# Motivic cohomology actions and the geometry of eigenvarieties

David Hansen*

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These informal notes discuss a recent joint paper of Jack Thorne and myself.

## A basic fact of life

Let $\mathbf{G}$ be a connected reductive group over $\mathbf{Q}$. Set $G_\infty = \mathbf{G}(\mathbf{R})$, and let $K_\infty \subset G_\infty$ be a maximal compact-mod-center subgroup, so $D_\infty = G_\infty / K_\infty$ is the usual symmetric space for $\mathbf{G}$. Following the notation in Borel and Wallach's book, we set

$$l_0 = \mathrm{rank}(G_\infty) - \mathrm{rank}(K_\infty)$$

and

$$q_0 = \tfrac{1}{2}(\mathrm{dim} D_\infty - l_0).$$

These are both nonnegative integers.

For any open compact subgroup $K \subset \mathbf{G}(\mathbf{A}_f)$ we have the usual locally symmetric quotient

$$\begin{aligned} Y_K &= \mathbf{G}(\mathbf{Q}) \backslash \left( D_\infty \times \mathbf{G}(\mathbf{A}_f) \right) / K \\ &= \coprod_i \Gamma_{K,i} \backslash D_\infty. \end{aligned}$$

It's a basic fact of life that the integer $l_0$ controls, to a remarkable degree, the geometry and arithmetic of the $Y_K$'s and associated automorphic representations of $\mathbf{G}(\mathbf{A})$. Here are some examples of this principle:

1) (Harish-Chandra) The semisimple group $G_\infty^{\mathrm{ad}}$ has a discrete series if and only if $l_0 = 0$. If $D_\infty$ is Hermitian symmetric (i.e. the $Y_K$'s are Shimura varieties), then $l_0 = 0$.

2) (Bergeron–Venkatesh) The integral cohomology groups

$$H^*(Y_K, \mathbf{Z})$$

conjecturally have a large (relative to $\mathrm{vol}(Y_K)$) torsion subgroup if and only if $l_0 = 1$.

3) (Borel–Wallach, Zuckerman) Let $\mathcal{L}_{\lambda,\mathbf{C}}$ be an irreducible algebraic representation of $\mathbf{G}(\mathbf{C})$. If $\pi$ is a cuspidal automorphic representation of $\mathbf{G}(\mathbf{A})$ such that $\pi_\infty$ is tempered and cohomological of weight $\lambda$, then the $\pi$-part of $H^n(Y_K, \mathcal{L}_{\lambda,\mathbf{C}})$ is nonvanishing only for $n \in [q_0, q_0 + l_0]$, in which case

$$\mathrm{dim} H^n(Y_K, \mathcal{L}_{\lambda,\mathbf{C}})_\pi = m_{\pi,K} \cdot \binom{l_0}{n - q_0}.$$

Here $m_{\pi,K}$ is an integer (possibly zero).

4) Let $\pi$ be a tempered cohomological cusp form as in the previous example. Then conjecturally we have

$$\mathrm{ord}_{s=0} L(s, \mathrm{ad}^0 \pi) = l_0.$$

This is known in many cases, e.g. when $\mathbf{G} = \mathrm{Res}_{F/\mathbf{Q}} \mathrm{GL}_n$ for some number field $F/\mathbf{Q}$. (Note: my convention for the adjoint representation $\mathrm{ad}^0$ is that $L(s, \pi \otimes \pi^\vee) = \zeta_F(s) L(s, \mathrm{ad}^0 \pi)$ when $\mathbf{G} = \mathrm{Res}_{F/\mathbf{Q}} \mathrm{GL}_n$, i.e. I'm stripping off the trivial part of the adjoint representation of ${}^L\mathbf{G}$. )

*Department of Mathematics, Columbia University, 2990 Broadway, New York NY 10027; hansen@math.columbia.edu

5) (Hida, Urban) Let $\mathfrak{X} = \mathfrak{X}_{\mathbf{G},K}$ denote the eigenvariety for $\mathbf{G}$ (of some tame level $K$), with its weight map $w : \mathfrak{X} \to \mathcal{W} = \mathcal{W}_{\mathbf{G},K}$. Let $x \in \mathfrak{X}$ be a "noncritical" point associated with a tempered cohomological cusp form $\pi$. Then any irreducible component of $\mathfrak{X}$ containing $x$ has conjectural dimension $\dim\mathcal{W} - l_0$.

Venkatesh has recently conjectured a remarkable "arithmetic enhancement" of 3) above. Our paper explores a surprising link between Venkatesh's conjecture and 5).

## Venkatesh's conjecture

For simplicity, let me restrict to the case where $\mathbf{G} = \mathrm{Res}_{F/\mathbf{Q}}\mathrm{GL}_n$ for some number field $F/\mathbf{Q}$ with $r_1$ (resp. $r_2$) real (resp. complex) places as usual. A fun calculation (left to the reader) shows that $l_0 = \lfloor \frac{n-1}{2} \rfloor r_1 + (n-1)r_2$ in this case, so e.g.:

- $l_0 = 0$ exactly when $F$ is totally real and $n = 2$,
- $l_0 = 1$ exactly when $F = \mathbf{Q}$ and $n \in \{3, 4\}$, or when $F$ has one complex place and $n = 2$.

Let $\pi$ be a regular algebraic cusp form on $\mathbf{G}$. According to a fundamental conjecture of Clozel, there should be an irreducible rank $n$ Grothendieck motive $M_\pi$ over $F$ with coefficients in some number field $E$ containing the field of Hecke eigenvalues of $\pi$, such that the Frobenius eigenvalues on the $\lambda$-adic realizations $M_{\pi,\lambda}$ match up with the Hecke eigenvalues of $\pi$ in the usual way. Now, the cohomology groups in 3) make equally good sense with coefficients in $E$ rather than $\mathbf{C}$, and Venkatesh's conjecture is then the following:

**Conjecture.** *There is a canonical $E$-vector space $V_\pi$ of dimension $l_0$ together with a canonical (degree-lowering) action of the exterior algebra $\wedge_E^* V_\pi$ on*

$$H^*(Y_K, \mathcal{L}_{\lambda,E})_\pi$$

*making the latter finite free and generated in degree $q_0 + l_0$ as a graded module over the former. The vector space $V_\pi$ is given explicitly by*

$$V_\pi = \mathrm{Ext}^1_{\mathcal{MM}\mathrm{ot}_{\mathcal{O}_F}}(\mathbf{1}, \mathrm{ad}^0 M_\pi(1))$$

*where $\mathcal{MM}\mathrm{ot}_{\mathcal{O}_F}$ denotes a putative category of mixed motives over $\mathcal{O}_F$ (with coefficients in $E$).*

Note that this conjecture would "explain" the dimensions of the $\pi$-parts in 3) above in a purely arithmetic fashion.

Why might you believe this? As a first sanity check, note that for any finite place $\lambda$ of $E$, the $\lambda$-adic realization functor on mixed motives should induce an isomorphism

$$V_\pi \otimes_E E_\lambda \cong H^1_f(F, \mathrm{ad}^0 M_{\pi,\lambda}(1)),$$

and the Bloch-Kato conjecture predicts $\mathrm{ord}_{s=0} L(s, \mathrm{ad}^0\pi) = l_0$ as the dimension of this $H^1_f$, so the conjectural dimension of $V_\pi$ is at least correct.

A more serious shadow of the conjecture is that, if this conjecture were true, we would get in particular an isomorphism

$$H^{q_0}(Y_K, \mathcal{L}_{\lambda,E})_\pi \cong H^{q_0+l_0}(Y_K, \mathcal{L}_{\lambda,E})_\pi \otimes_E \det{}_E V_\pi.$$

Now in many cases, the "comparison" of rational structures on $H^{q_0}$ and $H^{q_0+l_0}$ can be related to periods of $\pi$ and then to adjoint $L$-values; on the other hand, comparison of rational structures on $\det_{\mathbf{R}} V_\pi \otimes \mathbf{R}$ should give rise to the periods intervening in Beilinson's conjecture for $L(1, \mathrm{ad}^0\pi)$.

## A summary of our results

We now restrict further to the case $F = \mathbf{Q}$ (the actual setting considered in our paper, although our ideas work equally well over a general $F$). So, let $\pi$ be a regular algebraic cuspidal automorphic representation on $\mathbf{G} = \mathrm{GL}_n/\mathbf{Q}$ as above, contributing to some $H^*(Y_K, \mathcal{L}_{\lambda,\mathbf{C}})$ as in 3). Let $\lambda = (k_1 \geq k_2 \geq \cdots \geq k_n)$ be the highest weight of $\mathcal{L}_{\lambda,\mathbf{C}}$, and let $E \subset \mathbf{C}$ be the number field generated by the Hecke eigenvalues of $\pi$. Choose a prime $p$ such that $\pi_p$ is unramified with regular semisimple Satake parameter, and let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a fixed ordering on the eigenvalues of $\mathrm{rec}(\pi_p \otimes |\det|^{\frac{1-n}{2}})(\mathrm{Frob}_p)$. (These eigenvalues give the Satake parameter of $\pi_p$ up to scalar multiple, and there are $n!$ such orderings. When $n = 2$ and $\pi$ is generated by a newform of weight $k$ and level prime to $p$, the $\alpha_i$'s are the roots of the usual polynomial $X^2 - a_f(p)X + \varepsilon_f(p)p^{k-1}$.) Fix an isomorphism $\iota : \mathbf{C} \xrightarrow{\sim} \overline{\mathbf{Q}_p}$, and let $L \subset \overline{\mathbf{Q}_p}$ be the finite extension of $\mathbf{Q}_p$ generated by $\iota(E)$ and the $\iota(\alpha_i)$'s. From now on, I'll typically suppress $\iota$.

Let $N$ be the conductor of $\pi$. Let $\Gamma_1(N) \subset \mathrm{GL}_n(\mathbf{Z})$ be the usual group of matrices with lowest row $\equiv (0, \ldots, 0, 1) \bmod N$, and let $\Gamma_1(N; p) \subset \Gamma_1(N)$ be the subgroup of matrices which are upper-triangular modulo $p$. For brevity, we set

$$H^i_\pi = H^i\left(\Gamma_1(N), \mathcal{L}_{\lambda,L}\right)_\pi$$

and

$$H^i_{\pi,\alpha} = H^i\left(\Gamma_1(N;p), \mathcal{L}_{\lambda,L}\right)_{\pi,\alpha}.$$

Here the subscripts $(-)_\pi$ (resp. $(-)_{\pi,\alpha}$) denote the $\pi$-part (resp. the $(\pi,\alpha)$-part) of this cohomology (see the paper for details). By a direct calculation using the aforementioned results of Borel–Wallach together with results of Clozel, Franke–Schwermer, Jacquet–Piatetski-Shapiro–Shalika, etc., we prove that the $L$-vector spaces $H^i_\pi$ and $H^i_{\pi,\alpha}$ vanish for $i \notin [q_0, q_0 + l_0]$ and have $L$-dimension exactly $\binom{l_0}{n - q_0}$ otherwise. Our goal is to say something about Venkatesh's conjecture for the cohomologies $H^*_{\pi,\alpha}$ and $H^*_\pi$.

Let $\mathfrak{X}$ denote the eigenvariety for $\mathrm{GL}_n/\mathbf{Q}$ of tame level $\Gamma_1(N)$ with its weight map $w : \mathfrak{X} \to \mathcal{W}$. By the construction of $\mathfrak{X}$, there is a natural algebra map $\phi : \mathcal{T} \to \mathcal{O}(\mathfrak{X})$ where $\mathcal{T}$ is the usual "abstract" Hecke algebra generated over $\mathbf{Q}_p$ by the usual operators $T_{\ell,i}$ ($\ell \nmid Np$) and $U_{p,i}$ for $1 \le i \le n$. Assuming that the refinement $\alpha$ satisfies a "small slope" condition, we construct a canonical point $x = x(\pi,\alpha) \in \mathfrak{X}(L)$ of weight $\lambda$ attached to the pair $(\pi,\alpha)$. Let $\mathbf{T}_x = \widehat{\mathcal{O}_{\mathfrak{X},x}}$ and $\Lambda = \widehat{\mathcal{O}_{\mathcal{W},\lambda}}$ be the completed local rings of the eigenvariety and of the weight space at $x$ and $\lambda = w(x)$, respectively, so $\mathbf{T}_x$ is naturally a finite $\Lambda$-algebra. We set things up in such a way that these are naturally complete local Noetherian $L$-algebras with residue field $L$ (in particular $\Lambda \simeq L[[t_1, \ldots, t_n]]$). We also construct a finite faithful $\mathbf{T}_x$-module $H^*_x = \oplus_{0 \le i \le \dim D_\infty} H^i_x$ of $p$-adic automorphic forms, with the further property that

$$H^{q_0+l_0}_x \otimes_\Lambda L \cong H^{q_0+l_0}_{\pi,\alpha}$$

canonically and Hecke-equivariantly.

In this setting, Hida and Urban conjecture the equality

$$\dim \mathbf{T}_x = \dim \Lambda - l_0 (= 1 + \lfloor n/2 \rfloor),$$

which we shall refer to as *the dimension conjecture (at x)*. For $n = 2$ this is classical; when $n \in \{3, 4\}$, I proved this result in my thesis. In general, Newton proved (using ideas from my thesis) the inequality $\dim \mathbf{T}_x \geq \dim \Lambda - l_0$.

Our first main result is the following theorem.

**Theorem A.** *Let notation and assumptions be as above, and assume the dimension conjecture holds at* $x = x(\pi,\alpha)$. *Then*

*(a) The module* $H^i_x$ *vanishes for* $i \neq q_0 + l_0$, *and* $H_x := H^{q_0+l_0}_x$ *is free of rank one over* $\mathbf{T}_x$.

*(b) There exist canonical isomorphisms*

$$\mathrm{Tor}^\Lambda_i(H_x, L) \cong H^{q_0+l_0-i}(\Gamma_1(N;p), \mathcal{L}_{\lambda,L})_{\pi,\alpha} = H^{q_0+l_0-i}_{\pi,\alpha}$$

*for all* $i \geq 0$.

*(c) The map* $\Lambda \to \mathbf{T}_x$ *is surjective, and the ring* $\mathbf{T}_x$ *is a complete intersection.*

*(d) Set* $V_x = (\ker(\Lambda \to \mathbf{T}_x)) \otimes_\Lambda L$, *an* $l_0$*-dimensional* $L$*-vector space. Then there is a* canonical *degree-lowering action of* $\wedge^*_L V_x$ *on* $H^*_{\pi,\alpha}$ *which makes the latter free of rank one as a graded module over the former.*

Here is a sketch of the proof. The vanishing result for $H^i_x$, assuming the dimension conjecture, is essentially immediate from a lemma in commutative algebra; the relevant lemma was observed separately by Calegari-Geraghty and myself in the context of the Taylor-Wiles method. Since $H_x \otimes_\Lambda L \simeq L$, $H_x$ is a quotient of $\Lambda$

by Nakayama's lemma, say with $H_x \simeq \Lambda/I$. Granted the vanishing of $H^i_x$ for $i \neq q_0 + l_0$, we see that $H_x$ is also faithful over $\mathbf{T}_x$. Since $\mathbf{T}_x$ is a $\Lambda$-subalgebra of $\mathrm{End}_\Lambda(H_x) = \Lambda/I$, we get $\mathbf{T}_x \cong \Lambda/I$ as well. This shows (a) together with the first part of (c).

For (b), we construct a canonical spectral sequence

$$\mathrm{Tor}^{\Lambda}_{-i}\left(H^j_x, L\right) \Rightarrow H^{i+j}_{\pi,\alpha}$$

by adapting a spectral sequence for overconvergent cohomology from my thesis. By the vanishing result in (a), this degenerates to the claimed isomorphisms.

To see that $\mathbf{T}_x$ is a complete intersection, it suffices (by our knowledge of dimensions plus the fact that $\Lambda$ is complete regular local) to show that $I$ can be generated by $l_0$ elements. For this, we examine the isomorphism in (b) for $i = 1$: since $H_x \simeq \mathbf{T}_x \cong \Lambda/I$ as $\Lambda$-modules, we get

$$H^{q_0+l_0-1}_{\pi,\alpha} \cong \mathrm{Tor}^{\Lambda}_1(H_x, L) \simeq \mathrm{Tor}^{\Lambda}_1(\Lambda/I, L) \cong I \otimes_\Lambda L,$$

but we already know $\dim H^{q_0+l_0-1}_{\pi,\alpha} = l_0$, so we conclude by Nakayama.

For (d), we observe that $\mathrm{Tor}^{\Lambda}_*(\Lambda/I, L)$ (which is naturally a skew-commutative graded ring) acts on the graded module

$$\mathrm{Tor}^{\Lambda}_*(H_x, L) \cong H^{q_0+l_0-*}_{\pi,\alpha}$$

with the latter free of rank one over the former; on the other hand, we prove that

$$\mathrm{Tor}^{\Lambda}_*(\Lambda/I, L) \cong \wedge^*_L(I \otimes_\Lambda L) = \wedge^*_L V_x.$$

(Here we use that $I$ is generated by a regular sequence.)

The space $V_x$ is somewhat mysterious. Can we relate it to $V_\pi$? There is a big hint here, as we've already said: one expects the $p$-adic realization functor on mixed motives to induce a canonical isomorphism

$$V_\pi \otimes_{E,\iota} L \quad \cong \quad H^1_f\left(\mathbf{Q}, \mathrm{ad}\rho_\pi(1)\right)$$

where $\rho_\pi : G_{\mathbf{Q}} \to \mathrm{GL}_n(L)$ denotes the Galois representation associated with $\pi$ (and $\iota$) by the work of very many mathematicians: we mention in particular Eichler, Shimura, Deligne, Clozel, Kottwitz, Harris-Taylor, Morel, Shin, HLTT, and Scholze (please note that in what follows, we shall assume that $\rho_\pi$ satisfies full local-global compatibility at all places, including $p$; this is known when $\pi$ is essentially self-dual, and in some cases beyond). But in order to see the appearance of an $H^1_f$, we need another idea.

The idea now is to consider a Galois deformation problem $\mathcal{D}_{\pi,\alpha}$ on Artinian local $L$-algebras, consisting of deformations of $\rho_\pi$ which are minimally ramified at primes away from $p$ and *trianguline at $p$ with a triangulation lifting the triangulation of $\rho_\pi|G_{\mathbf{Q}_p}$ determined by $\alpha$*. This deformation problem, appropriately defined, is pro-represented by a complete local Noetherian $L$-algebra $R_{\pi,\alpha}$, which moreover is canonically a $\Lambda$-algebra. We can then take advantage of the following circumstances:

- On the one hand, we believe in a natural "$R = \mathbb{T}$"-type conjecture in this setting.
- On the other hand, the tangent space of $\mathcal{D}_{\pi,\alpha}$ (and its local-at-$p$ analogue) is naturally related to $H^1_f$'s.

The necessary local-at-$p$ trianguline deformation functor here was studied in detail by Bellaïche-Chenevier, and we draw heavily on their results. In particular, they prove that the tangent space $H^1_\alpha(\mathbf{Q}_p, \mathrm{ad}\rho_\pi) \subset H^1(\mathbf{Q}_p, \mathrm{ad}\rho_\pi)$ sits in a canonical short exact sequence

$$0 \to H^1_f(\mathbf{Q}_p, \mathrm{ad}\rho_\pi) \to H^1_\alpha(\mathbf{Q}_p, \mathrm{ad}\rho_\pi) \xrightarrow{dw} L^n \to 0.$$

Using this together with Poitou-Tate duality and some other results of Bellaïche-Chenevier, we show the following theorem.

**Theorem B.**

(a) *The tangent space* $\mathcal{D}_{\pi,\alpha}(L[\epsilon])$ *is naturally identified with a certain global Selmer group* $H^1_\alpha(\mathbf{Q}, \mathrm{ad}\rho_\pi)$*, which sits in a canonical five-term exact sequence*

$$0 \to H^1_f(\mathbf{Q}, \mathrm{ad}\rho_\pi) \to H^1_\alpha(\mathbf{Q}, \mathrm{ad}\rho_\pi) \to L^n \overset{\mu_\alpha^\vee}{\to} H^1_f(\mathbf{Q}, \mathrm{ad}\rho_\pi(1))^\vee \to H^1_\alpha(\mathbf{Q}, \mathrm{ad}\rho_\pi(1))^\vee \to 0.$$

*Here* $(-)^\vee$ *denotes* $L$*-linear dual, and* $\mu_\alpha : H^1_f(\mathbf{Q}, \mathrm{ad}\rho_\pi(1)) \to L^n$ *is a certain canonically defined regulator map. The first three terms in this sequence are compatible (via localization at* $p$*) with the aforementioned short exact sequence.*

(b) *There is a canonical isomorphism* $R_{\pi,\alpha} \otimes_\Lambda L \cong R_{\pi,\mathrm{crys}}$*.*

(c) *The ring* $R_{\pi,\alpha}$ *has tangent space of dimension* $g = h^1_\alpha(\mathbf{Q}, \mathrm{ad}\rho_\pi)$ *and admits a presentation* $R_{\pi,\alpha} \simeq L[[x_1, \ldots, x_g]]/(f_1, \ldots, f_r)$ *where* $r \leq h^1_\alpha(\mathbf{Q}, \mathrm{ad}\rho_\pi(1))$*. Furthermore,* $h^1_\alpha(\mathbf{Q}, \mathrm{ad}\rho_\pi) - h^1_\alpha(\mathbf{Q}, \mathrm{ad}\rho_\pi(1)) = n - l_0$*.*

The relevant $R = \mathbb{T}$ conjecture is as follows:

**Conjecture C.** *There is an isomorphism* $R_{\pi,\alpha} \overset{\sim}{\to} \mathbf{T}_x$ *of* $\Lambda$*-algebras satisfying the usual compatibility between Frobenius and Hecke eigenvalues.*

Why should you believe in this conjecture? Most convincingly, it's true in the unitary group setting, thanks to results of Bellaïche-Chenevier, Chenevier, and Allen. There are also certain formal similarities on both sides: in particular, point (c) above implies that $R_{\pi,\alpha}$ has dimension $\geq n - l_0$ and is a complete intersection if equality holds, while recall we proved exactly the same result independently for $\mathbf{T}_x$!

Here is an arrangement of the rest of our results which differs a bit from the Theorem stated in our paper (but which I find compelling).

**Theorem D.** *Suppose there is a* surjection *of* $\Lambda$*-algebras*

$$R_{\pi,\alpha} \twoheadrightarrow \mathbf{T}_x$$

*satisfying the expected compatibilities, and that* $\mu_\alpha$ *is injective. Then:*

(a) *The map* $R_{\pi,\alpha} \to \mathbf{T}_x$ *is an isomorphism of* $n - l_0$*-dimensional regular local rings; in particular, the eigenvariety is smooth at* $x$*, and the dimension conjecture is true at* $x$*, so all the conclusions of Theorem A hold as well.*

(b) *We have* $H^1_f(\mathbf{Q}, \mathrm{ad}\rho_\pi) = 0$ *and* $\dim_L H^1_f(\mathbf{Q}, \mathrm{ad}\rho_\pi(1)) = l_0$*.*

(c) *The map* $\mu_\alpha$ *induces a canonical isomorphism* $H^1_f(\mathbf{Q}, \mathrm{ad}\rho_\pi(1)) \cong V_x$*, so we get a canonical action of* $\wedge^*_L H^1_f(\mathbf{Q}, \mathrm{ad}\rho_\pi(1))$ *on* $H^*_{\pi,\alpha}$ *with the latter free of rank one over the former.*

Here is an idea of the proof. By Newton's theorem mentioned above, we have $\dim \mathbf{T}_x \geq n - l_0$. On the other hand, the injectivity of $\mu_\alpha$ implies $h^1_\alpha(\mathbf{Q}, \mathrm{ad}\rho_\pi(1)) = 0$ and (by the Greenberg-Wiles duality formula) $h^1_\alpha(\mathbf{Q}, \mathrm{ad}\rho_\pi) = n - l_0$, so by part (b) of Theorem B we deduce $R_{\pi,\alpha} \simeq L[[x_1, \ldots, x_{n-l_0}]]$. Comparing dimensions, the map $R \to \mathbf{T}$ is an isomorphism. To deduce (b), we note that

$$R_{\pi,\mathrm{crys}} \simeq R_{\pi,\alpha} \otimes_\Lambda L \simeq \mathbf{T}_x \otimes_\Lambda L \simeq L,$$

by Theorems A and B together with our knowledge of $R = \mathbf{T}$; since $H^1_f(\mathbf{Q}, \mathrm{ad}\rho_\pi)$ is the tangent space of $R_{\pi,\mathrm{crys}}$, its vanishing follows. The remainder of (b) follows from Greenberg-Wiles duality again.

For (c), let $I$ be the kernel of the surjection $\Lambda \to \mathbf{T}_x$ as before; since $\mathbf{T}_x$ is regular, the generators of $I$ lie in $\mathfrak{m}_\Lambda \smallsetminus \mathfrak{m}_\Lambda^2$ and we get a natural injection $V_x = I \otimes_\Lambda L \hookrightarrow \mathfrak{m}_\Lambda/\mathfrak{m}_\Lambda^2$. Now, dualizing the five-term sequence from Theorem B and noting that the outermost terms vanish, we get a short exact sequence

$$0 \to H^1_f(\mathbf{Q}, \mathrm{ad}\rho_\pi(1)) \overset{\mu_\alpha}{\to} \mathfrak{m}_\Lambda/\mathfrak{m}_\Lambda^2 \overset{t}{\to} \mathfrak{m}_R/\mathfrak{m}_R^2 \cong \mathfrak{m}_\mathbf{T}/\mathfrak{m}_\mathbf{T}^2 \to 0.$$

Since $V_x \subseteq \ker t = \mathrm{im}\mu_\alpha$ and $\dim V_x = h^1_f(\mathbf{Q}, \mathrm{ad}\rho_\pi(1))$, $\mu_\alpha$ induces an isomorphism

$$H^1_f(\mathbf{Q}, \mathrm{ad}\rho_\pi(1)) \cong V_x$$

as desired.

We have one more conjecture. To state it, note that the natural restriction map $H^i(\Gamma_1(N), -) \to H^i(\Gamma_1(N;p), -)$ induces a graded isomorphism $r_\alpha : H^*_\pi \overset{\sim}{\to} H^*_{\pi,\alpha}$.

**Conjecture E.** *Suppose the hypotheses of Theorem D holds true for a fixed* $\pi$ *and multiple* $\alpha$*'s, so for each* $\alpha$ *we get an action*

$$\wedge^* H^1_f(\mathbf{Q}, \mathrm{ad}\rho_\pi(1)) \to \mathrm{End}\,(H^*_\pi)$$

*by intertwining the action from part (c) of Theorem D with the isomorphism* $r_\alpha$*. Then this action is* independent of $\alpha$.

This last action should be the $p$-adic completion of the "true" action of $V_\pi$ on $H^*(\Gamma_1(N), \mathcal{L}_{\lambda,E})_\pi$ predicted by Venkatesh's conjecture.

## More on the regulator $\mu_\alpha$

In this section, we give two definitions of the regulator map $\mu_\alpha$ from Theorem B. We hope to convince the reader that $\mu_\alpha$ is a *p-adic analogue of the Beilinson regulator for* $\mathrm{ad}M_\pi(1)$. This at least makes its injectivity morally reasonable.

The regulator map $\mu_\alpha : H^1_f(\mathbf{Q}, \mathrm{ad}\rho_\pi(1)) \to L^n$ is defined as a certain composite

$$H^1_f(\mathbf{Q}, \mathrm{ad}\rho_\pi(1)) \stackrel{\mathrm{res}_p}{\to} H^1_f(\mathbf{Q}_\pi, \mathrm{ad}\rho_\pi(1)) \stackrel{\nu_\alpha}{\to} L^n,$$

so we need to define the local regulator map $\nu_\alpha$. Our first definition of $\nu_\alpha$ goes as follows. Recall the short exact sequence

$$0 \to H^1_f(\mathbf{Q}_p, \mathrm{ad}\rho_\pi) \to H^1_\alpha(\mathbf{Q}_p, \mathrm{ad}\rho_\pi) \stackrel{dw}{\to} L^n \to 0$$

of Bellaïche-Chenevier mentioned above. An easy snake lemma argument gives an associated short exact sequence

$$0 \to L^n = H^1_\alpha/H^1_f \to H^1/H^1_f \to H^1/H^1_\alpha \to 0,$$

where $H^1_\bullet := H^1_\bullet(\mathbf{Q}_p, \mathrm{ad}\rho_\pi)$. By the self-adjointness of the crystalline Selmer condition under Tate local duality, taking the $L$-linear dual gives a short exact sequence

$$0 \to H^1_\alpha(\mathbf{Q}_p, \mathrm{ad}\rho_\pi(1)) \to H^1_f(\mathbf{Q}_p, \mathrm{ad}\rho_\pi(1)) \to L^n \to 0,$$

(this is the *definition* of the dual local condition $H^1_\alpha(\mathbf{Q}_p, \mathrm{ad}\rho_\pi(1))$), and we define $\nu_\alpha$ as the third arrow in this sequence.

To give our second definition of $\nu_\alpha$, we temporarily work in a slightly more general setting. Let $V$ be an $n$-dimensional $L$-linear crystalline representation of $G_{\mathbf{Q}_p}$. We may identify $\mathbf{D}_{\mathrm{crys}}(V) \cong \mathbf{D}_{\mathrm{dR}}(V)$, so $\mathbf{D}_{\mathrm{crys}}(V)$ has a canonical Hodge filtration by subspaces $\mathrm{Fil}^i = \mathrm{Fil}^i\mathbf{D}_{\mathrm{dR}}(V)$. We shall assume that $V$ has $n$ distinct Hodge-Tate weights $w_1 < w_2 < \cdots < w_n$, and that $\varphi$ acting on $\mathbf{D}_{\mathrm{crys}}(V)$ has $n$ distinct eigenvalues $\varphi_1, \ldots, \varphi_n$ which furthermore satisfy $\varphi_i\varphi_j^{-1} \notin \{1, p^{\pm 1}\}$ for $i \neq j$. [1] Fix an ordering $\alpha = (\alpha_1, \ldots, \alpha_n)$ on the $\varphi$-eigenvalues. By our multiplicity-freeness assumption, our chosen ordering $\alpha$ determines a unique refinement

$$\mathcal{F}(\alpha) : \{0 \subsetneq \mathcal{F}_1 \subsetneq \mathcal{F}_2 \subsetneq \cdots \subsetneq \mathcal{F}_n = \mathbf{D}_{\mathrm{crys}}(V)\}$$

of $\mathbf{D}_{\mathrm{crys}}(V)$ by the usual rule $\det(X - \varphi)|\mathcal{F}_i = \prod_{1 \leq j \leq i}(X - \alpha_j)$. We suppose that this refinement is *noncritical*, i.e. that $\mathcal{F}_i \oplus \mathrm{Fil}^{w_i+1} = \mathbf{D}_{\mathrm{crys}}(V)$ for all $1 \leq i \leq n-1$. This assumption is absolutely essential in what follows. In the context of Theorems B and D, this noncriticality is guaranteed by our "small slope" assumption.

Our second construction of $\nu_\alpha$ goes via the Bloch-Kato logarithm, with the target $L^n$ now realized as a quotient of

$$\mathbf{D}_{\mathrm{dR}}\,(\mathrm{ad}V(1))\,/\mathbf{D}^+_{\mathrm{dR}}\,(\mathrm{ad}V(1))\,.$$

Compare this structure to the target of the Beilinson regulator

$$\begin{aligned} r : \mathrm{Ext}^1_{\mathcal{MM}_{\mathbf{Q}}}(\mathbf{Q}(0), M) &\to \mathrm{Ext}^1_{\mathcal{MH}^+_{\mathbf{R}}}(\mathbf{R}(0), M_B \otimes \mathbf{R}) \\ &\cong (\mathrm{Fil}^0 M_{\mathrm{dR}} \otimes \mathbf{R})\backslash(M_{\mathrm{dR}} \otimes \mathbf{R})/(M_B^+ \otimes \mathbf{R}). \end{aligned}$$

[1] When $V$ is the restriction of a representation of $G_{\mathbf{Q}}$ coming from a pure motive, the possibility $\varphi_i\varphi_j^{-1} = p^{\pm 1}$ is automatically ruled out, since the $\varphi_i$'s are then $p$-Weil numbers of some weight independent of $i$.

Set $D = \mathbf{D}_{\mathrm{crys}}(V)$ for brevity, and set $\mathrm{ad}D = \mathrm{Hom}_L(D, D)$ with the induced $\varphi$-module structure and Hodge filtration. By the functoriality of $\mathbf{D}_{\mathrm{crys}}$ we have $\mathrm{ad}D \cong \mathbf{D}_{\mathrm{crys}}(\mathrm{ad}V)$. Set

$$\mathrm{ad}_{\alpha,0}D = \{f \in \mathrm{Hom}_L(D, D) \mid f(\mathcal{F}_i) \subseteq \mathcal{F}_i \,\forall 1 \leq i \leq n\}$$

and

$$\mathrm{ad}_{\alpha,-1}D = \{f \in \mathrm{Hom}_L(D, D) \mid f(\mathcal{F}_i) \subseteq \mathcal{F}_{i-1} \,\forall 1 \leq i \leq n\}.$$

These are $\varphi$-stable subspaces of $\mathrm{ad}(D)$, and

$$\begin{aligned} \mathrm{ad}_{\alpha,0}D/\mathrm{ad}_{\alpha,-1}D &\cong \prod_{i=1}^{n} \mathrm{End}_L(\mathcal{F}_i/\mathcal{F}_{i-1}) \\ &\cong L^n. \end{aligned}$$

Let $q_\alpha : \mathrm{ad}_{\alpha,0}D \twoheadrightarrow L^n$ denote the composite of this isomorphism with the projection $\mathrm{ad}_{\alpha,0}D \to \mathrm{ad}_{\alpha,0}D/\mathrm{ad}_{\alpha,-1}D$.

**Fact one:** If $W$ is any de Rham representation of $G_{\mathbf{Q}_p}$ with $\mathbf{D}_{\mathrm{crys}}(W)^{\varphi=1} = 0$, the Bloch-Kato exponential induces an isomorphism

$$\exp_W : \mathbf{D}_{\mathrm{dR}}(W)/\mathbf{D}^+_{\mathrm{dR}}(W) \xrightarrow{\sim} H^1_f(\mathbf{Q}_p, W).$$

We write $\log_W$ for the inverse isomorphism as usual.

One checks that under the assumptions above, the previous fact applies in the case $W = \mathrm{ad}V(1)$, so we get

$$\log_{\mathrm{ad}V(1)} : H^1_f(\mathbf{Q}_p, \mathrm{ad}V(1)) \xrightarrow{\sim} \mathbf{D}_{\mathrm{dR}}(\mathrm{ad}V(1))/\mathbf{D}^+_{\mathrm{dR}}(\mathrm{ad}V(1)).$$

**Fact two (key observation):** Under the isomorphism $\mathbf{D}_{\mathrm{crys}}(\mathrm{ad}V(1)) \cong \mathbf{D}_{\mathrm{dR}}(\mathrm{ad}V(1))$, the noncriticality assumption implies the direct sum decomposition

$$\mathbf{D}_{\mathrm{dR}}(\mathrm{ad}V(1)) = \mathbf{D}^+_{\mathrm{dR}}(\mathrm{ad}V(1)) \oplus (\mathrm{ad}_{\alpha,0}D)(1),$$

i.e. $(\mathrm{ad}_{\alpha,0}D)(1)$ splits the inclusion $\mathbf{D}^+_{\mathrm{dR}}(\mathrm{ad}V(1)) \subset \mathbf{D}_{\mathrm{dR}}(\mathrm{ad}V(1))$. In particular, we get a canonical isomorphism

$$\mathrm{pr}_\alpha : \mathbf{D}_{\mathrm{dR}}(\mathrm{ad}V(1))/\mathbf{D}^+_{\mathrm{dR}}(\mathrm{ad}V(1)) \xrightarrow{\sim} (\mathrm{ad}_{\alpha,0}D)(1) \cong \mathrm{ad}_{\alpha,0}D.$$

**Theorem F.** *The composite map*

$$q_\alpha \circ \mathrm{pr}_\alpha \circ \log_{\mathrm{ad}V(1)} : H^1_f(\mathbf{Q}_p, \mathrm{ad}(V)(1)) \to L^n$$

*coincides with* $\nu_\alpha$.

It seems worth noting that in the global context above, the map

$$\log_{\mathrm{ad}\rho_\pi(1)} \circ \mathrm{res}_p : H^1_f(\mathbf{Q}, \mathrm{ad}\rho_\pi(1)) \to \mathbf{D}_{\mathrm{dR}}(\mathrm{ad}\rho_\pi(1))/\mathbf{D}^+_{\mathrm{dR}}(\mathrm{ad}\rho_\pi(1))$$

is conjecturally injective; indeed, this map is expected to coincide with the syntomic regulator, whose injectivity for pure motives of weight $\leq -2$ seems to be a folklore conjecture.

After playing with duality, one reduces Theorem F to the following lemma.

**Lemma G.** *For $V$ as above, the Bloch-Kato dual exponential* $\exp^* = \exp^*_{\mathrm{ad}(V)(1)}$ *induces a short exact sequence*

$$0 \to H^1_f(\mathbf{Q}_p, \mathrm{ad}V) \to H^1(\mathbf{Q}_p, \mathrm{ad}V) \xrightarrow{\exp^*} \mathbf{D}^+_{\mathrm{dR}}(\mathrm{ad}V) \to 0,$$

*and* $H^1_\alpha(\mathbf{Q}_p, \mathrm{ad}V) \subset H^1(\mathbf{Q}_p, \mathrm{ad}V)$ *is the preimage of* $\mathbf{D}^+_{\mathrm{dR}}(\mathrm{ad}V) \cap \mathrm{ad}_{\alpha,0}D$. *Furthermore, there is a canonical isomorphism* $\mathbf{D}^+_{\mathrm{dR}}(\mathrm{ad}V) \cap \mathrm{ad}_{\alpha,0}D \cong L^n$ *such that the induced map* $\exp^* : H^1_\alpha(\mathbf{Q}_p, \mathrm{ad}V) \to L^n$ *coincides with the map* $dw$.

The proof of this lemma requires actually knowing something about the definition of $H^1_\alpha(\mathbf{Q}_p, \mathrm{ad}V)$ in terms of the cohomology of $(\varphi, \Gamma)$-modules.

**Question.** Suppose $n = 4$ and $\pi$ is such that $\rho_\pi \simeq \mathrm{Ind}_{G_F}^{G_{\mathbf{Q}}}\chi$, where $F/\mathbf{Q}$ is an abelian quartic CM field and $\chi$ is a $p$-adic Hecke character of $F$. Suppose for simplicity that $p$ is split completely in $F$, so there is an "ordinary" refinement $\alpha$. Can the injectivity of $\mu_\alpha$ in this case be reduced to more familiar problems in the algebraic number theory of $F$, or perhaps even be shown unconditionally?