

# Mordell-Weil growth for GL<sub>2</sub>-type abelian varieties over Hilbert class fields of CM fields

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June 10, 2010

## Abstract

Let  $A$  be a modular abelian variety of GL<sub>2</sub>-type over a totally real field  $F$  of class number one. Under some mild assumptions, we show that the Mordell-Weil rank of  $A$  grows polynomially over Hilbert class fields of CM extensions of  $F$ .

## 1 Introduction and statement of results

Let  $E$  be an elliptic curve defined over a number field  $F$ . The Mordell-Weil group  $E(\overline{F})$  is one of the most mysterious groups in arithmetic. By now, there are many theorems giving partial or complete descriptions of  $E(K)$  for various classes of extensions  $K/F$  which are abelian or nearly abelian. The growth of  $E(K)$  as  $K$  ascends through some sequence of nearly abelian extensions tends to be controlled by root numbers:

- When  $F = \mathbf{Q}$  and  $K_\infty$  is the anticyclotomic  $\mathbf{Z}_p$ -extension of a fixed imaginary quadratic field  $K$ , Vatsal and Cornut ([V], [CV]) show that as one ascends up the cyclic layers of  $K_\infty$ , the rank of  $E$  is controlled entirely by the root number of  $E/K$ .
- When  $F = \mathbf{Q}$  and  $K_d$  is the Hilbert class field of an imaginary quadratic field  $\mathbf{Q}(\sqrt{-d})$ , Templier shows that the rank of  $E$  over  $K_d$  is at least  $\gg d^\delta$  for some small but positive fixed  $\delta$ , provided that  $\varepsilon(E) = -\varepsilon(E \otimes \chi_{-d})$ . In fact, Templier has given two distinct proofs of this theorem: a short proof [Te2] built on the Gross-Zagier theorem and equidistribution theorems for Galois orbits, and an analytic proof [Te1] which analyzes an average value of L-functions directly, using tools from analytic number theory.

Our aim in this paper is to generalize Templier's analytic proof to totally real base fields. This is not a triviality, and leads us to solve an interesting auxiliary problem concerning the meromorphic continuation and growth of a certain Dirichlet series.

To state our theorems, we introduce a little notation. Let  $F$  be a totally real number field of degree  $d$ ; we assume for simplicity that  $F$  has class number one. Let  $f$  be a non-CM holomorphic Hilbert modular form over  $F$  whose weights are all even, with trivial central character. For any

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$\alpha \in \mathcal{O}_F$  which is totally positive, the field  $F(\sqrt{-\alpha})$  is a CM extension. Write  $H_\alpha$  for the Hilbert class field of this extension. Write  $\chi_\alpha$  for the quadratic idele class character of  $F$  cut out by  $F(\sqrt{-\alpha})$ . Now, let  $\mathcal{C}(\alpha)$  be the group of everywhere unramified idele class characters of  $F(\sqrt{-\alpha})$ , or equivalently the character group of the class group of  $F(\sqrt{-\alpha})$ . By the Brauer-Siegel theorem, this group is of size  $|\mathcal{C}(\alpha)| \gg_\varepsilon (\mathbf{N}\alpha)^{\frac{1}{2}-\varepsilon}$  as  $\mathbf{N}\alpha \rightarrow \infty$ . By quadratic automorphic induction, any  $\chi \in \mathcal{C}(\alpha)$  gives rise to a holomorphic Hilbert modular form  $\theta_\chi$  over  $F$ , of parallel weight one. The root number of the Rankin-Selberg L-series  $L(s, f \otimes \theta_\chi)$  is  $\pm 1$ , and is independent of  $\chi$ ; it equals  $\varepsilon(f)\varepsilon(f \otimes \chi_\alpha)$ .

Our main result is the following theorem.

**Theorem 1.1.** *Notation as above, if  $\alpha \in \mathcal{O}_F^+$  is such that  $\varepsilon(f)\varepsilon(f \otimes \chi_\alpha) = -1$ , then we have*

$$\frac{1}{|\mathcal{C}(\alpha)|} \sum_{\chi \in \mathcal{C}(\alpha)} L'(\tfrac{1}{2}, f \otimes \theta_\chi) \gg L(1, \text{sym}^2 f) \log(\mathbf{N}\alpha) L(1, \chi_\alpha)$$

as  $\mathbf{N}\alpha \rightarrow \infty$ , where the implied constant depends only on  $F$ .

In fact we give a precise asymptotic for this average; see Theorem 3.2 (when  $F = \mathbf{Q}$ , Theorem 3.2 is one of the main results of [Te1]). The subconvexity results of [MV] give  $L'(\tfrac{1}{2}, f \otimes \theta_\chi) \ll \mathbf{N}\alpha^{\frac{1}{2}-\delta}$  for some fixed positive  $\delta$ , so via the Brauer-Siegel theorem we immediately deduce

**Corollary 1.2.** *Notation and assumptions as above, there exists some  $\delta > 0$  such that at least  $\gg (\mathbf{N}\alpha)^{\delta-\varepsilon}$  of the central derivatives  $L'(\tfrac{1}{2}, f \otimes \theta_\chi)$  are nonvanishing.*

The zeta function of  $H_\alpha$  factors as

$$\zeta_{H_\alpha}(s) = \prod_{\chi \in \mathcal{C}(\alpha)} L(s, \theta_\chi).$$

Hence, feeding Corollary 1.2 into the results of [TZ] (see e.g. Theorem 4.3.1 of [Zh]) yields

**Corollary 1.3.** *Let  $A/F$  be a modular abelian variety of  $\text{GL}_2$ -type, with associated Hilbert modular form  $f$ . Then  $\text{rank}A(H_\alpha) \gg \dim A \cdot (\mathbf{N}\alpha)^{\delta-\varepsilon}$  as  $\mathbf{N}\alpha \rightarrow \infty$  along any sequence with  $\varepsilon(f \otimes \chi_\alpha) = -\varepsilon(f)$ .*

We turn to an overview of the proof of Theorem 1.1. The first step is to give an expression for  $L'(s, f \otimes \theta_\chi)$  as a short Dirichlet polynomial essentially of length  $\mathbf{N}\alpha$ . Averaging over  $\chi$  yields an expression of shape

$$\frac{1}{|\mathcal{C}(\alpha)|} \sum_{\chi \in \mathcal{C}(\alpha)} L'(\tfrac{1}{2}, f \otimes \theta_\chi) \approx \sum_{x, y \in (\mathcal{O}_F \times \mathcal{O}_F) / \Delta U_F} \frac{\lambda_f(x^2 + \alpha y^2)}{(\mathbf{N}(x^2 + \alpha y^2))^{\frac{1}{2}}} V(\mathbf{N}(x^2 + \alpha y^2)).$$

Here  $\lambda_f$  are the Hecke eigenvalues of  $f$ , indexed by ideals of  $\mathcal{O}_F$ , and  $V(x)$  is a smooth function  $\mathbf{R}_{>0} \rightarrow \mathbf{R}$  which decays rapidly for  $x \gg (\mathbf{N}\alpha)^{1+\varepsilon}$  and diverges near the origin like  $V(x) \sim c \log x$ . Splitting off the  $y = 0$  term yields a main term. Our problem then reduces to estimating sums of the form

$$\sum_{\gamma \in \mathcal{O}_F} \lambda_f(\gamma^2 + \beta) W(\gamma^2 + \beta) \tag{1}$$

for  $\beta \in \mathcal{O}_F^+$  fixed and  $W$  smooth, in various ranges.

So far we have followed, in this reduction, Templier's analytic proof [Te1] of Theorem 1.1 over  $\mathbf{Q}$ . Templier treats the sums (1) over  $F = \mathbf{Q}$  by a delicate and ingenious application of the  $\delta$ -symbol method of Duke-Friedlander-Iwaniec, which is in turn an elaboration of the original circle method

of Hardy-Littlewood-Ramanujan. Rather than trying to make the circle method work over number fields, we analyze the sums (1) by the spectral theory of Hilbert modular forms of half-integral weight. Our main result in this direction is the following theorem.

**Theorem 1.4.** *Notation as above, fix  $\beta \in \mathcal{O}_F^+$  and define the Dirichlet series*

$$\mathcal{D}_f(s; \beta) = \sum_{\gamma \in \mathcal{O}_F} \frac{\lambda_f(\gamma^2 + \beta)}{(\mathbf{N}(\gamma^2 + \beta))^s}.$$

*Then  $\mathcal{D}_f(s; \beta)$  admits a meromorphic continuation to the whole complex plane. Furthermore,  $\mathcal{D}_f(s; \beta)$  is entire in the halfplane  $\operatorname{Re}(s) > \frac{1}{4}$ , with the exception of at most finitely many poles in the interval  $s \in [\frac{1}{4}, \frac{1}{4} + \frac{\theta}{2}]$ , and satisfies the bound  $\mathcal{D}_f(s; \beta) \ll e^{\pi d|s|} (1 + |s|)^A (\mathbf{N}\beta)^{\frac{1}{2}-s - \frac{1}{16}(1-2\theta)}$  in that same halfplane.*

Actually we prove a slightly more general result dealing with general quadratic polynomials, see Proposition 3.1. This result seems to be new even over  $\mathbf{Q}$ . Here  $\theta = \frac{7}{64}$  is the best known bound towards the Ramanujan conjecture for  $\mathrm{GL}_2$  over number fields [BB]. If we could only show this theorem with the exponent  $\frac{1}{2} - \frac{1}{16}(1 - 2\theta)$  replaced by  $\frac{1}{2}$ , this would just barely fail to be strong enough to imply Theorem 1.1. After giving a spectral expansion of  $D_f(s; \beta)$ , we eventually deduce this crucial savings from a beautiful theorem of Baruch-Mao, relating Fourier coefficients on  $\widetilde{\mathrm{SL}}_2$  to twisted L-values, which we control in turn via a subconvex bound due to Blomer and Harcos [BH].

This paper is organized as follows. In section two we review holomorphic Hilbert modular forms and their L-functions, as well as the spectral theory of Hilbert modular forms of half-integral weight. In section three, we show that Theorem 1.1 is implied by Proposition 3.1, which we prove in turn in section 4.

## Acknowledgements

This material had its genesis in my 2010 Brown University senior honors thesis [Ha], where I proved Theorem 1.4 in the case  $F = \mathbf{Q}$ . I would like to thank Jeff Hoffstein for many extremely helpful discussions during the writing of my thesis, and for reading an earlier draft of this paper. I am also grateful to Keith Conrad for several enlightening conversations on the arithmetic of CM extensions, to Jordan Ellenberg for answering a key question, and to Nicolas Templier for very helpful comments on an earlier draft of this paper.

During the writing of this paper, I was supported by a Barry M. Goldwater scholarship, and by the Josephine de Kármán Fellowship Trust; it is a pleasure for me to acknowledge their generosity.

## 2 Background and lemmas

### 2.1 Hilbert modular forms

Fix  $F/\mathbf{Q}$  totally real of degree  $d$  and class number one. Fix an ordering  $\sigma_1, \dots, \sigma_d$  on the embeddings of  $F$  into  $\mathbf{R}$ . Write  $\mathcal{O}_F$  for the ring of integers of  $F$ ,  $\mathcal{O}_F^+$  for the totally positive integers,  $U_F$  for the unit group, and  $U_F^+$  for the totally positive units. We shall frequently use the fact that ideals in  $\mathcal{O}_F$  are parametrized by the set  $\mathcal{O}_F^+/U_F^+$ . Fix  $\delta$  a totally positive generator of the different ideal of  $F$ . Set  $\Delta_F$  the absolute discriminant of  $F$ .

Let  $g \in \mathrm{SL}_2(\mathcal{O}_F)$  act on  $z = (z_1, \dots, z_d) \in \mathfrak{H}^d$  in the usual way, i.e. via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \left( \frac{\sigma_1(a)z_1 + \sigma_1(b)}{\sigma_1(c)z_1 + \sigma_1(d)}, \dots, \frac{\sigma_d(a)z_d + \sigma_d(b)}{\sigma_d(c)z_d + \sigma_d(d)} \right).$$

For  $\mathfrak{b}, \mathfrak{c} \subset \mathcal{O}_F$ , we define congruence subgroups by

$$\Gamma_0(\mathfrak{b}, \mathfrak{c}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}_F), b \in \mathfrak{b}, c \in \mathfrak{c} \right\}.$$

A Hilbert modular form of weight  $(k_1, \dots, k_d)$  and level  $\mathfrak{a}$  is a function  $f : \mathfrak{H}^d \rightarrow \mathbf{C}$  which transforms as

$$f(\gamma z) = f(z) \cdot \prod_{j=1}^d (\sigma_j(c)z_j + \sigma_j(d))^{k_j}$$

for  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(1, \mathfrak{a})$ . We shall restrict our attention to Hilbert modular forms of parallel weight two. It will be abundantly clear at every step of the proof that the general weight case is no harder, and in fact that we could treat forms which are spherical at infinity if we so desired; we have made this choice to simplify our notation.

A Hilbert modular form of parallel weight two has a Fourier expansion

$$f(z_1, \dots, z_d) = \sum_{\alpha \in \mathcal{O}_F^+} \lambda_f(\alpha) (\mathbf{N}\alpha)^{\frac{1}{2}} e(\sigma_1(\delta^{-1}\alpha)z_1 + \dots + \sigma_d(\delta^{-1}\alpha)z_d).$$

Here  $\delta$  is a totally positive generator of the different ideal of  $F$ , and the coefficients  $\lambda_f(\alpha)$  depend solely on the ideal generated by  $\alpha$ . The conductor of  $f$  is the unique ideal  $\mathfrak{n}_f \subset \mathcal{O}_F$  such that  $f$  is a new vector for  $\Gamma_0(1, \mathfrak{n}_f)$ . Hereafter, for any  $\alpha \in F$  and any  $z \in \mathfrak{H}^d$ , we abbreviate  $\mathrm{tr}(\alpha z) = \sigma_1(\alpha)z_1 + \dots + \sigma_d(\alpha)z_d$ . The L-function attached to  $f$  has Dirichlet series

$$L(s, f) = \sum_{\alpha \in \mathcal{O}_F^+ / U_F^+} \frac{\lambda_f(\alpha)}{\mathbf{N}\alpha^s}.$$

Given  $\chi \in \mathcal{C}(\alpha)$ , define

$$r_\chi(a) = \sum_{\mathfrak{a} \subset \mathcal{O}_{F(\sqrt{-\alpha})}, \mathfrak{a}\bar{\mathfrak{a}}=(a)} \chi(\mathfrak{a}).$$

The function

$$\theta_\chi(z) = \sum_{a \in \mathcal{O}_F^+} r_\chi(a) e(\mathrm{tr}(\delta^{-1}az))$$

is a Hilbert modular form of parallel weight one and level  $\Gamma_0(1, (\alpha))$  with central character  $\chi_\alpha$ . The Rankin-Selberg L-function attached to  $f \otimes \theta_\chi$  has Dirichlet series

$$L(s, f \otimes \theta_\chi) = L^{(\mathfrak{n}_f)}(2s, \chi_\alpha) \sum_{a \in \mathcal{O}_F^+ / U_F^+} \frac{\lambda_f(a)r_\chi(a)}{\mathbf{N}a^s}.$$

Here the superscript indicates removal of the Euler factors at primes dividing  $\mathfrak{n}_f$ . We are assuming for simplicity that  $\alpha$  and  $\mathfrak{n}_f$  are coprime. The completed L-function is given by

$$\Lambda(s, f \otimes \theta_\chi) = (\mathbf{N}\alpha\mathbf{N}\mathfrak{n}_f)^s (2\pi)^{-2ds} \Gamma\left(s + \frac{1}{2}\right)^d \Gamma\left(s + \frac{3}{2}\right)^d L(s, f \otimes \theta_\chi).$$

This function satisfies  $\Lambda(1-s, f \otimes \theta_\chi) = \varepsilon(f)\varepsilon(f \otimes \chi_\alpha)\Lambda(s, f \otimes \theta_\chi)$ .

We shall also require Hilbert modular forms of half-integral weight. For a more detailed exposition of these see [Sh1] or [Ko]. Set

$$\theta_F(z) = \sum_{\alpha \in \mathcal{O}_F} e(\operatorname{tr}(\delta^{-1}\alpha^2 z)).$$

This is an analogue of Jacobi's theta function, and is a Hilbert modular form of parallel weight  $\frac{1}{2}$  for  $\Gamma_0((2), (2))$ ; c.f. [Sh1]. Define  $j(\gamma, z)$  by

$$j(\gamma, z) = \frac{\theta_F(\gamma z)}{\theta_F(z)|cz + d|^{\frac{1}{2}}}.$$

Note that  $|j(\gamma, z)| = 1$ . Now, for  $\nu = \frac{1}{2}, \frac{3}{2}$ , and any ideal with  $\mathfrak{n} \subset (2)$ , define  $\mathbf{H}_\nu(\mathfrak{n})$  to be the space of functions on  $\mathfrak{H}^d$  which transform as

$$\phi(\gamma z) = j(\gamma, z)^{2\nu} \phi(z), \quad \forall \gamma \in \Gamma_0((2), \mathfrak{n}).$$

We could of course allow more general vector weights, but for our purposes this is enough. This is a Hilbert space under the inner product

$$\langle \phi_1, \phi_2 \rangle = \int_{\Gamma_0(\mathfrak{n}) \backslash \mathfrak{H}^d} \phi_1(z) \overline{\phi_2(z)} d\mu.$$

There is a large collection of commuting self-adjoint operators acting on  $\mathbf{H}_\nu(\mathfrak{n})$ : the weight  $\nu$  Laplace operator  $\Delta_\nu$  acts on each  $z_i$ -variable separately. Under the action of these operators,  $\mathbf{H}_\nu(\mathfrak{n})$  breaks up as a direct sum of two orthogonal subspaces spanned by unitary Eisenstein series and cusp forms, respectively. Suppose  $\phi$  is such that  $\Delta_\nu^{(j)} \phi = \lambda_\phi^{(j)} \phi$  for  $j = 1, \dots, d$ , where the superscript indicates which  $z_i$ -variable we are acting on. Define  $t_\phi^{(j)}$  by  $\lambda_\phi^{(j)} = \frac{1}{4} + (t_\phi^{(j)})^2$ . Then  $\phi$  has a Fourier expansion

$$\phi(z) = \phi_0(y) + \sum_{\alpha \in \mathcal{O}_F \setminus \{0\}} \frac{\rho_\phi(\alpha)}{\sqrt{|\mathbf{N}\alpha|}} \prod_{j=1}^d W_{\frac{\nu}{2} \operatorname{sign}(\sigma_j(\alpha)), it_\phi^{(j)}}(4\pi |\sigma_j(\delta^{-1}\alpha)| y_j) e(\operatorname{tr}(\delta^{-1}\alpha x)).$$

If  $\phi$  is a cusp form the term  $\phi_0(y)$  vanishes.

Within each cuspidal  $(\Delta^{(1)}, \dots, \Delta^{(d)})$ -eigenspace we take an orthonormal basis which furthermore is diagonalized for all the  $T_{\mathfrak{p}^2}$ -Hecke operators, for all  $\mathfrak{p}$  prime to  $2\mathfrak{n}$ . We then have the following crucial

**Lemma 2.1.** *Under the above assumptions, the coefficients  $\rho_\phi(\alpha)$  satisfy the bound*

$$\rho_\phi(\alpha) \ll \mathbf{N}\alpha^{\frac{1}{4} - \frac{1}{16}(1-2\theta)} \prod_{j=1}^d e^{\frac{\nu}{2} t_\phi^{(j)}} (1 + |t_\phi^{(j)}|)^A,$$

where  $A$  is some large but fixed positive constant, and  $\theta = \frac{7}{64}$  is the best known exponent toward the Ramanujan-Petersson conjecture on  $\mathrm{GL}_2$ .

*Proof.* This follows from a theorem of Baruch-Mao [BM], which gives a relation of the form

$$\frac{|\rho_\phi(\alpha)|^2}{\|\phi\|^2} e(\phi) = \frac{L(\frac{1}{2}, \Phi \otimes \chi_\alpha)}{\|\Phi\|^2}.$$

for  $\alpha$  squarefree. Here  $\Phi$  is the integral-weight Shimura correspondant of  $\phi$ , and  $e(\phi)$  is an archimedean integral, which in our case can be computed as a ratio of  $\Gamma$ -functions. The twisted L-values satisfy the bound  $L(\frac{1}{2}, \Phi \otimes \chi_\alpha) \ll (\mathbf{N}\alpha)^{\frac{1}{2} - \frac{1}{8}(1-2\theta)} \prod_{j=1}^d (1 + |t_\phi^{(j)}|)^A$  by the main theorem of [BH]. For  $\alpha$  non-squarefree, the  $\rho_\phi(\alpha)$  can be expressed recursively in terms of their values on squarefree divisors of  $\alpha$ .

We shall in fact require slightly more general theta functions than  $\theta_F$ .

**Lemma 2.2.** *For any  $\beta \in \mathcal{O}_F$ , the function*

$$\theta_F^\beta(z) = \sum_{\alpha \in \mathcal{O}_F} e(\mathrm{tr}(\delta^{-1}(\beta + 2\alpha)^2 z))$$

is a Hilbert modular form of parallel weight  $\frac{1}{2}$  and bounded level.

### 3 Reduction to an analytic problem

Let  $f$  and  $\theta_\chi$  be as in section 2. From now on we assume that the global root number of  $f \otimes \theta_\chi$  is  $-1$ . Our goal is an asymptotic evaluation of the series

$$S = \frac{1}{|\mathcal{C}(\alpha)|} \sum_{\chi \in \mathcal{C}(\alpha)} L'(\frac{1}{2}, f \otimes \theta_\chi).$$

Our first step is to derive an exact formula for  $\Lambda'(\frac{1}{2}, f \otimes \theta_\chi)$ . Consider the integral

$$I = \frac{1}{2\pi i} \int_{(3)} \Lambda(s + \frac{1}{2}, f \otimes \theta_\chi) \cos\left(\frac{\pi s}{200}\right)^{-200} \frac{ds}{s^2}.$$

By construction the integrand has a simple pole at  $s = 0$  of residue  $\Lambda'(\frac{1}{2}, f \otimes \theta_\chi)$ . On the other hand, moving the contour of integration to  $(-3)$  we derive

$$\Lambda'(\frac{1}{2}, f \otimes \theta_\chi) = (2\pi)^{-d} \sqrt{\mathbf{N}\alpha c_f} \cdot \sum_{(b, \mathbf{n}_f)=1} \chi_\alpha(b) \sum_a \frac{\lambda_f(a) r_\chi(a)}{\sqrt{\mathbf{N}a \mathbf{N}b}} V\left(\frac{\mathbf{N}ab^2}{\mathbf{N}\alpha c_f}\right),$$

where

$$V(x) = \frac{1}{4\pi i} \int_{(3)} \Gamma(s+1)^d \Gamma(s+2)^d \cos\left(\frac{\pi s}{200}\right)^{-200} (2\pi)^{-2ds} x^{-s} \frac{ds}{s^2}.$$

This gives

$$L'(\frac{1}{2}, f \otimes \theta_\chi) = \frac{1}{2} \sum_b \sum_a \frac{\lambda_f(a) r_\chi(a)}{\sqrt{\mathbf{N}a \mathbf{N}b}} V\left(\frac{\mathbf{N}ab^2}{\mathbf{N}\alpha c_f}\right).$$

Next we sum over  $\chi \in \mathcal{C}(\alpha)$ , giving by orthogonality

$$S = \frac{1}{\mathcal{C}(\alpha)} \sum_{\chi \in \mathcal{C}(\alpha)} L'(\tfrac{1}{2}, f \otimes \theta_\chi) = \frac{1}{2} \sum_{b \in \mathcal{O}_F^+ / U_F^+} \sum_{\mathfrak{a} \subset \mathcal{O}_{F(\sqrt{-\alpha})}, \mathfrak{a} \text{ principal}} \frac{\lambda_f(\mathfrak{a}\bar{\alpha})}{\sqrt{\mathbf{N}(\mathfrak{a}\bar{\alpha})\mathbf{N}b}} V\left(\frac{\mathbf{N}(\mathfrak{a}\bar{\alpha})\mathbf{N}b^2}{\mathbf{N}\alpha c_f}\right)$$

Let  $\xi \in \mathcal{O}_{F(\sqrt{-\alpha})}$  be such that  $\mathcal{O}_{F(\sqrt{-\alpha})} = \mathcal{O}_F + \mathcal{O}_F\xi$ ; our class number assumption guarantees the existence of such a ‘‘relative integral basis’’ in our situation. Then the norms of principal ideals in the ring  $\mathcal{O}_{F(\sqrt{-\alpha})}$  are parametrized precisely by the quadratic form  $(\gamma + \xi\beta)(\gamma + \bar{\xi}\beta) = \gamma^2 + (\xi + \bar{\xi})\beta + \xi\bar{\xi}\beta^2$ , where  $\gamma$  and  $\beta$  run over  $\mathcal{O}_F \times \mathcal{O}_F$  modulo the diagonal action of the unit group. We now split the sum  $S$  into two sums,  $S_{\text{main}}$  and  $S_0$  according to whether  $\beta = 0$  or not. Thus we have

$$S = S_{\text{main}} + S_0,$$

with

$$S_{\text{main}} = \frac{1}{2} \sum_{b \in \mathcal{O}_F^+ / U_F^+, (b, \mathfrak{n}_f)=1} \chi_\alpha(b) \sum_{\gamma \in \mathcal{O}_F / U_F} \frac{\lambda_f(\gamma^2)}{\mathbf{N}\gamma\mathbf{N}b} V\left(\frac{\mathbf{N}\gamma^2\mathbf{N}b^2}{\mathbf{N}(\alpha\mathfrak{n}_f)}\right)$$

and

$$S_0 = \frac{1}{2} \sum_{b \in \mathcal{O}_F^+ / U_F^+, (b, \mathfrak{n}_f)=1} \chi_\alpha(b) \sum_{\beta \in (\mathcal{O}_F \setminus \{0\}) / U_F} \sum_{\gamma \in \mathcal{O}_F} \frac{\lambda_f(\gamma^2 + (\xi + \bar{\xi})\beta + \xi\bar{\xi}\beta^2)}{\sqrt{\mathbf{N}(\gamma^2 + (\xi + \bar{\xi})\beta + \xi\bar{\xi}\beta^2)\mathbf{N}b}} V\left(\frac{\mathbf{N}(\gamma^2 + (\xi + \bar{\xi})\beta + \xi\bar{\xi}\beta^2)\mathbf{N}b^2}{\mathbf{N}(\alpha\mathfrak{n}_f)}\right).$$

To evaluate  $S_{\text{main}}$ , note that it is given identically by the contour integral

$$S_{\text{main}} = \frac{1}{2\pi i} \int_{(3)} \frac{L^{(\mathfrak{n}_f)}(2s+1, \chi_\alpha) L(2s+1, \text{sym}^2 f)}{\zeta_F(4s+2)} \Gamma(s+1)^d \Gamma(s+2)^d \cos\left(\frac{\pi s}{200}\right)^{-200} (2\pi)^{-2ds} (\mathbf{N}\alpha\mathfrak{n}_f)^s \frac{ds}{s^2}.$$

Pushing the contour to  $(-\frac{1}{4})$  we pick up a pole at  $s = 0$ , of residue

$$r(f, \alpha) = \frac{L^{(\mathfrak{n}_f)}(1, \chi_\alpha) L(1, \text{sym}^2 f)}{\zeta_F(2)} \left( \frac{1}{2} \log \mathbf{N}\mathfrak{n}_f + \frac{1}{2} \log \mathbf{N}\alpha + \frac{L^{(\mathfrak{n}_f)}(1, \chi_\alpha)}{L^{(\mathfrak{n}_f)}(1, \text{sym}^2 f)} + \frac{L'}{L}(1, \text{sym}^2 f) + c_F \right).$$

To estimate the integrand along the contour  $\text{Re}(s) = (-\frac{1}{4})$ , we use the subconvexity bound  $L(\frac{1}{2} + it, \chi_\alpha) \ll (\mathbf{N}\alpha)^{\frac{1}{4} - \frac{1}{16}(1-2\theta)}$ . Invoking the bound  $|\zeta_F(1+it)| \gg_F (\log(|t|+3))^{-d}$  ([IK], Ch. 5) we derive

$$S_{\text{main}} = r(f, \alpha) + O\left(\mathbf{N}\alpha^{-\frac{1}{16}(1-2\theta)}\right).$$

Finally, we show the estimate  $r(f, \alpha) \gg_{F,\varepsilon} L(1, \text{sym}^2 f) (\mathbf{N}\alpha)^{-\varepsilon}$ . To see this, we define a function on ideals in  $\mathcal{O}_F$  by

$$\tau_\alpha(\mathfrak{a}) = \sum_{\mathfrak{b} \supset \mathfrak{a}} \chi(\mathfrak{b}).$$

The generating function of this is simply

$$\sum_{\mathfrak{a} \subset \mathcal{O}_F} \tau_\alpha(\mathfrak{a}) \mathbf{N}\mathfrak{a}^{-s} = \zeta_F(s) L(s, \chi_\alpha).$$

This immediately implies  $\tau_\alpha(\mathbf{a}) \geq 0$ , and if  $\mathbf{a}$  is a square (i.e. it has even valuation at every finite place) then  $\tau_\alpha(\mathbf{a}) = 1$ . Now we compute the sum

$$S(X, \alpha) = \sum_{\mathbf{a} \subset \mathcal{O}_F} \frac{\tau_\alpha(\mathbf{a})}{\mathbf{N}\mathbf{a}} \exp\left(-\frac{\mathbf{N}\mathbf{a}}{X}\right)$$

in two different ways. On one hand, we may write

$$S(X, \alpha) = \frac{1}{2\pi i} \int_{(3)} \zeta_F(s) L(s, \chi_\alpha) \Gamma(s-1) X^{s-1} ds;$$

moving the contour to  $\operatorname{Re}(s) = \frac{1}{2}$ , we pass a pole at  $s = 1$ , and estimating the integral along  $\operatorname{Re}(s) = \frac{1}{2}$  using the subconvex bound for  $L(\frac{1}{2} + it, \chi_\alpha)$  and the rapid decay of the gamma function yields the asymptotic

$$S(X, \alpha) = L(1, \chi_\alpha)(\log X + \gamma_F) + L'(1, \chi_\alpha) + O(\mathbf{N}\alpha^{\frac{1}{4} - \frac{1}{16}(1-2\theta)} X^{-\frac{1}{2}}).$$

On the other hand,  $S(X, \alpha) \geq 0$ , so choosing  $X = e^{-\gamma_F} \mathbf{N}\alpha^{\frac{1}{2} - \delta}$  with  $0 < \delta < \frac{1}{8}(1-2\theta)$  gives

$$\left(\frac{1}{2} - \delta\right) \log \mathbf{N}\alpha \cdot L(1, \chi_\alpha) + L'(1, \chi_\alpha) \geq 0,$$

so

$$\frac{1}{2} \log \mathbf{N}\alpha + \frac{L'}{L}(1, \chi_\alpha) \gg \log \mathbf{N}\alpha$$

and hence

$$\begin{aligned} r(f, \alpha) &\gg_F L(1, \operatorname{sym}^2 f) L(1, \chi_\alpha) \log \mathbf{N}\alpha. \\ &\gg_{F, \varepsilon} L(1, \operatorname{sym}^2 f) (\mathbf{N}\alpha)^{-\varepsilon} \end{aligned}$$

by the Brauer-Siegel theorem.

There are two notable cases where the clean estimate  $r(f, \alpha) \gg_F L(1, \operatorname{sym}^2 f)$  holds. If  $F(\sqrt{-\alpha})$  does not contain a quadratic extension of  $\mathbf{Q}$ , this follows immediately from Lemma 8 of [St]. If  $d > 2$  and the Galois closure of  $F$  has degree  $d!$  over  $\mathbf{Q}$ , this is a consequence of the beautiful results in [HJ].

To bound the error term  $S_0$ , first complete the square, writing

$$\begin{aligned} \gamma^2 + (\xi + \bar{\xi})\beta + \xi\bar{\xi}\beta^2 &= \left(\gamma + \frac{1}{2}(\xi + \bar{\xi})\beta\right)^2 + \left(\xi\bar{\xi} - \frac{1}{4}(\xi + \bar{\xi})^2\right)\beta^2 \\ &= \left(\gamma + \frac{1}{2}(\xi + \bar{\xi})\beta\right)^2 - \frac{1}{4}(\xi - \bar{\xi})^2\beta^2. \end{aligned}$$

Recall that  $\xi$  is a relative integral generator for the ring of integers of a CM extension  $F(\sqrt{-\alpha})/F$ , and as such we may write

$$\xi = \frac{1}{2}(x + y\sqrt{-\alpha})$$

for some  $x, y \in \mathcal{O}_F$ . Hence  $\xi + \bar{\xi} = x$  is an element of  $\mathcal{O}_F$ , and  $\xi - \bar{\xi} = y\sqrt{-\alpha}$ , so  $-\frac{1}{4}(\xi - \bar{\xi})^2 = \frac{1}{4}y^2\alpha$  is a totally positive element of  $F$ , of absolute norm  $\geq 4^{-d}\mathbf{N}\alpha$ .

Now, to evaluate  $S_0$ , we first execute the  $b$ -sum inside the definition of  $V$ , giving

$$S_0 = \sum_{\beta \in (\mathcal{O}_F \setminus \{0\})/U_F} \sum_{\gamma \in \mathcal{O}_F} \frac{\lambda_f\left(\left(\gamma + \frac{1}{2}(\xi + \bar{\xi})\beta\right)^2 - \frac{1}{4}(\xi - \bar{\xi})^2\beta^2\right)}{\sqrt{\mathbf{N}\left(\left(\gamma + \frac{1}{2}(\xi + \bar{\xi})\beta\right)^2 - \frac{1}{4}(\xi - \bar{\xi})^2\beta^2\right)}} W\left(\frac{\mathbf{N}\left(\left(\gamma + \frac{1}{2}(\xi + \bar{\xi})\beta\right)^2 - \frac{1}{4}(\xi - \bar{\xi})^2\beta^2\right)}{\mathbf{N}(\alpha n_f)}\right)$$



where we have set

$$\begin{aligned} W(x) &= \sum_{b \in \mathcal{O}_F^+ / U_F^+, (b, \mathfrak{n}_f) = 1} \chi_\alpha(b) (\mathbf{N}b)^{-1} V(x\mathbf{N}b^2) \\ &= \frac{1}{4\pi i} \int_{(3)} L^{(\mathfrak{n}_f)}(2s+1, \chi_\alpha) \Gamma\left(s + \frac{1}{2}\right)^d \Gamma\left(s + \frac{3}{2}\right)^d \cos\left(\frac{\pi s}{200}\right)^{-200} (2\pi)^{-2ds} x^{-s} \frac{ds}{s^2}. \end{aligned}$$

The  $W$ -function satisfies the crude estimate  $W(x) \ll_A x^{-A}$  for any fixed  $A < 100$ .

The key ingredient in estimating  $S_0$  is the following proposition, which is a slight generalization of Theorem 1.4; we defer the proof until section four.

**Proposition 3.1.** *Let  $P(x) = x^2 + ax + b$  be a polynomial with  $a, b \in \mathcal{O}_F$  and  $D = b - \frac{1}{4}a^2 \in \frac{1}{4}\mathcal{O}_F$  totally positive. Then the Dirichlet series*

$$\mathcal{D}_f(s; D) = \sum_{\gamma \in \mathcal{O}_F} \frac{\lambda_f(\gamma^2 + a\gamma + b)}{(\mathbf{N}(\gamma^2 + a\gamma + b))^s}$$

admits a meromorphic continuation to the entire complex plane. Furthermore, it is holomorphic in the half-plane  $\operatorname{Re}(s) \geq \frac{1}{4}$  with the exception of at most finitely many poles in the interval  $[\frac{1}{4}, \frac{1}{4} + \frac{\theta}{2}]$ , and it satisfies the bound

$$\mathcal{D}_f(s; D) \ll (1 + |s|)^A e^{\pi d|s|} (\mathbf{N}D)^{\frac{1}{2} - s - \frac{1}{16}(1-2\theta)}$$

in that same half-plane, where the implied constant depends polynomially on  $f$ .

For  $\beta$  fixed, the  $\gamma$ -sum is given exactly by the integral

$$I(\beta) = \frac{1}{2\pi i} \int_{(2)} \mathcal{D}_f(s + \frac{1}{2}; \frac{1}{4}\alpha y^2 \beta^2) L^{(\mathfrak{n}_f)}(2s+1, \chi_\alpha) \Gamma(s+1)^d \Gamma(s+2)^d \cos\left(\frac{\pi s}{200}\right)^{-200} (2\pi)^{-2ds} (\mathbf{N}\alpha \mathfrak{n}_f)^s \frac{ds}{s^2};$$

moving the contour to  $\operatorname{Re}(s) = 50$  is justified by the absence of poles in that region, and the rapid decay of the integrand. To estimate the integral along this contour we use the bound of Proposition 3.1, giving  $\mathcal{D}_f(50 + \frac{1}{2} + it, \frac{1}{4}\alpha y^2 \beta^2) \ll e^{\pi d|t|} \cdot \mathbf{N}(\alpha \beta^2)^{-50 - \frac{1}{16}(1-2\theta)}$ . The product of  $\Gamma$ -functions decays like  $e^{-\pi d|t|}$  by Stirling's formula, and the cosine decays like  $e^{-\pi|t|}$ , so upon using the trivial bound  $L(101 + it, \chi_\alpha) \asymp 1$ , the integral converges absolutely and is bounded by  $(\mathbf{N}\alpha)^{-\frac{1}{16}(1-2\theta)} (\mathbf{N}\beta)^{-100}$ . Inserting this bound into the definition of  $S_0$ , the  $\beta$ -sum converges absolutely, giving

$$S_0 \ll \mathbf{N}\alpha^{-\frac{1}{16}(1-2\theta)}.$$

Gathering results, we have proven

**Theorem 3.2.** *Notation and assumptions as in Theorem 1.1, and assuming Proposition 3.1, we have*

$$\frac{1}{|\mathcal{C}(\alpha)|} \sum_{\chi \in \mathcal{C}(\alpha)} L'(\frac{1}{2}, f \otimes \theta_\chi) = r(f, \alpha) + O_F((\mathbf{N}\alpha)^{-\frac{1}{16}(1-2\theta)})$$

where

$$\begin{aligned} r(f, \alpha) &= \pi^d \frac{L^{(\mathfrak{n}_f)}(1, \chi_\alpha) L(1, \operatorname{sym}^2 f)}{\zeta_F(2)} \left( \frac{1}{2} \log \mathbf{N}\mathfrak{n}_f + \frac{1}{2} \log \mathbf{N}\alpha + \frac{L'(\mathfrak{n}_f)}{L^{(\mathfrak{n}_f)}}(1, \chi_\alpha) + \frac{L'}{L}(1, \operatorname{sym}^2 f) + c_F \right) \\ &\gg_F L(1, \operatorname{sym}^2 f) L(1, \chi_\alpha) \log \mathbf{N}\alpha. \end{aligned}$$

## 4 Sums of Hecke eigenvalues along quadratic sequences

Fix  $a, b \in \mathcal{O}_F$  with  $D = b - \frac{1}{4}a^2$  totally positive. Consider the integral

$$I_f(s; D) = \int_{\mathcal{O}_F \setminus \mathbf{R}^d \times (i\mathbf{R}_{>0})^d} |y|^{\frac{1}{4}} \theta_F^a(z) e(4D\delta^{-1}z) \overline{|y|f(4z)} |y|^s d\mu.$$

Inserting the Fourier expansions of  $\theta_F^a$  and  $f$  yields

$$\begin{aligned} I_f(s; D) &= \Delta_F \cdot \sum_{\gamma \in \mathcal{O}_F} \mathbf{N}(\gamma^2 + a\gamma + b)^{\frac{1}{2}} \lambda_f(\gamma^2 + a\gamma + b) \int_{(i\mathbf{R}_{>0})^d} e^{-16\pi \text{tr}(\delta^{-1}(\gamma^2 + a\gamma + b)y)} |y|^{s+\frac{1}{4}} d^\times |y| \\ &= \Delta_F^{s+\frac{5}{4}} (4\pi)^{-d(s+\frac{1}{4})} \Gamma(s+\frac{1}{4})^d \sum_{\gamma \in \mathcal{O}_F} \frac{\lambda_f(\gamma^2 + a\gamma + b)}{(\mathbf{N}(\gamma^2 + a\gamma + b))^{s-\frac{1}{4}}}. \end{aligned}$$

On the other hand, the function  $|y|^{\frac{5}{4}} f(4z) \overline{\theta_F^b(z)}$  is an automorphic form of weight  $\frac{3}{2}$  for some arithmetic group  $\Gamma(b, \mathbf{n}_f) \subset \text{SL}_2(\mathcal{O}_F)$  whose index is bounded polynomially in  $\mathbf{N}\mathbf{n}_f$ . Write  $\mathbf{H}_{\frac{3}{2}}(b, \mathbf{n}_f)$  for the group of automorphic forms of weight  $\frac{3}{2}$  and level  $\Gamma(b, \mathbf{n}_f)$ . We may spectrally expand the function  $|y|^{\frac{5}{4}} f(4z) \overline{\theta_F^b(z)}$  over an orthonormal basis of this space, giving

$$|y|^{\frac{5}{4}} f(4z) \overline{\theta_F^b(z)} = \sum_{\phi \in \mathbf{H}_{\frac{3}{2}}^{cusp}(b, \mathbf{n}_f)} \langle f, \theta_F^b \phi \rangle \phi(z) + \sum_{\gamma \in \text{cusps} \cdot \mathbf{R}} \langle f, \theta_F E_\gamma(\bullet, \frac{1}{2} + it) \rangle E_\gamma(z, \frac{1}{2} + it) dt.$$

This spectral expansion converges absolutely, and thus we may insert it into  $I(s)$  and interchange the order of integration and summation, giving

$$\begin{aligned} I_f(s; D) &= \Delta_F (\mathbf{N}D)^{-\frac{1}{2}} \sum_{\phi \in \mathbf{H}_{\frac{3}{2}}^{cusp}(4\mathbf{n}_f)} \frac{\langle f, \theta_F \phi \rangle}{\langle \phi, \phi \rangle} \rho_\phi(4D) \int_{(i\mathbf{R}_{>0})^d} |y|^{s-1} e^{-2\pi \text{tr}(\delta^{-1}4Dy)} \prod_{j=1}^d W_{\frac{3}{4}, it_\phi^{(j)}}(4\pi |\sigma_j(\delta^{-1}4D)| y_j) d^\times |y| \\ &\quad + \text{Eis}. \end{aligned}$$

Strictly speaking, there is a contribution from weight- $\frac{3}{2}$  single-variable theta functions as well, but for  $\theta$  such a function, the inner product  $\langle f, \theta_F^b \theta \rangle$  is a linear combination of nonzero multiples of  $\text{res}_{s=1} L(s, \text{sym}^2 f \otimes \eta)$  for  $\eta$  some finite-order Hecke characters (cf. [Sh2]), and any twist of  $L(s, \text{sym}^2 f)$  is entire since we are assuming that  $f$  is not CM. By the Mellin transform formula

$$\int_0^\infty e^{-2\pi y} W_{\alpha, \beta}(4\pi y) y^s d^\times y = (4\pi)^{-s} \frac{\Gamma(s + \frac{1}{2} - \beta) \Gamma(s + \frac{1}{2} + \beta)}{\Gamma(s + 1 - \alpha)},$$

the  $d$ -fold integral evaluates to

$$(\mathbf{N}D)^{1-s} (4\pi)^{-ds} \Gamma(s - \frac{3}{4})^{-d} \prod_{j=1}^d \Gamma(s + it_\phi^{(j)} - \frac{1}{2}) \Gamma(s - it_\phi^{(j)} - \frac{1}{2}).$$

Comparing these two expansions yields

$$\mathcal{D}_f(s; D) = C \cdot \Delta_F \cdot (\mathbf{N}D)^{\frac{1}{4}-s} \sum_{\phi \in \mathbf{H}_{\frac{3}{2}}^{cusp}(b, \mathbf{n}_f)} \frac{\rho_\phi(4D) \langle f, \theta_F \phi \rangle}{\langle \phi, \phi \rangle} \prod_{j=1}^d \frac{\Gamma(s + it_\phi^{(j)} - \frac{1}{4}) \Gamma(s - it_\phi^{(j)} - \frac{1}{4})}{\Gamma(s - \frac{1}{2}) \Gamma(s + \frac{1}{2})} + \text{Eis}(s).$$

We shall only treat the cuspidal part of this expansion, the Eisenstein terms being a great deal simpler.

By Stirling's formula  $\Gamma(\sigma + it) \asymp |t|^{\sigma - \frac{1}{2}} \exp(-\frac{\pi}{2}|t|)$ , the individual quotients of gamma functions for  $s = \sigma + it$  are bounded away from their poles by

$$(1 + |t|)^A \exp\left(-\frac{\pi}{2}(|t + t_\phi^{(j)}| + |t - t_\phi^{(j)}| - 2|t|)\right).$$

Using the identity  $|a + b| + |a - b| - 2|a| = 2 \max(|b| - |a|, 0)$ , the  $d$ -fold product of gamma functions is bounded by

$$(1 + |t|)^A \exp\left(-\pi \sum_{j=1}^d \max(|t_j| - |t|, 0)\right).$$

The triple product  $\langle f, \theta_F \phi \rangle$  is bounded as  $\phi$  varies, by Cauchy-Schwarz, so using Lemma 2.1 we find

$$\mathcal{D}_f(s; D) \ll (\mathbf{N}D)^{\frac{1}{2} - s - \frac{1}{16}(1-2\theta)} \sum_{\phi \in \mathbf{H}_{\frac{3}{2}}^{\text{cusp}}(b, \mathbf{n}_f)} (1 + |t|)^A \exp\left(\pi \sum_{j=1}^d \left(\frac{|t_j|}{2} - \max(|t_j| - |t|, 0)\right)\right)$$

Only  $\phi$ 's with  $|t_\phi^{(j)}| \leq 2t$ ,  $j = 1..d$  contribute to this sum, and the number of such eigenvalues is bounded polynomially in  $|t|$  and  $\mathbf{N}\mathbf{n}_f$  by a weak form of Weyl's law, so summing their contribution trivially we conclude

$$\mathcal{D}_f(s; D) \ll (\mathbf{N}D)^{\frac{1}{2} - s - \frac{1}{16}(1-2\theta)} (1 + |s|)^A e^{\pi d |s|},$$

away from the poles of the quotients of gamma functions, which occur at the points  $s = \frac{1}{4} \pm it_\phi^{(j)}$ .

By the Shimura correspondence and [BB], the numbers  $t_\phi^{(j)}$  lie in  $\mathbf{R} \cup i[-\frac{\theta}{2}, \frac{\theta}{2}]$ . This concludes the proof of Proposition 3.1.

The exponential factor  $e^{\pi d |s|}$  appearing in the bound of Proposition 3.1 can likely be removed with a little more work. The key to doing this would be to prove a triple product bound

$$\langle f, \theta_F \phi \rangle \ll \prod (1 + |t_\phi^{(j)}|)^A e^{-\frac{\pi}{2}|t_\phi^{(j)}|}.$$

Here is one possible way to show this bound. Let  $\pi$  and  $\sigma$  be unitary automorphic representations of  $\text{PGL}_2/F$ , not both Eisenstein, and let  $\tilde{\sigma}$  be an automorphic representation of  $\widetilde{\text{SL}}_2/F$  which lifts to  $\sigma$  under the Shimura-Shintani-Waldspurger correspondence. Let  $\chi$  be a quadratic idele class character, with associated one-variable theta function  $\theta_\chi$ , and let  $\pi_\chi$  be the automorphic representation of  $\widetilde{\text{SL}}_2$  generated by the adelic lift of  $\theta_\chi$ . Choose factorizable vectors  $\varphi_\pi \in \pi$ ,  $\varphi_{\tilde{\sigma}} \in \tilde{\sigma}$ ,  $\varphi_\chi \in \pi_\chi$ , and let  $S$  denote the set of places where at least one of the three local vectors is ramified. Then we conjecture a formula of the form

$$\frac{|\langle \varphi_{\tilde{\sigma}} \varphi_\chi, \varphi_\pi \rangle|^2}{\langle \varphi_{\tilde{\sigma}}, \varphi_{\tilde{\sigma}} \rangle \langle \varphi_\pi, \varphi_\pi \rangle} = C \frac{L(\frac{1}{2}, \sigma \otimes \chi \otimes \text{sym}^2 \pi)}{L(1, \text{ad}\pi) L(1, \text{ad}\sigma)} \prod_{v \in S} \beta_v(\varphi_{\tilde{\sigma}, v}, \varphi_{\pi, v}, \varphi_{\chi, v}).$$

Here the L-functions appearing on the right are *completed* with their archimedean gamma factors, and the  $\beta_v$ 's are local integrals. This is a simultaneous generalization of two important formulas of Shimura: when  $\pi$  arises from an Eisenstein series, the period integral on the left was key in

Shimura's original lifting construction from half-integral weight forms to integral weight forms; and when  $\sigma$  arises from an Eisenstein series, this is the integral representation for the symmetric square L-function given in [Sh2]. For certain very special pairs  $\pi, \sigma$  this conjecture is in fact a theorem of Ichino [Ich], and it seems quite reasonable to adapt his technique for a general proof. Anyway, assuming this formula, the convexity bound for L-functions combined with the exponential decay of the archimedean gamma factors yields an immediate proof of the purported triple product bound.

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