Perfectoid quotients of the Lubin-Tate tower, revisited

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0.1 Statement of results

As in the main text[1] of the paper, fix a finite extension $K/\mathbf{Q}_p$ with residue field $k = \mathbf{F}_q$. Let $\breve{K}$ be the completed maximal unramified extension of K, and fix some complete algebraically closed extension $C/\breve{K}$. Let $P_{n-d,d} \subset \mathrm{GL}_n$ denote the usual maximal parabolic with upper left (resp. lower right) diagonal block sizes $n-d$ (resp. d), and set $P = P_{n-1,1}$. For $U \subset \mathrm{GL}_n(K)$ any open compact subgroup, let $\mathcal{M}_U$ denote the base change along $\breve{K} \to C$ of the generic fiber of the Lubin-Tate space with U-level structure. By the results in [SW13], there is a unique perfectoid space $\mathcal{M}_\infty$ over $\mathrm{Spa}\, C$ with a $\mathrm{GL}_n(K)$-action such that $\mathcal{M}_\infty \sim \lim_{U \to \{1\}} \mathcal{M}_U$.[2]

In this appendix we study the quotient $\mathcal{M}_{P(K)} = \mathcal{M}_\infty/P(K)$, and variants for other parabolics, by purely local methods. In paricular, when $n = 2$, we give a direct proof that $\mathcal{M}_{P(K)}$ is a perfectoid space, avoiding any reference to the p-adic geometry of modular curves. Our main tool is a p-adic Hodge-theoretic description of $\mathcal{M}_\infty$ in terms of vector bundles on the Fargues-Fontaine curve, due to Weinstein and Scholze-Weinstein.

Our first result is a moduli-theoretic description of these quotients. For this, recall that for any characteristic p perfectoid space S/k there is an associated (adic, relative) Fargues-Fontaine curve $\mathcal{X}_S = \mathcal{X}_{S,K}$ defined functorially in S. Moreover for any reduced rational number $\lambda = d/r$, this curve comes with a canonical vector bundle $\mathcal{O}(\lambda)$ of rank r and degree d, defined functorially in S.

Theorem 0.1. *As a functor on perfectoid spaces over* $\mathrm{Spa}\, C$*, the diamond quotient* $\mathcal{M}_\infty/\underline{P(K)}$ *is canonically equivalent to the functor sending any* $S \to \mathrm{Spa}\, C$ *to the set of sub-vector bundles* $\mathcal{E} \subset \mathcal{O}(1/n)$ *over* $\mathcal{X}_{S^\flat}$ *such that after pullback along any geometric point* $\overline{x} \to S^\flat$*, the map* $\mathcal{E}_{\overline{x}} \to \mathcal{O}(1/n)_{\overline{x}}$ *is injective and* $\mathcal{E}_{\overline{x}} \simeq \mathcal{O}^{n-1}$.

Curiously, this description shows that $\mathcal{M}_\infty/\underline{P(K)}$ is the base change along $\mathrm{Spd}\, C \to \mathrm{Spd}\, k$ of a naturally defined functor on all perfectoid spaces over k. We also note that, by Proposition 4.2.7 in the main text, the diamond quotient $\mathcal{M}_\infty/\underline{P(K)}$ coincides with the perfectoid space $\mathcal{M}_{P(K)}$, so there is little harm in switching between these points of view.

Corollary 0.2. *As a diamond over* $\mathrm{Spd}\, C$*,* $\mathcal{M}_\infty/\underline{P(K)}$ *is proper and* ℓ*-cohomologically smooth for any* $\ell \neq p$.

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[1] The final version of this note will appear as an appendix to the paper "A quotient of the Lubin-Tate tower II" by C. Johansson and J. Ludwig. This explains the references to "the main text of the paper", etc.

[2] In the main text of the paper, $\mathcal{M}_\infty$ is denoted $\mathcal{M}_{\mathbf{1}}$.

By [Sch17, Theorem 1.12], this implies very strong finiteness properties for the $\mathbf{F}_\ell$-étale cohomology of $\mathcal{M}_\infty/\underline{P(K)}$. On the other hand, $\mathcal{M}_\infty/\underline{P(K)}$ is a perfectoid space by the main results of the paper. Thus $\mathcal{M}_\infty/\underline{P(K)}$ is an example of an interesting and naturally occuring perfectoid space with reasonable étale cohomology.

When $n = 2$, the description of $\mathcal{M}_\infty/\underline{P(K)}$ can be unwound further.

Theorem 0.3. *If $n = 2$, then $\mathcal{M}_\infty/\underline{P(K)}$ can be presented as the quotient*

$$\left(\operatorname{Spd} L^\flat \times_{\operatorname{Spd} \mathbf{F}_{q^2}} \operatorname{Spd} C\right)/(\varphi \times \mathrm{id})^{\mathbf{Z}}$$

for a certain perfectoid field $L^\flat/\mathbf{F}_{q^2}$, where φ is the q^2-power Frobenius on $L^\flat$.

Combining this description with some formalism of diamonds, we obtain a purely local proof of the main result of [L], avoiding the global p-adic geometry of modular curves.

Corollary 0.4. *When $n = 2$, the quotient $\mathcal{M}_\infty/\underline{P(K)}$ is a perfectoid space over C.*

In fact, it turns out that our methods give some information about the more general quotients $\mathcal{M}_\infty/\underline{P_{n-d,d}(K)}$. In particular, we prove the following result, which shows that Theorem A in the paper is essentially optimal.

Theorem 0.5. *As a diamond over $\operatorname{Spd} C$, $\mathcal{M}_\infty/\underline{P_{n-d,d}(K)}$ is proper and ℓ-cohomologically smooth for any $1 \leq d < n$. However, when $d > 1$, this quotient is* never *a perfectoid space.*

Again, we deduce this from a moduli-theoretic description of these more general quotients in terms of vector bundles on the curve, which recovers Theorem 0.1 when $d = 1$. When $d > 1$ this description is slightly more complicated, cf. Proposition 0.10 below.

It's unclear to me how far these ideas can be extended beyond the specific case of the Lubin-Tate tower. As an illustrative example, let $\mathcal{N}_\infty$ be the infinite-level perfectoid space over $\operatorname{Spa} C$ associated with the Rapoport-Zink tower for an isoclinic ϖ-divisible $\mathcal{O}_K$-module of height 5 and dimension 2. There is a natural action of $\mathrm{GL}_5(K)$ on $\mathcal{N}_\infty$, and one can check (by adapting the arguments below) that the quotients $\mathcal{N}_\infty/\underline{P_{i,5-i}(K)}$ are ℓ-cohomologically smooth over $\operatorname{Spd} C$ for $i \in \{1, 2, 4\}$. However, for $i = 3$, the method breaks down, and I don't know whether the quotient is smooth in that case.

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I'm very grateful to Christian Johansson and Judith Ludwig for their invitation to write this appendix, and for some very interesting conversations about this circle of ideas. This appendix grew out of the (re)proof of Corollary 0.4 given below, and I'd like to thank Jared Weinstein for some stimulating initial conversations around the question of whether this result could be proved by purely local methods.

0.2 Preliminaries

For any perfectoid space S/k, we write $\mathcal{X}_S = \mathcal{X}_{S,K}$ for the associated relative Fargues-Fontaine curve, regarded as an adic space over $\operatorname{Spa} K$. If $S = T^\flat$ arises as the tilt of some perfectoid space T/K, $\mathcal{X}_{T^\flat}$ comes equipped with a canonical closed immersion $\iota : T \to \mathcal{X}_{T^\flat}$. Aside from the original reference [FF], some relevant background on the curve is given in [KL15, Ch. 8] and [Han, §2.3]. One might also look at [CS15, §3.2-3.3] or at some portions of [BFHHLWY].

When S is a point, $\mathcal{X}_S$ is a Noetherian adic space of dimension one [Ked], with a good theory of slopes and Harder-Narasimhan filtrations. Moreover, it is reduced and all of its local rings are DVRs, so any coherent $\mathcal{O}_\mathcal{X}$-module $\mathcal{F}$ has a canonical filtration $0 \to \mathcal{F}_{\mathrm{tors}} \to \mathcal{F} \to \mathcal{F}_{\mathrm{free}} \to 0$ where $\mathcal{F}_{\mathrm{tors}}$ is a torsion coherent sheaf and $\mathcal{F}_{\mathrm{free}}$ is a vector bundle. In particular, given any vector bundle $\mathcal{F}$, any coherent subsheaf $\mathcal{E} \subset \mathcal{F}$ is also a vector bundle, and admits a canonical *saturation* $\mathcal{E}^{\mathrm{sat}} \subset \mathcal{F}$, defined as the preimage of $(\mathcal{F}/\mathcal{E})_{\mathrm{tors}} \subset \mathcal{F}/\mathcal{E}$ in $\mathcal{F}$. This is the minimal subbundle of $\mathcal{F}$ containing $\mathcal{E}$ such that $\mathcal{F}/\mathcal{E}^{\mathrm{sat}}$ is also a vector bundle. Note also that $\deg(\mathcal{E}^{\mathrm{sat}}) = \deg(\mathcal{E}) + \mathrm{length}(\mathcal{F}/\mathcal{E})_{\mathrm{tors}}$. If $\mathcal{E} = \mathcal{E}^{\mathrm{sat}}$, we say that $\mathcal{E}$ is saturated. With these preparations, we can state a trivial lemma, which nevertheless is frequently very useful.

Lemma 0.6. *Suppose that S is a point, and that $\mathcal{E} \to \mathcal{F}$ is an injective map of vector bundles on $\mathcal{X}_S$. Suppose that the point $(\mathrm{rank}(\mathcal{E}), \deg(\mathcal{E})+1)$ lies above the Harder-Narasimhan polygon of $\mathcal{F}$. Then $\mathcal{E}$ is automatically saturated.*

In particular, if $\mathcal{F}$ is semistable and $\frac{\deg(\mathcal{E})+1}{\mathrm{rank}(\mathcal{E})} > \mu(\mathcal{F})$, then $\mathcal{E}$ is automatically saturated.

If $i : \mathcal{E} \to \mathcal{F}$ is any injective map of vector bundles (or arbitrary $\mathcal{O}_{\mathcal{X}_S}$-modules) over a relative curve $\mathcal{X}_S$, we say that i is *stably injective* if it remains injective after base change along $\mathcal{X}_T \to \mathcal{X}_S$ for $T \to S$ any map of perfectoid spaces. This is equivalent to the a priori weaker condition that $\mathcal{E} \to \mathcal{F}$ remains injective after base change along $\mathcal{X}_{\overline{x}} \to \mathcal{X}_S$ for any geometric point $\overline{x} \to S$. This condition automatically holds if the quotient $\mathcal{F}/\mathcal{E}$ is a vector bundle, but in general it is weaker.

Lemma 0.7. *Let $\mathcal{E} \to \mathcal{F}$ be a stably injective map of vector bundles over a relative curve $\mathcal{X}_S$, such that $\mathcal{E} \simeq \mathcal{O}^m$ and $\mathcal{F} \simeq \mathcal{O}(1/n)$ at all geometric points of S, for some fixed integers $m < n$. Then the quotient $\mathcal{F}/\mathcal{E}$ is a vector bundle, with $\mathcal{F}/\mathcal{E} \simeq \mathcal{O}(1/(n-m))$ at all geometric points of S.*

Proof. When S is a geometric point, the claim follows from the Fargues-Fontaine classification of vector bundles on $\mathcal{X}_S$. Indeed, consider an injective map $i : \mathcal{O}^m \to \mathcal{O}(1/n)$ with $m < n$. Since $\mathcal{O}(1/n)$ is stable, the previous Lemma implies that $\operatorname{coker} i$ is automatically a vector bundle, which necessarily has rank $n-m$ and degree 1. Moreover, all the Harder-Narasimhan slopes of $\operatorname{coker} i$ are $\geq 1/n$ (using the stability of $\mathcal{O}(1/n)$ again), so in particular, they are all positive, so the degree of $\operatorname{coker} i$ is bounded below by its number of distinct Harder-Narasimhan slopes. Thus $\operatorname{coker} i$ has a unique slope, which must be $1/(n-m)$, so $\operatorname{coker} i \simeq \mathcal{O}(1/(n-m))$.

The result when S is a (not necessarily geometric) point now follows by an easy descent. To check that $\mathcal{F}/\mathcal{E}$ is a vector bundle in general, note that our arguments so far imply that for any S and any point $x \in |\mathcal{X}_S|$, the $k(x)$-rank of the fiber $(\mathcal{F}/\mathcal{E}) \otimes_{\mathcal{O}_{\mathcal{X}_S}} k(x)$ is $n-m$. Indeed, let $y \in |S|$ be the image of x under the map $|\mathcal{X}_S| \to |S|$; then formation of the $k(x)$-fiber factors over the pullback of $\mathcal{E} \to \mathcal{F}$ along $\mathcal{X}_y = \mathcal{X}_{\mathrm{Spa}(k_y, k_y^+)} \to \mathcal{X}_S$, in the sense that $(\mathcal{F}/\mathcal{E})_x \cong (\mathcal{F}_y/\mathcal{E}_y) \otimes_{\mathcal{O}_{\mathcal{X}_y}} k(x)$. By our previous arguments, $\mathcal{F}_y/\mathcal{E}_y$ is a vector bundle of rank $n-m$, so $\mathrm{rank}_{k(x)}(\mathcal{F}/\mathcal{E}) \otimes_{\mathcal{O}_{\mathcal{X}_S}} k(x)$ is constant as a function of x. Since $\mathcal{X}_S$ is a stably uniform adic space, we then deduce from [KL15, Prop. 2.8.4] that $\mathcal{F}/\mathcal{E}$ is a finite locally free $\mathcal{O}_{\mathcal{X}_S}$-module. □

Remark. The argument in the preceding proof shows more generally that if $i : \mathcal{E} \to \mathcal{F}$ is any stably injective map of vector bundles over a relative curve $\mathcal{X}_S$ such that $\operatorname{coker} i_{\overline{x}}$ is torsion-free after pullback along any geometric point $\overline{x} \to S$, then $\operatorname{coker} i$ is a vector bundle.

0.3 General results

In this section we prove Theorems 0.1 and 0.5, and Corollary 0.2.

Our starting point is the following result of Scholze-Weinstein, which is a special case of [SW17, Cor. 23.2.3 and Theorem 24.3.5] (cf. also [SW13]).

Proposition 0.8. *As a functor on perfectoid spaces over* $\operatorname{Spa} C$*,* $\mathcal{M}_\infty$ *is canonically identified with the functor sending any* $S \to \operatorname{Spa} C$ *to the set of stably injective maps* $\alpha : \mathcal{O}^n \to \mathcal{O}(1/n)$ *over* $\mathcal{X}_{S^\flat}$ *such that* $\operatorname{coker} \alpha \simeq \iota_* W$ *for some rank one projective* $\mathcal{O}_S$*-module* W*.*

Next, we note that for a closed subgroup $H \subset \mathrm{GL}_n(K)$, it is easy to tell whether $\mathcal{M}_\infty/\underline{H} \to \operatorname{Spd} C$ is proper.

Proposition 0.9. *If* $H \subset \mathrm{GL}_n(K)$ *is any closed subgroup, the structure map* $\mathcal{M}_\infty/\underline{H} \to \operatorname{Spd} C$ *is separated; moreover, it is proper if and only if* $\mathrm{GL}_n(K)/H$ *is compact. In particular, any quotient* $\mathcal{M}_\infty/\underline{P_{n-d,d}(K)}$ *is proper over* $\operatorname{Spd} C$*.*

Proof. For any such quotient, the structure map to $\operatorname{Spd} C$ factors over a (surjective!) map $q : \mathcal{M}_\infty/\underline{H} \to \mathbf{P}_C^{n-1,\diamond}$ induced by the Gross-Hopkins period map. The pullback of q along the v-cover $\mathcal{M}_\infty \to \mathbf{P}_C^{n-1,\diamond}$ is then canonically identified with the projection map $\tilde{q} : \underline{\mathrm{GL}_n(K)/H} \times \mathcal{M}_\infty \to \mathcal{M}_\infty$. The latter map is always separated, so q is separated by [Sch17, Proposition 10.11.ii]; since the target of q is separated over $\operatorname{Spd} C$, this shows that the source is too. Likewise, q is quasicompact if and only if $\tilde{q}$ is quasicompact, and the latter clearly holds if and only if $\mathrm{GL}_n(K)/H$ is compact. □

We begin by analyzing the general quotients $\mathcal{M}_\infty/\underline{P_{n-d,d}(K)}$.

Proposition 0.10. *Fix any* $1 \leq d < n$*. Then the diamond quotient* $\mathcal{M}_\infty/\underline{P_{n-d,d}(K)}$ *is canonically identified with the functor* $X_{n,d}$ *on perfectoid spaces over* $\operatorname{Spa} C$ *sending any* S *to the set of (isomorphism classes of) diagrams*

$$\mathcal{O}(1/n) \twoheadrightarrow \mathcal{E} \hookleftarrow \mathcal{F}$$

of vector bundles over $\mathcal{X}_{S^\flat}$ *such that* $\mathcal{E} \simeq \mathcal{O}(1/d)$ *and* $\mathcal{F} \simeq \mathcal{O}^d$ *at all geometric points and such that* $\operatorname{coker}(\mathcal{F} \to \mathcal{E}) \simeq \iota_* W$ *for some projective rank one* $\mathcal{O}_S$*-module* W*.*

Proof. First, observe that there is a natural map $\mathcal{M}_\infty \to X_{n,d}$, given by sending any $\{\alpha : \mathcal{O}^n \to \mathcal{O}(1/n)\} \in \mathcal{M}_\infty(S)$ to the diagram

$$\mathcal{O}(1/n) \twoheadrightarrow \mathcal{O}(1/n)/\alpha(\mathcal{O}^{n-d} \oplus 0) \hookleftarrow \alpha(\mathcal{O}^n)/\alpha(\mathcal{O}^{n-d} \oplus 0)$$

of vector bundles over $\mathcal{X}_{S^\flat}$. For this, observe that the quotient $\mathcal{O}(1/n)/\alpha(\mathcal{O}^{n-d} \oplus 0)$ is isomorphic to $\mathcal{O}(1/d)$ at all geometric points by Lemma 0.7, and the remaining conditions are clearly satisfied. The datum of this diagram only depends on the $\underline{P_{n-d,d}(K)}(S)$-orbit of α, so this map factors over a natural transformation $\mathcal{M}_\infty/\underline{P_{n-d,d}(K)} \to X_{n,d}$, and we claim this transformation is actually an isomorphism.

It clearly suffices to check that $\mathcal{M}_\infty \to X_{n,d}$ is a $\underline{P_{n-d,d}(K)}$-torsor. For this, let $\mathcal{O}(1/n) \twoheadrightarrow \mathcal{E} \hookleftarrow \mathcal{F}$ be any S-point of $X_{n,d}$. Let $\mathcal{V} \subset \mathcal{O}(1/n)$ be the rank n sub-vector bundle defined by the cartesian diagram

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{\gamma} & \mathcal{F} \\ \downarrow & & \downarrow \\ \mathcal{O}(1/n) & \longrightarrow & \mathcal{E} \end{array}$$

so $\mathcal{V} \to \mathcal{O}(1/n)$ is stably injective and $\mathcal{O}(1/n)/\mathcal{V} = \mathcal{E}/\mathcal{F} \simeq \iota_* W$. Moreover, at any geometric point of S, $\mathcal{V}$ has degree zero and all HN slopes $\leq 1/n$, so in fact $\mathcal{V} \simeq \mathcal{O}^n$ at all geometric points. Now, the ambiguity in lifting our given S-point of $X_{n,d}$ to an S-point of $\mathcal{M}_\infty$ is exactly the ambiguity of choosing a trivialization $\mathcal{O}^n \xrightarrow{\sim} \mathcal{V}$ which maps $\mathcal{O}^{n-d} \oplus 0$ isomorphically onto $\ker \gamma$, and the space of such trivializations is clearly a $\underline{P_{n-d,d}(K)}$-torsor over $X_{n,d}$, as desired. □

Proposition 0.11. *Fix any $1 \le d < n$. Then the diamond $\mathcal{M}_\infty/\underline{P_{n-d,d}(K)} \cong X_{n,d}$ is isomorphic to the quotient*

$$\left(\mathcal{S}\mathrm{urj}(\mathcal{O}(1/n),\mathcal{O}(1/d)) \times_{\mathrm{Spd}\, C} \mathbf{P}_C^{d-1,\lozenge}\right)/\underline{D_{1/d}^\times}.$$

Here $\mathcal{S}\mathrm{urj}(\mathcal{O}(1/n),\mathcal{O}(1/d))$ is the functor on perfectoid spaces over C parametrizing surjective maps $\mathcal{O}(1/n) \to \mathcal{O}(1/d)$, and $D_{1/d}$ is the division algebra over K of invariant $1/d$, with $\underline{D_{1/d}^\times}$ acting diagonally on the two factors.

Proof. Let $\tilde{X}_{n,d}$ be the $\underline{D_{1/d}^\times}$-torsor over $X_{n,d}$ which (in the notation of the previous proposition) parametrizes trivializations $\mathcal{O}(1/d) \xrightarrow{\sim} \mathcal{E}$. Then $\tilde{X}_{n,d}$ clearly decomposes as

$$\mathcal{S}\mathrm{urj}(\mathcal{O}(1/n),\mathcal{O}(1/d)) \times_{\mathrm{Spd}\, C} Y$$

where Y is the functor whose S-points parametrize subbundles $\mathcal{F} \subset \mathcal{O}(1/d)$ such that $\mathcal{O}(1/d)/\mathcal{F} \simeq \iota_* W$ for some projective rank one $\mathcal{O}_S$-module W. The data of such an $\mathcal{F}$ is obviously equivalent to the data of a rank one projective $\mathcal{O}_S$-module quotient $\iota^*\mathcal{O}(1/d) \to W$: the functor in one direction is obvious, and the functor in the other direction sends $\iota^*\mathcal{O}(1/d) \to W$ to $\ker(\mathcal{O}(1/d) \to \iota_*\iota^*\mathcal{O}(1/d) \to \iota_* W)$. Finally, $\iota^*\mathcal{O}(1/d)$ is *canonically* identified with $\mathcal{O}_S^d$. Putting these observations together, Y identifies with the functor sending S to the set of rank one locally free $\mathcal{O}_S$-module quotients $\mathcal{O}_S^d = \iota^*\mathcal{O}(1/d) \to W$. The latter functor is obviously represented by $\mathbf{P}_C^{d-1,\lozenge}$, as desired. □

Proof of Theorem 0.5. We've already dealt with properness. For cohomological smoothness, combining the previous proposition with [Sch17, Prop. 24.2] reduces us to showing that

$$\mathcal{S}\mathrm{urj}(\mathcal{O}(1/n),\mathcal{O}(1/d)) \times_{\mathrm{Spd}\, C} \mathbf{P}_C^{d-1,\lozenge} \to \mathrm{Spd}\, C$$

is cohomologically smooth. This reduces to the smoothness of each factor over $\mathrm{Spd}\, C$. The projective space factor is immediately handled by [Sch17, Prop. 24.4]. For the first factor, we note that $\mathcal{S}\mathrm{urj}(\mathcal{O}(1/n),\mathcal{O}(1/d))$ is an open subfunctor of $\mathcal{H}^0(\mathcal{O}(1/d) \otimes \mathcal{O}(-1/n))$, cf. [BFHHLWY, Prop. 3.3.6]. Since $\mathcal{O}(1/d)\otimes\mathcal{O}(-1/n)$ has slopes strictly between 0 and 1, the latter functor is representable by an open perfectoid ball in $n-d$ variables over C, so now smoothness follows from [Sch17, Prop. 24.1].

Finally, suppose that $\mathcal{M}_\infty/\underline{P_{n-d,d}(K)}$ is a perfectoid space. As in the previous proposition, we have a natural $\underline{D_{1/d}^\times}$-torsor

$$\mathcal{S}\mathrm{urj}(\mathcal{O}(1/n),\mathcal{O}(1/d)) \times_{\mathrm{Spd}\, C} \mathbf{P}_C^{d-1,\lozenge} \to \mathcal{M}_\infty/\underline{P_{n-d,d}(K)}.$$

By assumption, the target is perfectoid, so then the source is perfectoid as well by [Sch17, Prop. 10.11]. Intuitively, we now expect a contradiction if $d > 1$, because the projective space factor should contribute "non-perfectoid directions" to the source.

To make this precise, choose some perfectoid field C'/C and a map $\mathrm{Spd}\, C' \to \mathcal{S}\mathrm{urj}(\mathcal{O}(1/n),\mathcal{O}(1/d))$. We've already observed that $\mathcal{S}\mathrm{urj}(\mathcal{O}(1/n),\mathcal{O}(1/d))$ is perfectoid, so

$$\mathrm{Spd}\, C' \times_{\mathcal{S}\mathrm{urj}(\mathcal{O}(1/n),\mathcal{O}(1/d))} \left(\mathcal{S}\mathrm{urj}(\mathcal{O}(1/n),\mathcal{O}(1/d)) \times_{\mathrm{Spd}\, C} \mathbf{P}_C^{d-1,\lozenge}\right)$$

is a fiber product of perfectoid spaces, and thus is perfectoid. On the other hand, this fiber product is just $\mathbf{P}_{C'}^{d-1,\lozenge}$. Putting things together, we've shown that if $\mathcal{M}_\infty/\underline{P_{n-d,d}(K)}$ is perfectoid, then $\mathbf{P}_{C'}^{d-1,\lozenge}$ is necessarily perfectoid, which forces $d = 1$, as desired. □

Proof of Theorem 0.1. Specializing Proposition 0.11 to the situation where $d = 1$, we get a canonical identification

$$\mathcal{M}_\infty/\underline{P(K)} \cong \mathcal{S}\mathrm{urj}(\mathcal{O}(1/n), \mathcal{O}(1))/\underline{K^\times}.$$

This is nothing more than the functor parametrizing quotients $\mathcal{O}(1/n) \twoheadrightarrow \mathcal{L}$ where $\mathcal{L}$ is a line bundle of degree one. It remains to identify this functor with the functor parametrizing subbundles $\mathcal{E} \subset \mathcal{O}(1/n)$ as specified in Theorem 0.1.

For this, note that sending any such $\mathcal{E} \subset \mathcal{O}(1/n)$ to the quotient $\mathcal{O}(1/n) \twoheadrightarrow \mathcal{O}(1/n)/\mathcal{E}$ defines a natural transformation in one direction, since $\mathcal{O}(1/n)/\mathcal{E}$ is a line bundle of degree one by Lemma 0.7. We also have a transformation in the other direction, sending any $q : \mathcal{O}(1/n) \twoheadrightarrow \mathcal{L}$ to the inclusion $\ker q \subset \mathcal{O}(1/n)$: one easily checks that, at any geometric point, $\ker q$ has rank $n-1$, degree zero, and all HN slopes $\leq 1/n$, so $\mathrm{ker} q \simeq \mathcal{O}^{n-1}$ at any geometric point. These two natural transformations are mutually inverse to each other, as desired. □

0.4 The case $n = 2$

In this section we prove Theorem 0.3 and Corollary 0.4.

As in the body of the paper, fix a uniformizer $\varpi \in \mathcal{O}_K$. Let E be the unramified quadratic extension of K, and let $\mathcal{G} = \mathcal{G}_E$ be the unique Lubin-Tate formal $\mathcal{O}_E$-module for which multiplication by ϖ is given by the polynomial $f(T) = T^{q^2} + \varpi T$. Let $\tilde{E}/K$ be the completion of the extension obtained by adjoining all ϖ-division points of $\mathcal{G}$ to E. By Lubin-Tate theory, $\tilde{E}$ is Galois over E with Galois group $\mathcal{O}_E^\times$, and $\tilde{E}$ is a perfectoid field.

Lemma 0.12. *The fixed field $L = \tilde{E}^{\mathcal{O}_K^\times}$ is a perfectoid field.*

Proof. By the basic definitions, L is the completion of a Galois extension of E with Galois group $\mathcal{O}_E^\times/\mathcal{O}_K^\times$. This is an abelian p-adic Lie group of dimension $[K : \mathbf{Q}_p] > 0$, so L is perfectoid by a theorem of Sen. (Alternately, up to a finite extension, L is the completion of a compositum of totally ramified $\mathbf{Z}_p$-extensions of E, so we could appeal to Tate's original results.) □

Let $\tilde{\mathcal{G}} = \lim_{\substack{\leftarrow \\ [\varpi]}} \mathcal{G}$ be the universal cover of $\mathcal{G}$, and let $\tilde{\mathcal{G}}_0$ be its reduction mod ϖ. As in [Wei], there is an identification $\tilde{\mathcal{G}}_0 = \mathrm{Spf}\mathbf{F}_{q^2}[[T^{1/p^\infty}]]$. This is a formal E-vector space in the category of formal schemes over $\mathbf{F}_{q^2}$.

To relate this object to vector bundles, let Y be the functor on perfectoid spaces over $\mathbf{F}_{q^2}$ sending any S to $H^0(\mathcal{X}_{S,K}, \mathcal{O}(1/2)) = H^0(\mathcal{X}_{S,E}, \mathcal{O}(1))$, and let $Y^\times \subset Y$ be the open subfunctor of nowhere-vanishing sections. By Theorem 0.1, there is a natural identification

$$\mathcal{M}_\infty/\underline{P(K)} \cong Y^\times/\underline{K^\times} \times_{\mathrm{Spd}\,\mathbf{F}_{q^2}} \mathrm{Spd}\, C.$$

Proposition 0.13. *There are compatible $K^\times$-equivariant isomorphisms $Y \cong \tilde{\mathcal{G}}_0$ and $Y^\times \cong \tilde{\mathcal{G}}_0 \smallsetminus \{0\}$.*

Proof. The first isomorphism follows from the fact that Y and $\tilde{\mathcal{G}}_0$ are both naturally identified with the functor $\mathbf{B}_{\mathrm{crys},E}^{+,\varphi_{q^2}=\varpi}$. For Y this identification is immediate from the definition of $\mathcal{O}(1)$ over $\mathcal{X}_{S,E}$ as the descent of a φ_{q^2}-equivariant bundle on the usual cover $\mathcal{Y}_{S,E} \to \mathcal{X}_{S,E}$. For $\tilde{\mathcal{G}}_0$ this identification follows from [SW13], and in fact can be given by an explicit formula: an (R, R^+)-point of $\tilde{\mathcal{G}}_0$ is the same as an element $x \in R^{\circ\circ}$, and we map x to the element $\log_{\mathcal{G}}(\{x\})$, where $\{x\} = \lim_{n\to\infty}[\varpi^n](\varphi_{q^2}^{-n}(\tilde{x}))$ with $\tilde{x} \in W_{\mathcal{O}_E}(R^+) = W(R^+) \otimes_{W(\mathbf{F}_{q^2})} \mathcal{O}_E$ any lift of x. The second isomorphism is then immediate. □

On the other hand, we have

Proposition 0.14. *There are compatible $E^\times$-equivariant isomorphisms $\tilde{\mathcal{G}}_0 \cong \mathrm{Spa}\,\mathcal{O}_{\tilde{E}}^\flat$ and $\tilde{\mathcal{G}}_0 \smallsetminus \{0\} \cong \mathrm{Spa}\,\tilde{E}^\flat$, where on the right-hand sides $\mathcal{O}_E^\times \subset E^\times$ acts through its natural identification with $\mathrm{Gal}(\tilde{E}/E)$ and ϖ acts as the q^2-power Frobenius.*

Proof. This follows from Proposition 3.5.3 in [Wei]. □

Putting these two propositions together, we get a $K^\times$-equivariant isomorphism

$$Y^\times \times_{\mathrm{Spd}\,\mathbf{F}_{q^2}} \mathrm{Spd}\,C \cong \mathrm{Spd}\,\tilde{E}^\flat \times_{\mathrm{Spd}\,\mathbf{F}_{q^2}} \mathrm{Spd}\,C.$$

Passing to the quotient by the action of $\mathcal{O}_K^\times$ gives

$$\begin{aligned} Y^\times/\underline{\mathcal{O}_K^\times} \times_{\mathrm{Spd}\,\mathbf{F}_{q^2}} \mathrm{Spd}\,C &\cong (\mathrm{Spd}\,\tilde{E}^\flat)/\underline{\mathcal{O}_K^\times} \times_{\mathrm{Spd}\,\mathbf{F}_{q^2}} \mathrm{Spd}\,C \\ &\cong \mathrm{Spd}\,(\tilde{E}^\flat)^{\mathcal{O}_K^\times} \times_{\mathrm{Spd}\,\mathbf{F}_{q^2}} \mathrm{Spd}\,C \\ &\cong \mathrm{Spd}\,L^\flat \times_{\mathrm{Spd}\,\mathbf{F}_{q^2}} \mathrm{Spd}\,C \end{aligned}$$

where in the second line we've used Lemma 0.12. Note that this diamond is the fiber product of two characteristic p perfectoid spaces over a *discrete* field. Nevertheless we have the following result.

Proposition 0.15. *Let $k/\mathbf{F}_p$ be a field with the discrete topology, and let X and Y be perfectoid spaces over k. Then the product $X \times Y$ is representable by a perfectoid space over k, where the product is taken in the category of sheaves of sets on Perf_k.*

By this result, $\mathrm{Spd}\,L^\flat \times_{\mathrm{Spd}\,\mathbf{F}_{q^2}} \mathrm{Spd}\,C$ is representable by a perfectoid space over $\mathbf{F}_{q^2}$, which moreover comes equipped with a canonical map to $\mathrm{Spd}\,C = \mathrm{Spa}\,C^\flat$. Moreover, writing $\varphi : \mathrm{Spd}\,L^\flat \to \mathrm{Spd}\,L^\flat$ for the q^2-power Frobenius, we easily see that $\varphi \times \mathrm{id}$ acts properly discontinuously on this product, so the quotient

$$(\mathrm{Spd}\,L^\flat \times_{\mathrm{Spd}\,\mathbf{F}_{q^2}} \mathrm{Spd}\,C)/(\varphi \times \mathrm{id})^{\mathbf{Z}}$$

is representable by a perfectoid space over k with a map to $\mathrm{Spd}\,C$. This has a unique untilt to a perfectoid space over $\mathrm{Spa}\,C$. On the other hand, summarizing the analysis above, we have canonical isomorphisms

$$(\mathrm{Spd}\,L^\flat \times_{\mathrm{Spd}\,\mathbf{F}_{q^2}} \mathrm{Spd}\,C)/(\varphi \times \mathrm{id})^{\mathbf{Z}} \cong Y^\times/\underline{K^\times} \times_{\mathrm{Spd}\,\mathbf{F}_{q^2}} \mathrm{Spd}\,C \cong \mathcal{M}_\infty/\underline{P(K)},$$

so $\mathcal{M}_\infty/\underline{P(K)}$ is a perfectoid space, as desired.

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