Sketch of the main geometric argument in HKW

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On a previous episode

Recall the setup: $F/\mathbf{Q}_p$ a finite extension, G/F a connected reductive group, μ a conjugacy class of geometric cocharacters, $b \in B(G,\mu)$ basic, G_b the associated inner form.
Recall from Lecture 2: have an explicit Jacquet-Langlands transfer operator $T_{b,\mu}^{G_b \to G} : C(G_b(F)_{\mathrm{sr}}//G_b(F)) \to C(G(F)_{\mathrm{sr}}//G(F))$ given by

$$[T_{b,\mu}^{G_b \to G} f](g) = (-1)^{\langle 2\rho,\mu \rangle} \sum_{(g,g',\lambda)\in \mathrm{Rel}_b} \mathrm{dim} r_\mu[\lambda] f(g').$$

Theorem (Lecture 2)

Assume the refined LLC + etc. Let ϕ be a discrete L-parameter. Then for any $\rho \in \Pi_\phi(G_b)$, have an equality

$$[T_{b,\mu}^{G_b \to G} \Theta_\rho](g) = \sum_{\pi \in \Pi_\phi(G)} [\dim \mathrm{Hom}_{S_\phi}(\delta_{\pi,\rho}, r_\mu)]\Theta_\pi(g).$$

This is half of the puzzle...

Today's episode

Recall from Lecture 1 the complex $R\Gamma(G, b, \mu)[\rho]$ appearing in the Kottwitz conjecture. This is a bounded complex whose cohomologies are *finite-length* admissible representations of $G(F)$. $\rightsquigarrow$ Can form the finite-length virtual representation $\mathrm{Mant}_{G,b,\mu}(\rho) = \sum_i (-1)^i H^i(R\Gamma(G, b, \mu)[\rho])$, which then has a Harish-Chandra character $\Theta_{\mathrm{Mant}_{G,b,\mu}(\rho)}$.

Theorem (Today's main theorem)

We have an equality

$$\Theta_{\mathrm{Mant}_{G,b,\mu}(\rho)}(g) = [T_{b,\mu}^{G_b \to G} \Theta_\rho](g)$$

for any **elliptic** $g \in G(F)$.

This theorem holds for **any** finite-length ρ, independent of any knowledge of LLC. Today: Detailed sketch of the argument.

Overview

Key steps in the proof:

1. Switch to distributions.
2. Reduction to ℓ-torsion coefficients.
3. Invocation of the Lefschetz-Verdier trace formula.
4. Decoupling the contributions of ρ and $\mathcal{S}_\mu$.
5. Explicit calculation of local terms associated with $\mathcal{S}_\mu$.

Steps 1. and 2. are preliminary reductions. Steps 3. and 4. are the heart of the argument. Step 5. can be taken as a black box.

Step 1: Switch to distributions

Recall from Lecture 3: For Λ any $\mathbf{Z}[1/p]$-algebra, can form

$$\mathrm{Dist}(G(F),\Lambda)^{G(F)} = \mathrm{Hom}_{G(F)}(C_c(G(F),\Lambda)\otimes \mathrm{Haar}(G,\Lambda),\Lambda).$$

Can also define an elliptic variant

$$\mathrm{Dist}(G(F)_{\mathrm{ell}},\Lambda)^{G(F)} = \mathrm{Hom}_{G(F)}(C_c(G(F)_{\mathrm{ell}},\Lambda)\otimes \mathrm{Haar}(G,\Lambda),\Lambda),$$

so have a restriction map $\mathrm{Dist}(G(F),\Lambda)^{G(F)} \to \mathrm{Dist}(G(F)_{\mathrm{ell}},\Lambda)^{G(F)}$.
Any admissible $A \in D(G(F),\Lambda)$ has a trace distribution $\mathrm{tr.dist}(A) \in \mathrm{Dist}(G(F),\Lambda)^{G(F)}$. Write $\mathrm{tr.dist}_{\mathrm{ell}}(A)$ for its image in $\mathrm{Dist}(G(F)_{\mathrm{ell}},\Lambda)^{G(F)}$.
Want to reinterpret $T_{b,\mu}^{G_b\to G}$ in terms of distributions.

Step 1: Switch to distributions cont'd

Using the geometry of shtuka spaces, we will define a linear map

$$\mathcal{T}_{b,\mu}^{G_b \to G} : \mathrm{Dist}(G_b(F)_{\mathrm{ell}}, \Lambda)^{G_b(F)} \to \mathrm{Dist}(G(F)_{\mathrm{ell}}, \Lambda)^{G(F)}$$

such that, if $\Lambda = \overline{\mathbf{Q}_\ell}$, the diagram

$$\begin{array}{ccc} C(G_b(F)_{\mathrm{sr}}//G_b(F)) & \longrightarrow & \mathrm{Dist}(G_b(F)_{\mathrm{ell}}, \Lambda) \\ \downarrow {\scriptstyle T_{b,\mu}^{G_b \to G}} & & \downarrow {\scriptstyle \mathcal{T}_{b,\mu}^{G_b \to G}} \\ C(G(F)_{\mathrm{sr}}//G(F)) & \longrightarrow & \mathrm{Dist}(G(F)_{\mathrm{ell}}, \Lambda) \end{array}$$

commutes. $\rightsquigarrow$ We are reduced (by easy formal arguments) to proving that for all $\rho \in \mathrm{Irr}_{\overline{\mathbf{Q}_\ell}}(G_b(F))$, there is an equality

$$\mathcal{T}_{b,\mu}^{G_b \to G}(\mathrm{tr.dist}_{\mathrm{ell}}\rho) = \mathrm{tr.dist}_{\mathrm{ell}} R\Gamma(G, b, \mu)[\rho] \qquad (\dagger)$$

in $\mathrm{Dist}(G(F)_{\mathrm{ell}}, \overline{\mathbf{Q}_\ell})^{G(F)}$.

Step 2: Reduction to ℓ-torsion coefficients

Need to show: For all $\rho \in \mathrm{Irr}_{\overline{\mathbf{Q}_\ell}}(G_b(F))$, there is an equality

$$\mathcal{T}_{b,\mu}^{G_b \to G}(\mathrm{tr.dist}_{\mathrm{ell}}\rho) = \mathrm{tr.dist}_{\mathrm{ell}} R\Gamma(G, b, \mu)[\rho] \qquad (\dagger)$$

in $\mathrm{Dist}(G(F)_{\mathrm{ell}}, \overline{\mathbf{Q}_\ell})^{G(F)}$.

Key claim. If the equality $(\dagger)$ holds for all $\rho \in \mathrm{Irr}_{\overline{\mathbf{Q}_\ell}}(G_b(F))$ which admit an invariant $\overline{\mathbf{Z}_\ell}$-lattice, then it holds in general.

Sketch. Evaluate both sides of $(\dagger)$ on some $\phi dg \in C_c(G(F)_{\mathrm{ell}}, \Lambda) \otimes \mathrm{Haar}(G)$, regarding ρ as **variable**. LHS can be written as $\rho \mapsto \mathrm{tr}(\phi' dg' | \rho)$ for some $\phi' dg' \in C_c(G_b(F)_{\mathrm{ell}}, \Lambda) \otimes \mathrm{Haar}(G_b)$. ($\sim$ obvious from the definition of $\mathcal{T}_{b,\mu}^{G_b \to G}$.)
RHS can **also** be written in the form $\rho \mapsto \mathrm{tr}(\phi'' dg' | \rho)$ for some $\phi'' dg' \in C_c(G_b(F), \Lambda) \otimes \mathrm{Haar}(G_b)$! **Not** obvious; conjectured by Taylor, proved in HKW.
$\rightsquigarrow$ RHS-LHS: $\rho \mapsto \mathrm{tr}(\phi''' dg' | \rho)$ for some ϕ''', and $= 0$ by assumption when ρ admits a lattice. $\rightsquigarrow$ RHS-LHS $= 0$ for all ρ. Since ϕ was arbitrary, this gives what we want.

Step 2: Reduction to ℓ-torsion coefficients cont'd

By previous slide, reduced to showing: For all $\rho \in \mathrm{Irr}_{\overline{\mathbf{Q}_\ell}}(G_b(F))$ admitting an invariant lattice, there is an equality

$$\mathcal{T}_{b,\mu}^{G_b \to G}(\mathrm{tr.dist}_{\mathrm{ell}}\rho) = \mathrm{tr.dist}_{\mathrm{ell}} R\Gamma(G, b, \mu)[\rho] \qquad (\dagger)$$

in $\mathrm{Dist}(G(F)_{\mathrm{ell}}, \overline{\mathbf{Q}_\ell})^{G(F)}$.

Now exploit the fact that $R\Gamma(G, b, \mu)[-]$, $\mathcal{T}_{b,\mu}^{G_b \to G}(-)$, $\mathrm{tr.dist}$, etc. can be defined with coefficients in any $\mathbf{Z}_\ell$-algebra Λ, compatibly with extension of scalars. This reduces our goal to:

For $\Lambda = \overline{\mathbf{Z}_\ell}/\ell^n$ and any admissible $\rho \in D(G_b(F), \Lambda)$, we have an equality

$$\mathcal{T}_{b,\mu}^{G_b \to G}(\mathrm{tr.dist}_{\mathrm{ell}}\rho) = \mathrm{tr.dist}_{\mathrm{ell}} R\Gamma(G, b, \mu)[\rho]$$

in $\mathrm{Dist}(G(F)_{\mathrm{ell}}, \Lambda)^{G(F)}$.

Now we are ready to use the trace formula.

Interlude: Definition of the distributional transfer

To keep going, we need some p-adic geometry. Recall the usual diagram of diamonds over $\mathbf{C}_p$:

$$\begin{array}{ccc} & \mathrm{Sht}_{G,\mu,b} \circlearrowleft G_b(F) \times G(F) & \\ \swarrow^{\pi_{\mathrm{GM}}} & & \searrow^{\pi_{\mathrm{HT}}} \\ \mathrm{Gr}_{G,\leq\mu} \circlearrowleft G_b(F) & & \mathrm{Gr}_{G,\leq-\mu} \circlearrowleft G(F) \end{array}$$

From this we get some locally profinite sets:
$X_b = \{(x, g') \in \mathrm{Gr}_{G,\leq\mu}(\mathbf{C}_p) \times G_b(F)_{\mathrm{ell}} \mid xg' = x\} \circlearrowleft G_b(F)$,
$X_1 = \{(x, g) \in \mathrm{Gr}_{G,\leq-\mu}(\mathbf{C}_p) \times G(F)_{\mathrm{ell}} \mid xg = x\} \circlearrowleft G(F)$,
$\tilde{X} = \{(x, g', g) \in \mathrm{Sht}_{G,\mu,b}(\mathbf{C}_p) \times G_b(F)_{\mathrm{ell}} \times G(F)_{\mathrm{ell}} \mid x.(g', g) = x\} \circlearrowleft G_b(F) \times G(F)$.
These sit in a diagram

$$\begin{array}{ccccc} X_b & \xleftarrow{p_1} & \tilde{X} & \xrightarrow{p_2} & X_1 \\ \downarrow{\scriptstyle q_1} & & & & \downarrow{\scriptstyle q_2} \\ G_b(F)_{\mathrm{ell}} & & & & G(F)_{\mathrm{ell}} \end{array}$$

where q_1 and q_2 are finite étale, p_1 is a $G(F)$-torsor, and p_2 is a $G_b(F)$-torsor.

Interlude cont'd

On $\tilde{X}$ we have the (locally constant $G_b \times G$-invariant $\mathbf{Z}$-valued) function K_μ sending (x, g, g') to $(-1)^{\langle 2\rho, \mu \rangle} \dim r_\mu[\lambda_x]$, where λ_x measures the relative position of the modification parametrized by x.

Definition. $\mathcal{T}_{b,\mu}^{G_b \to G}$ is defined as the composition

$$\begin{aligned}
\mathrm{Dist}(G_b(F)_{\mathrm{ell}}, \Lambda)^{G_b(F)} &\cong H^0([G_b(F)_{\mathrm{ell}}/G_b(F)], K_{[G_b(F)_{\mathrm{ell}}/G_b(F)]}) \\
&\xrightarrow{(q_1/G_b(F))^*} H^0([X_b/G_b(F)], K_{[X_b/G_b(F)]}) \\
&\cong H^0([\tilde{X}/G_b(F) \times G(F)], K_{[\tilde{X}/G_b(F) \times G(F)]}) \\
&\xrightarrow{\cdot K_\mu} H^0([\tilde{X}/G_b(F) \times G(F)], K_{[\tilde{X}/G_b(F) \times G(F)]}) \\
&\cong H^0([X_1/G(F)], K_{[X_1/G(F)]}) \\
&\xrightarrow{(q_2/G(F))_*} H^0([G(F)_{\mathrm{ell}}/G(F)], K_{[G(F)_{\mathrm{ell}}/G(F)]}) \\
&\cong \mathrm{Dist}(G(F)_{\mathrm{ell}}, \Lambda)^{G(F)}.
\end{aligned}$$

Commutation of the square on slide no. 6 follows from this definition by a direct calculation. One key point: adjointness of $T_{b,\mu}^{G_b \to G}$ and $T_{b,\mu}^{G \to G_b}$) w/r/t stable Weyl integration pairing, as in Lecture 2).

Step 3: Invocation of the trace formula

Recall our goal: we want to prove $\mathcal{T}_{b,\mu}^{G_b \to G}(\mathrm{tr.dist}_{\mathrm{ell}}\rho) = \mathrm{tr.dist}_{\mathrm{ell}} R\Gamma(G, b, \mu)[\rho]$ for any admissible $\rho \in D(G_b(F), \overline{\mathbf{Z}_\ell}/\ell^n)$. Let's contemplate the diagram

$$\begin{array}{ccccc}
\mathrm{Bun}_G^{\mathbf{1}} = [*/G(F)] & \xrightarrow{i_{\mathbf{1}}} & \mathrm{Bun}_G & & \\
\uparrow{\scriptstyle h_2'} & & \uparrow{\scriptstyle h_2} & & \\
\mathrm{Hck}_{G,\leq\mu}^{\mathbf{1}} & \xrightarrow{j} & \mathrm{Hck}_{G,\leq\mu} & \xrightarrow{\epsilon} & [\mathrm{Gr}_{G,\leq\mu}/L_n^+ G] \\
 & & \downarrow{\scriptstyle h_1} & & \downarrow \\
\mathrm{Bun}_G^b = [*/G_b(F)] & \xrightarrow{i_b} & \mathrm{Bun}_G & \longrightarrow & [*/L_n^+ G]
\end{array}$$

of small v-stacks over $* = \mathrm{Spd}\mathbf{C}_p$. Here $L_n^+ G = G(\mathbb{B}_{\mathrm{dR}}^+/\mathrm{Fil}^n)$ for some large n, the two squares are Cartesian, h_1, h_2 and h_2' are proper, and i_b, $i_{\mathbf{1}}$ and j are open immersions.

First key fact: Can write $R\Gamma(G, b, \mu)[\rho] = h_{2*}' j^*(\epsilon^* \mathcal{S}_\mu \otimes h_1^* i_{b*}\rho)$. Six functor calisthenics. Here $\mathcal{S}_\mu \in D_{\text{ét}}([\mathrm{Gr}_{G,\leq\mu}/L_n^+ G], \mathbf{Z}_\ell)$ comes from geometric Satake.

Next key fact: Various things are ULA. In particular, $i_{b*}\rho$ is ULA (for the structure map) and $\epsilon^*\mathcal{S}_\mu$ is h_1-ULA $\rightsquigarrow$ $\epsilon^*\mathcal{S}_\mu \otimes h_1^* i_{b*}\rho$ is ULA, and then also $j^*(\epsilon^*\mathcal{S}_\mu \otimes h_1^* i_{b*}\rho)$ is ULA. In particular, we can contemplate its characteristic class, and the characteristic class of its proper pushforward by h_2'.

Step 3: Invocation of the trace formula cont'd

From the two Key Facts, we see that we need to understand

$$\mathrm{tr.dist} R\Gamma(G,b,\mu)[\rho] = \mathrm{cc}_{\mathrm{Bun}_G^{\mathbf{1}}}(h'_{2*}j^*(\epsilon^*\mathcal{S}_\mu \otimes h_1^* i_{b*}\rho)).$$

NOW WE INVOKE THE TRACE FORMULA: h'_2 is proper, so the trace formula (as described in Lecture 3) says that we have an equality

$$\mathrm{cc}_{\mathrm{Bun}_G^{\mathbf{1}}}(h'_{2*}A) = \mathrm{In}(h'_2)_*\mathrm{cc}_{\mathrm{Hck}^{\mathbf{1}}_{G,\leq\mu}}(A)$$

in $H^0(\mathrm{In}(\mathrm{Bun}_G^{\mathbf{1}}), K) \cong \mathrm{Dist}(G(F),\Lambda)^{G(F)}$ for any ULA sheaf A.
So we want to compute

$$\mathrm{In}(h'_2)_*\mathrm{cc}_{\mathrm{Hck}^{\mathbf{1}}_{G,\leq\mu}}(j^*(\epsilon^*\mathcal{S}_\mu \otimes h_1^* i_{b*}\rho)).$$

This is still a complicated piece of data, because the map $\mathrm{In}(h'_2)$ is rather crazy. But remember! We only care about the elliptic part $\mathrm{tr.dist}_{\mathrm{ell}} R\Gamma(G,b,\mu)[\rho]$, or equivalently about the restriction of $\mathrm{In}(h'_2)_*\mathrm{cc}_{\mathrm{Hck}^{\mathbf{1}}_{G,\leq\mu}}(j^*(\epsilon^*\mathcal{S}_\mu \otimes h_1^* i_{b*}\rho))$ to (distributions on) the open substack $\mathrm{In}(\mathrm{Bun}_G^{\mathbf{1}})^{\mathrm{ell}} = [G(F)_{\mathrm{ell}}/G(F)] \subset [G(F)/G(F)] = \mathrm{In}(\mathrm{Bun}_G^{\mathbf{1}})$.
Miracle: The fiber product $\mathrm{In}(\mathrm{Hck}^{\mathbf{1}}_{G,\leq\mu})^{\mathrm{ell}} = \mathrm{In}(\mathrm{Bun}_G^{\mathbf{1}})^{\mathrm{ell}} \times_{\mathrm{In}(\mathrm{Bun}_G^{\mathbf{1}})} \mathrm{In}(\mathrm{Hck}^{\mathbf{1}}_{G,\leq\mu})$ is nothing more than $[X_1/G(F)]$, and the map $\mathrm{In}(h'_2)^{\mathrm{ell}} : \mathrm{In}(\mathrm{Hck}^{\mathbf{1}}_{G,\leq\mu})^{\mathrm{ell}} \to \mathrm{In}(\mathrm{Bun}_G^{\mathbf{1}})^{\mathrm{ell}}$ identifies with the map $q_2/G(F)$ discussed earlier. In particular, $\mathrm{In}(h'_2)^{\mathrm{ell}}$ is finite étale. Moreover, the map $\mathrm{In}(\mathrm{Hck}^{\mathbf{1}}_{G,\leq\mu})^{\mathrm{ell}} \to \mathrm{In}(\mathrm{Bun}_G)$ induced by $h_1 \circ j$ factors over the open substack $\mathrm{In}(\mathrm{Bun}_G^b)^{\mathrm{ell}}$, and agrees with the map $q_1/G_b(F)$ discussed earlier.

Step 3: Invocation of the trace formula cont'd

To recap: The arguments so far show that

$$\mathrm{tr.dist}_{\mathrm{ell}} R\Gamma(G,b,\mu)[\rho] = \mathrm{In}(h_2')_*^{\mathrm{ell}} \mathrm{cc}_{\mathrm{Hck}^{\mathbf{1}}_{G,\leq\mu}}(j^*(\epsilon^*\mathcal{S}_\mu \otimes h_1^* i_{b*}\rho))^{\mathrm{ell}},$$

where $\mathrm{cc}_{\mathrm{Hck}^{\mathbf{1}}_{G,\leq\mu}}(j^*(\epsilon^*\mathcal{S}_\mu \otimes h_1^* i_{b*}\rho))^{\mathrm{ell}}$ denotes the restriction of the characteristic class $\mathrm{cc}_{\mathrm{Hck}^{\mathbf{1}}_{G,\leq\mu}}(j^*(\epsilon^*\mathcal{S}_\mu \otimes h_1^* i_{b*}\rho))$ from $\mathrm{In}(\mathrm{Hck}^{\mathbf{1}}_{G,\leq\mu})$ to the open substack $\mathrm{In}(\mathrm{Hck}^{\mathbf{1}}_{G,\leq\mu})^{\mathrm{ell}} \cong [X_1/G(F)]$, and $\mathrm{In}(h_2')^{\mathrm{ell}} = q_2/G(F)$ as in the definition of $\mathcal{T}^{G_b\to G}_{b,\mu}$.
It remains to match the rest of (the RHS of) this formula with the remaining pieces of the definition of $\mathcal{T}^{G_b\to G}_{b,\mu}$.
Rough idea for how this goes: We will decompose $\mathrm{cc}_{\mathrm{Hck}^{\mathbf{1}}_{G,\leq\mu}}(j^*(\epsilon^*\mathcal{S}_\mu \otimes h_1^* i_{b*}\rho))$, by some kind of Künneth formula, into separate contributions from ρ and $\mathcal{S}_\mu$. Then on the elliptic locus, ρ will contribute $(q_1/G_b(F))^*\mathrm{tr.dist}_{\mathrm{ell}}\rho$ (using that $\mathrm{In}(h_1\circ j)^{\mathrm{ell}} = q_1/G_b(F)$ as mentioned on the previous slide), and $\mathcal{S}_\mu$ will contribute the kernel function K_μ.

Step 4: Decoupling ρ and $\mathcal{S}_\mu$

Setup. Consider a Cartesian diagram of small v-stacks

$$\begin{array}{ccc} Y & \xrightarrow{p_2} & X_2 \\ \downarrow{\scriptstyle p_1} & & \downarrow \\ X_1 & \longrightarrow & S \end{array}$$

smooth-locally nice over $* = \mathrm{Spd}\mathbf{C}_p$. Suppose $A_i \in D_{\text{ét}}(X_i, \Lambda)$ is ULA.

Proposition (Exceptional Künneth formula)

In the above setup, suppose that $S \to *$ and $\Delta : S \to S \times_* S$ are both cohomologically smooth. Then there is a canonical map

$$\kappa : H^0(\mathrm{In}(X_1), K) \otimes H^0(\mathrm{In}(X_2), K) \to H^0(\mathrm{In}(Y), K),$$

the sheaf $p_1^* A_1 \otimes p_2^* A_2$ is ULA, and we have an equality

$$\mathrm{cc}_Y(p_1^* A_1 \otimes p_2^* A_2) = \kappa(\mathrm{cc}_{X_1}(A_1) \otimes \mathrm{cc}_{X_2}(A_2)).$$

Step 4: Decoupling cont'd

When $S = *$ this is an easy exercise, but the general case is much less obvious. The hypothesis on S is very strong, but is satisfied if $S = [*/\mathcal{G}]$ for some cohomologically smooth group diamond $\mathcal{G}$.

Essence of step 4: Apply the above proposition to the diagram

$$\begin{array}{ccc} \mathrm{Hck}_{G,\leq\mu} & \xrightarrow{\epsilon} & [\mathrm{Gr}_{G,\leq\mu}/L_n^+ G] \\ \downarrow{\scriptstyle h_1} & & \downarrow \\ \mathrm{Bun}_G & \longrightarrow & [*/L_n^+ G] \end{array}$$

choosing the sheaves to be $i_{b*}\rho$ and $\mathcal{S}_\mu$.

Modulo actually understanding what the map κ does in our specific situation, this (finally!) reduces us to computing $\mathrm{cc}_{[\mathrm{Gr}_{G,\leq\mu}/L_n^+ G]}(\mathcal{S}_\mu)$.

Step 5: Local terms on the B_{dR}-affine Grassmannian

Repeat: We want to compute $\mathrm{cc}_{[\mathrm{Gr}_{G,\leq\mu}/L_n^+G]}(\mathcal{S}_\mu)$.
Without further qualification this is a meaningless task, since the whole inertia stack $\mathrm{In}([\mathrm{Gr}_{G,\leq\mu}/L_n^+G])$ is complicated, and the space of distributions on it is intractable.
However, there is a large open substack which is purely "combinatorial", and we only need to understand the situation after restriction to this substack.
More precisely, let $L_n^{+,\mathrm{sr}}G$ be the (open conjugation-invariant) preimage of $G^{\mathrm{sr}} \subset G$ under the θ-map $L_n^+G \to G$, and let $\mathrm{In}([\mathrm{Gr}_{G,\leq\mu}/L_n^+G])^{\mathrm{sr}}$ be the preimage of the open substack $[L_n^{+,\mathrm{sr}}G/L_n^+G] \subset \mathrm{In}([*/L_n^+G])$ along the evident map. Then $|\mathrm{In}([\mathrm{Gr}_{G,\leq\mu}/L_n^+G])^{\mathrm{sr}}| \cong X_*(T)_{\leq\mu}/W$ and

$$H^0(\mathrm{In}([\mathrm{Gr}_{G,\leq\mu}/L_n^+G])^{\mathrm{sr}}, K) \cong C(X_*(T)_{\leq\mu}, \Lambda)^W.$$

Proposition

Under the previous identification, we have an equality

$$\mathrm{cc}_{[\mathrm{Gr}_{G,\leq\mu}/L_n^+G]}(\mathcal{S}_\mu)^{\mathrm{sr}} : \lambda \mapsto (-1)^{\langle 2\rho,\mu\rangle} \dim r_\mu[\lambda].$$

This is exactly what we want to see!

Step 5: Local terms on the B_{dR}-affine Grassmannian, cont'd

After a nontrivial unwinding, this reduces to the following stack-free statement. Let $V \in \mathrm{Rep}(\hat{G}_\Lambda)$ be any representation, corresponding to some object $\mathcal{S}_V \in D_{\text{ét}}(\mathrm{Gr}_G, \Lambda)^{L^+G}$ in the Satake category. Let $g \in G(\overline{F})$ be any strongly regular semisimple element.

Proposition

Under the assumptions above, g has only isolated fixed points on Gr_G, and for any such fixed point x there is an equality

$$\mathrm{loc}_x(g, \mathcal{S}_V) = (-1)^{\langle 2\rho, \lambda_x \rangle} \dim V[\lambda_x].$$

Here $\lambda_x \in X_*(T)/W$ records which open Schubert cell of Gr_G contains x, and $V[\lambda_x]$ denotes the λ_x-weight space of V.

In the more familiar setting of complex analytic / schematic / Witt vector affine Grassmannians, this proposition can be deduced from a recent theorem of Varshavsky (using a nontrivial global-to-local argument with the weight functors in geometric Satake). In the B_{dR} setting, a direct attack seems impossible. Instead, we degenerate from the B_{dR}-affine Grassmannian to the Witt vector affine Grassmannian, using a suitable Beilinson-Drinfeld affine Grassmannian over $\mathcal{O}_{\mathbf{C}_p}$ as an intermediary.

Thank you for listening!