential strength of this lemma is that we don't need to know anything about the complementary restriction $\mathcal{F}|_{X\smallsetminus U}$.

Proof of Theorem 1.1. Fix (G,μ,b) as in the statement of the theorem, and let ρ be an inert supercuspidal representation ρ of $J_b(\mathbf{Q}_p)$. Set $d = \dim \mathrm{Sh}(G,\mu,b)_K$. Fix a global Shimura datum $(\mathbf{G},X)$ such that $\mathrm{Sh}(G,\mu,b)_K$ occurs in the associated basic uniformization at p. Note that the Shimura varieties associated with this Shimura datum also have dimension d. Let $\mathbf{G}'$ be the inner form of $\mathbf{G}$ occurring in the basic uniformization. After replacing ρ by an unramified twist if necessary, Proposition 2.9 implies that we may choose an $\mathcal{O}$-lattice $\rho^\circ \subset \rho$ and K^p and $\mathcal{L}_{\xi,\mathcal{O}}$ as in the statement of Proposition 2.9 such that ρ° occurs as an isogeny direct summand of $\mathcal{A}_{\mathbf{G}'}(K^p, \mathcal{L}_{\xi,\mathcal{O}})$. Let $\Pi = \mathcal{A}_{\mathbf{G}'}(K^p, \mathcal{L}_{\xi,\mathcal{O}})^\wedge$ be the ℓ-adic completion, so $\widehat{\rho^\circ}$ is an isogeny direct summand of Π.

We are going to apply Lemma 3.8 to $X = [\mathscr{F}\ell_{G,\mu}/\underline{K_p}]$ and the map $f = q_{K_p} : [\mathscr{F}\ell_{G,\mu}/\underline{K_p}] \to \mathrm{Bun}_G$, with $\mathcal{F} = R\pi_{\mathrm{HT},K_p*}\mathcal{L}_{\xi,\mathcal{O}}$ and with $b \in B(G,\mu)$ the unique basic element. Then $U = [\mathscr{F}\ell^b_{G,\mu}/\underline{K_p}]$, and Corollary 3.7 implies that $\mathcal{F}|_U \cong q^{b*}_{K_p}\mathcal{G}_b$ upon setting $\mathcal{G}_b = \mathcal{F}_\Pi$. Moreover, $\widehat{\rho^\circ}$ is an isogeny direct summand of Π by design, so $\mathcal{F}_{\widehat{\rho^\circ}}$ is an isogeny direct summand of $\mathcal{F}_\Pi = \mathcal{G}_b$.

This verifies the hypotheses of Lemma 3.8. Applying that lemma, we deduce that there is an isogeny inclusion

$$\tilde{j}_! q^{b*}_{K_p}\mathcal{F}_{\widehat{\rho^\circ}} \to R\pi_{\mathrm{HT},K_p*}\mathcal{L}_{\xi,\mathcal{O}}$$

functorially in K_p, where

$$\tilde{j} : [\mathscr{F}\ell^b_{G,\mu}/\underline{K_p}] \to [\mathscr{F}\ell_{G,\mu}/\underline{K_p}]$$

is the natural open immersion. Applying $R\Gamma([\mathscr{F}\ell_{G,\mu}/\underline{K_p}], -)$, we deduce that there is an isogeny inclusion

$$R\Gamma([\mathscr{F}\ell_{G,\mu}/\underline{K_p}], \tilde{j}_! q^{b*}_{K_p}\mathcal{F}_{\widehat{\rho^\circ}}) \to R\Gamma(\mathcal{S}(\mathbf{G},X)_{K^pK_p}, \mathcal{L}_{\xi,\mathcal{O}})$$

functorially in K_p. In particular, inverting ℓ and applying Proposition 2.17, we conclude that

$$R\Gamma_c(\mathrm{Sh}(G,\mu,b)_{K_p}, E) \otimes^{\mathbf{L}}_{\mathcal{H}(J_b(\mathbf{Q}_p))} \rho \cong R\Gamma([\mathscr{F}\ell_{G,\mu}/\underline{K_p}], \tilde{j}_! q^{b*}_{K_p}\mathcal{F}_{\widehat{\rho^\circ}})[1/\ell]$$

is a direct summand of $R\Gamma(\mathcal{S}(\mathbf{G},X)_{K^pK_p}, \mathcal{L}_{\xi,E})$. On the other hand, ξ is regular, so the vanishing theorem

$$H^i(\mathcal{S}(\mathbf{G},X)_{K^pK_p}, \mathcal{L}_{\xi,E}) = 0 \text{ for all } i < \dim \mathcal{S}(\mathbf{G},X)_K = d$$

follows by the Artin comparison theorem and Li-Schwermer's vanishing theorem [LS04]. Since

$$H^i_c(\mathrm{Sh}(G,\mu,b))[\rho] = \mathrm{colim}_{K_p} H^i(R\Gamma_c(\mathrm{Sh}(G,\mu,b)_{K_p}, E) \otimes^{\mathbf{L}}_{\mathcal{H}(J_b(\mathbf{Q}_p))} \rho),$$

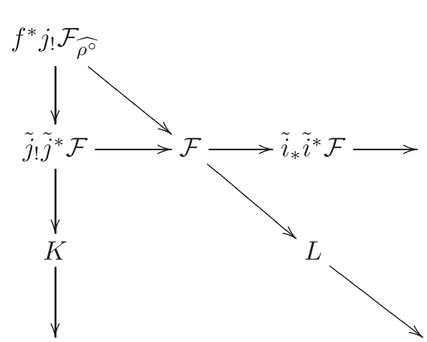
we conclude that $H^i_c(\mathrm{Sh}(G,\mu,b))[\rho] = 0$ for all $i < d$. Rerunning the entire argument with ρ replaced by ρ^* and applying Theorem 2.23, we deduce that

$$H^i_c(\mathrm{Sh}(G,\mu,b))[\rho]^* \cong H^{2d-i}_c(\mathrm{Sh}(G,\mu,b))[\rho^*](d) = 0$$

for all $i > d$. This completes the proof. □

Proof of Lemma 3.8. Let $i : \mathrm{Bun}_G \smallsetminus B\underline{J_b(\mathbf{Q}_p)} \to \mathrm{Bun}_G$ be the inclusion of the closed complement, and let $\tilde{i} : Z \to X$ be the pullback of i.

By assumption, we can pick an isogeny inclusion $\mathcal{F}_{\widehat{\rho^\circ}} \to \mathcal{G}_b$, which pulls back to an isogeny inclusion $f^*j_!\mathcal{F}_{\widehat{\rho^\circ}} \to f^*j_!\mathcal{G}_b = \tilde{j}_!\tilde{j}^*\mathcal{F}$. These can be embedded in a larger diagram



where by definition the small triangle commutes and each of the three sets of colinear arrows form a distinguished triangle. By Proposition 2.2.3, it suffices to show that the map $h : L \to f^*j_!\mathcal{F}_{\widehat{\rho^\circ}}[1]$ is killed by power of ℓ.

By the octahedral axiom, L sits in a distinguished triangle $K \to L \to \tilde{i}_* \tilde{i}^* \mathcal{F} \to$. Applying $\mathrm{Hom}(-, f^* j_! \mathcal{F}_{\widehat{\rho^\circ}}[1])$ to this triangle gives an exact sequence

$$\mathrm{Hom}(\tilde{i}_* \tilde{i}^* \mathcal{F}, f^* j_! \mathcal{F}_{\widehat{\rho^\circ}}[1]) \xrightarrow{\alpha} \mathrm{Hom}(L, f^* j_! \mathcal{F}_{\widehat{\rho^\circ}}[1]) \xrightarrow{\beta} \mathrm{Hom}(K, f^* j_! \mathcal{F}_{\widehat{\rho^\circ}}[1]).$$

The image of h under the map β is the connecting map $K \to f^* j_! \mathcal{F}_{\widehat{\rho^\circ}}[1]$ associated with the vertical triangle, which is killed by a power of ℓ since the map $f^* j_! \mathcal{F}_{\widehat{\rho^\circ}} \to \tilde{j}_! \tilde{j}^* \mathcal{F}$ is an isogeny inclusion (using Proposition 2.2.3 again). Therefore $\ell^n h \in \mathrm{im}\alpha$ for some n, so it suffices to show that $\mathrm{Hom}(\tilde{i}_* \tilde{i}^* \mathcal{F}, f^* j_! \mathcal{F}_{\widehat{\rho^\circ}}[1])$ is killed by a power of ℓ.

In fact, I claim that $R\mathrm{Hom}(\tilde{i}_* \tilde{i}^* \mathcal{F}, f^* j_! \mathcal{F}_{\widehat{\rho^\circ}})$ is killed by a power of ℓ. For this, note that $f^*(-) = \mathcal{L} \otimes Rf^!(-)$ for some shifted rank one local system $\mathcal{L}$, since f is ℓ-cohomologically smooth. Then

$$\begin{aligned} R\mathrm{Hom}(\tilde{i}_* \tilde{i}^* \mathcal{F}, f^* j_! \mathcal{F}_{\widehat{\rho^\circ}}) &\cong R\mathrm{Hom}(\tilde{i}_* \tilde{i}^* \mathcal{F}, \mathcal{L} \otimes Rf^! j_! \mathcal{F}_{\widehat{\rho^\circ}}) \\ &\cong R\mathrm{Hom}(Rf_!(\tilde{i}_* \tilde{i}^* \mathcal{F} \otimes \mathcal{L}^{-1}), j_! \mathcal{F}_{\widehat{\rho^\circ}}) \\ &\cong R\mathrm{Hom}(i_* i^* Rf_!(\mathcal{F} \otimes \mathcal{L}^{-1}), j_! \mathcal{F}_{\widehat{\rho^\circ}}) \\ &\cong R\mathrm{Hom}(i^* Rf_!(\mathcal{F} \otimes \mathcal{L}^{-1}), Ri^! j_! \mathcal{F}_{\widehat{\rho^\circ}}) \end{aligned}$$

where the second and fourth lines follow from adjointness and the third line follows from two applications of proper base change. Quite generally we have $Ri^! j_!(-) = i^* Rj_*(-)[-1]$, so $Ri^! j_! \mathcal{F}_{\widehat{\rho^\circ}} \cong i^* Rj_* \mathcal{F}_{\widehat{\rho^\circ}}[-1]$. Since ρ is inert by assumption, the cone C of the natural map $j_! \mathcal{F}_{\widehat{\rho^\circ}} \to Rj_* \mathcal{F}_{\widehat{\rho^\circ}}$ is killed by a power of ℓ after restriction to any quasicompact open substack $W \subset \mathrm{Bun}_G$. Applying i^* to this map and noting that $i^* j_! = 0$, we deduce that $Ri^! j_! \mathcal{F}_{\widehat{\rho^\circ}} \cong i^* Rj_* \mathcal{F}_{\widehat{\rho^\circ}}[-1] \cong i^* C[-1]$ is killed by a power of ℓ after restriction to any such W. Choosing W large enough to contain the image of f, the natural map

$$R\mathrm{Hom}(i^* Rf_!(\mathcal{F} \otimes \mathcal{L}^{-1}), Ri^! j_! \mathcal{F}_{\widehat{\rho^\circ}}) \to R\mathrm{Hom}(i^* Rf_!(\mathcal{F} \otimes \mathcal{L}^{-1})|_W, Ri^! j_! \mathcal{F}_{\widehat{\rho^\circ}}|_W)$$

is an isomorphism. Since $Ri^! j_! \mathcal{F}_{\widehat{\rho^\circ}}|_W$ is killed by a power of ℓ, this concludes the proof. □

3.3. **Remaining proofs.** In this section we prove the remaining results stated in the introduction.

Proof of Theorems 1.4 and 1.6. We first prove Theorem 1.4. Combining Theorem 1.1 and [KW19, Theorem 1.0.2], we deduce that $H^i(G, \mu, b)[\rho] = 0$ for all $i \neq d$, and hence the equality

$$\left[H^d(G, \mu, b)[\rho]\right] = \sum_{\pi \in \Pi_\phi(G)} \dim \mathrm{Hom}_{S_\phi}(\delta_{\pi,\rho}, r_\mu)[\pi]$$

in $\mathrm{Groth}(G(\mathbf{Q}_p))$, which we can reinterpret as an isomorphism

$$H^d(G, \mu, b) \simeq^{\mathrm{ss}} \oplus_{\pi \in \Pi_\phi(G)} \dim \mathrm{Hom}_{S_\phi}(\delta_{\pi,\rho}, r_\mu) \cdot \pi$$

where $\simeq^{\mathrm{ss}}$ denotes an isomorphism of semisimplified $G(\mathbf{Q}_p)$-representations. Let ω be the central character of ρ. A simple computation shows that $H^d(G, \mu, b)[\rho]$, and hence any subquotient, also has central character ω under the canonical identification $Z_G(\mathbf{Q}_p) \cong Z_{J_b}(\mathbf{Q}_p)$. This shows that $H^d(G, \mu, b)[\rho]$ is an iterated extension of the elements $\pi \in \Pi_\phi(G)$ (with multiplicities as given above) in the category $\mathrm{Rep}_E(G(\mathbf{Q}_p))_\omega$. Since all elements $\pi \in \Pi_\phi(G)$ are supercuspidal and any supercuspidal representation with central character ω is both projective and injective as an object of the category $\mathrm{Rep}_E(G(\mathbf{Q}_p))_\omega$ [AR04], all the relevant extension classes are zero. Therefore $H^d(G, \mu, b)[\rho]$ is semisimple, so

$$H^d(G, \mu, b)[\rho] = \oplus_{\pi \in \Pi_\phi(G)} \dim \mathrm{Hom}_{S_\phi}(\delta_{\pi,\rho}, r_\mu) \cdot \pi$$

as desired.

Theorem 1.6 now follows, since the Fargues-Scholze construction agrees with the known local Langlands correspondence for inner forms of restrictions of scalars of GL_n. This translates condition 2. of Theorem 1.4 into the stated condition on ρ. Condition 1. is a consequence of Theorem 3.5 and Remark 3.6. □

Proof of Theorem 1.5. Combine Theorem 1.1 with [Far04, Théorème 8.1.4] or [Shi12b, Corollary 1.3]. Again, Condition 1. follows from Theorem 3.5 and Remark 3.6. □

Proof of Theorem 1.8. Fix a finite extension $L/\mathbf{Q}_p$ and an embedding $L \to \overline{\mathbf{Q}_p}$. Set $\Sigma = \mathrm{Hom}_{\mathrm{alg}}(L, \overline{\mathbf{Q}_p})$. Let $G = \mathrm{Res}_{L/\mathbf{Q}_p}\mathrm{GL}_n$ be as in the theorem, and fix a supercuspidal Weil parameter $\varphi : W_{\mathbf{Q}_p} \to {}^LG(\overline{\mathbf{Q}_\ell})$. Note that by Proposition 8.4 in Borel's Corvallis article, isomorphism classes of such φ's are in bijection (in an obvious abuse of notation) with isomorphism classes of continuous irreducible $\varphi : W_L \to \mathrm{GL}_n(\overline{\mathbf{Q}_\ell})$. Note also that $\widehat{G} \cong \mathrm{GL}^{\Sigma}_{n,\overline{\mathbf{Q}_\ell}}$.

First, we note that $\mathscr{C}_\varphi$ can be described extremely explicitly in this situation. To give this description, recall that under the canonical identification $B(\mathrm{GL}_n, L) = B(G)_{\mathrm{bas}} \cong \mathbf{Z}$ given by sending an isocrystal to its degree, the composite map

$$\mathbf{Z} \cong B(G)_{\mathrm{bas}} \to H^1(\mathbf{Q}_p, G^{\mathrm{ad}})$$

sends d to the inner form $G_d = \mathrm{Res}_{L/\mathbf{Q}_p}\mathrm{GL}_m(D)$, where $m = \gcd(d, n)$ and D is the central division algebra over L with Hasse invariant $= \frac{d}{n} \bmod 1$. In this notation, we have a canonical identification

$$D_{\mathrm{lis}}(\mathrm{Bun}^{\mathrm{ss}}_G, \overline{\mathbf{Q}_\ell}) \cong \prod_{d\in\mathbf{Z}} D_{\mathrm{lis}}(B\underline{G_d(\mathbf{Q}_p)}, \overline{\mathbf{Q}_\ell}) = \prod_{d\in\mathbf{Z}} D(G_d(\mathbf{Q}_p), \overline{\mathbf{Q}_\ell}).$$

For each d, let $\pi_{\varphi,d}$ be the unique irreducible supercuspidal representation of $G_d(\mathbf{Q}_p)$ with Weil parameter φ, and let $\mathcal{F}_{\pi_{\varphi,d}}$ be the associated sheaf on $B\underline{G_d(\mathbf{Q}_p)}$ as before. Then an arbitrary object

$$\mathcal{F} = \prod \mathcal{F}_d \in D_{\mathrm{lis}}(\mathrm{Bun}^{\mathrm{ss}}_G, \overline{\mathbf{Q}_\ell})$$

lies in $\mathscr{C}_\varphi$ if and only if $H^i(\mathcal{F}_d)$ is an iterated self-extension of copies of $\mathcal{F}_{\pi_{\varphi,d}}$ for all d and all i.

Next we check that T_W is t-exact on $\mathscr{C}_\varphi$ for any irreducible minuscule $W \in \mathrm{Rep}(\widehat{G})$. For any irreducible representation, $(\overline{\mathbf{Q}_\ell}^{\times})^{\Sigma} \cong Z_{\widehat{G}}(\overline{\mathbf{Q}_\ell})$ acts on W via scaling by a character $(x_\sigma)_{\sigma\in\Sigma} \mapsto \prod x_\sigma^{d_\sigma}$. Set $d(W) = \sum d_\sigma$. Assuming W is minuscule and replacing W by a central twist, we can assume that the cocharacter μ_W corresponding to the highest weight character is as in the discussion before Theorem 1.5. Then unwinding all definitions and appealing to Theorem 1.6, we deduce that

$$T_W \mathcal{F}_{\pi_{\varphi,d}} = \mathcal{F}^{\oplus \dim W}_{\pi_{\varphi,d+d(W)}}$$

as elements of $\mathscr{C}_\varphi$. By the discussion in the previous paragraph, this proves the t-exactness of any such T_W. Since any representation is a summand of a tensor product of minuscule representations, we now conclude t-exactness of general Hecke operators by geometric Satake. Now, set $\mathscr{F}_\varphi = \prod_{d\in\mathbf{Z}} \mathcal{F}_{\pi_{\varphi,d}}$. Then $T_W \mathscr{F}_\varphi \simeq \mathscr{F}_\varphi^{\oplus \dim W}$ for any minuscule W.

We give the remainder of the proof in the case where $L = \mathbf{Q}_p$; the general case will be treated in the final draft of this paper. Let $\mathscr{G}_\varphi$ be the Hecke eigensheaf constructed by Anschütz-Le Bras [ALB20]. Writing $\mathscr{G}_{\varphi,d} = \mathscr{G}_\varphi | B\underline{G_d(\mathbf{Q}_p)}$, we need to prove that $\mathscr{G}_{\varphi,d} \simeq \mathcal{F}_{\pi_{\varphi,d}}$ for all d. By the analysis in [ALB20], we know that $\mathscr{G}_{\varphi,1} \simeq \mathcal{F}_{\pi_{\varphi,1}}$. Applying T_{Std} to this isomorphism and forgetting the Weil group action, the arguments in the previous paragraph show that

$$T_{\mathrm{Std}}\mathscr{G}_{\varphi,1} \simeq T_{\mathrm{Std}}\mathcal{F}_{\pi_{\varphi,1}} \simeq \mathcal{F}^{\oplus n}_{\pi_{\varphi,2}}.$$

On the other hand, $T_{\mathrm{Std}}\mathscr{G}_{\varphi,1} \simeq \mathscr{G}_{\varphi,2} \boxtimes \varphi$ by the Hecke eigensheaf property. Taken together, these observations show that $\mathscr{G}_{\varphi,2} \boxtimes \varphi \simeq \mathcal{F}_{\pi_{\varphi,2}} \boxtimes \varphi'$ for some n-dimensional $W_{\mathbf{Q}_p}$-representation φ'. Using the irreducibility of $\pi_{\varphi,2}$ and φ, this forces isomorphisms $\mathscr{G}_{\varphi,2} \simeq \mathcal{F}_{\pi_{\varphi,2}}$ and $\varphi \simeq \varphi'$. Applying T_{Std} to the isomorphism $\mathscr{G}_{\varphi,2} \simeq \mathcal{F}_{\pi_{\varphi,2}}$ and repeating this argument, we conclude by induction that $\mathscr{G}_{\varphi,j} \simeq \mathcal{F}_{\pi_{\varphi,j}}$ for all $j \geq 1$. For $j < 1$, we instead apply $T_{\mathrm{Std}^\vee}$ to the isomorphism $\mathscr{G}_{\varphi,1} \simeq \mathcal{F}_{\pi_{\varphi,1}}$ and argue by downwards induction. □

Remark 3.9. By combining the arguments in [ALB20] with [KW19, Theorem 1.0.2], it is not difficult to see that $\mathscr{G}_{\varphi,j} \simeq \mathcal{F}_{\pi_{\varphi,j}}[n_j]$ for all $j \in \mathbf{Z}$, but with some unspecified shifts n_j. When φ is self-dual, a simple duality argument shows that $n_j = 0$. However, for a general φ, it doesn't seem possible to show that $n_j = 0$ for all $j \in \mathbf{Z}$ by some "purely formal" argument. One really needs something like Theorem 1.1.

Proof of Theorem 1.9. This follows from unwinding all definitions. E.g., one checks that

$$R\Gamma_c(G,\mu,b,\mathcal{IC}_W)\otimes^{\mathbf{L}}_{\mathcal{H}(J_b(\mathbf{Q}_p))}\rho\cong (T_W j_!\mathcal{F}_\rho)|B\underline{G(\mathbf{Q}_p)}.$$

□

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