Degenerating vector bundles in p-adic Hodge theory

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July 3, 2017

Abstract

We compute the closure relations among the individual Harder-Narasimhan strata in the moduli stack of rank n vector bundles on the Fargues-Fontaine curve. The proof combines a dynamical argument on Banach-Colmez spaces with a precise existence theorem (proved in [BFH$^+$17]) for certain parabolic reductions of vector bundles.

Contents

1 Introduction

1.1 Background and main results

The fields of p-adic geometry and p-adic Hodge theory have undergone tremendous development in recent years, largely on account of two parallel developments: Fargues and Fontaine's discovery of the "fundamental curve of p-adic Hodge theory" (also known as the Fargues-Fontaine curve), and Scholze's discovery of the theory of perfectoid spaces. One of the most fascinating outcomes of this development is Fargues's conjectural "geometrization" of the local Langlands correspondence for a connected reductive group G over a non-archimedean local field E, in terms of ℓ-adic sheaves on the stack Bun_G of G-bundles on the Fargues-Fontaine curve [Far16]. Even more recently, Scholze

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has announced a natural construction associating a semisimple L-parameter φ_π with any smooth irreducible representation π of $G(E)$, which relies crucially on the geometry and étale sheaf theory of Bun_G.

In this article, we study some basic geometry of this stack in the case where $G = \mathrm{GL}_n$. To explain our main result, fix an algebraic closure $\overline{\mathbf{F}_p}$, and let $\mathrm{Perf}_{\overline{\mathbf{F}_p}}$ denote the site of perfectoid spaces over $\overline{\mathbf{F}_p}$ with its v-topology. For any characteristic p perfectoid space S, let $\mathcal{X}_S$ denote the relative Fargues-Fontaine curve over S. Let $\mathrm{Bun}_n \to \mathrm{Perf}_{\overline{\mathbf{F}_p}}$ denote the fibered category whose fiber over $S \in \mathrm{Perf}_{\overline{\mathbf{F}_p}}$ is the groupoid of rank n vector bundles on $\mathcal{X}_S$. This stack is a basic example of a *small v-stack* in the sense of [Sch17, Def. 12.4]. In particular, Bun_n has enough geometric structure that it comes with a naturally associated topological space $|\mathrm{Bun}_n|$. According to a fundamental theorem of Fargues and Fontaine, the underlying point set of $|\mathrm{Bun}_n|$ is canonically identified with the set $\mathcal{P}_n$ of Harder-Narasimhan polygons of width n.

For any $P \in \mathcal{P}_n$, let $\mathrm{Bun}_n^{\geq P}$ (resp. $\mathrm{Bun}_n^{\leq P}$) denote the substack parametrizing bundles $\mathcal{E}/\mathcal{X}_S$ such that for every geometric point $x \to S$, the Harder-Narasimhan polygon of $\mathcal{E}_x$ lies above or on (resp. below or on) P with the same endpoints as P. By results of Kedlaya-Liu, $\mathrm{Bun}_n^{\geq P}$ and $\mathrm{Bun}_n^{\leq P}$ are closed and open substacks of Bun_n, respectively, and so the individual Harder-Narasimhan strata $\mathrm{Bun}_n^P = \mathrm{Bun}_n^{\geq P} \cap \mathrm{Bun}_n^{\leq P}$ are locally closed substacks. (We will see that small v-stacks admit reasonable notions of open and (locally) closed substacks, cf. Definition 2.2.) Each individual stratum Bun_n^P is a gerbe, and the associated topological spaces $|\mathrm{Bun}_n^P| \subset |\mathrm{Bun}_n|$ are singletons.

Our main result computes the closure of Bun_n^P inside Bun_n. The precise statement is as follows.

Theorem 1.1. *For any $n \geq 2$ and any $P \in \mathcal{P}_n$, we have $\overline{\mathrm{Bun}_n^P} = \mathrm{Bun}_n^{\geq P}$ as substacks of Bun_n. More precisely, $\mathrm{Bun}_n^{\geq P}$ is the minimal closed substack of Bun_n containing Bun_n^P, and $\overline{|\mathrm{Bun}_n^P|} = |\mathrm{Bun}_n^{\geq P}|$ as subsets of $|\mathrm{Bun}_n|$.*

We note that in the classical setting of vector bundles on a connected smooth projective curve over an algebraically closed field, the analogue of Theorem 1.1 holds for curves of genus zero and one, but fails in higher genus [FM02, Sch15]. Theorem 1.1 is thus related to the heuristic idea that the Fargues-Fontaine curve has genus between zero and one.

Let us sketch the proof of Theorem 1.1 in some detail. As we've already mentioned, the inclusion $\overline{\mathrm{Bun}_n^P} \subseteq \mathrm{Bun}_n^{\geq P}$ is known, so it suffices to demonstrate the opposite inclusion. This is not formal, and roughly amounts to constructing well-behaved families of vector bundles whose HN polygons degenerate from a given polygon P to any specified $Q \geq P$.

To produce the necessary families, we introduce certain auxiliary moduli spaces $\mathcal{S}_Q/\mathrm{Spd}\,\overline{\mathbf{F}_p}$ parametrized by $Q \in \mathcal{P}_n$. Precisely, for any given Q, let $\lambda_1 < \cdots < \lambda_k$ denote the slopes of Q, and let $m_i \in \mathbf{N}_{>0}$ $(1 \leq i \leq k)$ be the multiplicities such that $Q = \mathrm{HN}(\oplus_{1 \leq i \leq k} \mathcal{O}(\lambda_i)^{m_i})$. We then define $\mathcal{S}_Q \to \mathrm{Perf}_{\overline{\mathbf{F}_p}}$ as the category fibered in groupoids whose fiber category over $T \in \mathrm{Perf}_{\overline{\mathbf{F}_p}}$ has objects given by tuples

$$\left(\mathcal{E}, F_\bullet\mathcal{E} = \{0 = F_0\mathcal{E} \subset F_1\mathcal{E} \subset \cdots \subset F_k\mathcal{E} = \mathcal{E}\}, r_\bullet = \{r_i : \mathcal{O}(\lambda_i)^{m_i} \xrightarrow{\sim} F_i\mathcal{E}/F_{i-1}\mathcal{E}\}_{1 \leq i \leq k}\right)$$

where $\mathcal{E}/\mathcal{X}_T$ is a rank n vector bundle and the remaining data has the evident meaning ("a filtration together with a rigidification of its graded pieces"), and whose morphisms $(\mathcal{E}, ...) \to (\mathcal{E}', ...)$ are given by isomorphisms $f : \mathcal{E} \xrightarrow{\sim} \mathcal{E}'$ which are compatible with the filtrations and such that $\mathrm{gr}^i f \circ r_i = r_i'$. One easily checks that this fibered category is a v-stack, and that objects of $\mathcal{S}_Q$ have no automorphisms, i.e. that $\mathcal{S}_Q \to \mathrm{Perf}_{\overline{\mathbf{F}_p}}$ is a category fibered in *setoids* in the terminology of [Sta17]. There is thus no harm in replacing $\mathcal{S}_Q$ with its associated sheaf of sets. Having done this, it turns out that $\mathcal{S}_Q$ is a small v-sheaf, and the map $\mathcal{S}_Q \to \mathrm{Spd}\,\overline{\mathbf{F}_p}$ is representable in locally spatial diamonds and moreover is partially proper, cf. Proposition 3.1. Furthermore, the natural map

$$\pi_Q : \mathcal{S}_Q \to \mathrm{Bun}_n$$

given by forgetting the filtration and rigidification induces a continuous map $|\mathcal{S}_Q| \to |\mathrm{Bun}_n|$; by general properties of slope filtrations, this map factors through the inclusion $\mathrm{Bun}_n^{\leq Q} \subset \mathrm{Bun}_n$. We now appeal to the following crucial theorem, which is more or less the main result of [BFH+17].

Theorem 1.2 ([BFH+17, Theorem 1.1.4])**.** *For any $Q \in \mathcal{P}_n$, the map $\pi_Q : \mathcal{S}_Q \to \mathrm{Bun}_n^{\leq Q}$ induces a* surjective *map $|\mathcal{S}_Q| \to |\mathrm{Bun}_n^{\leq Q}|$.*

In particular, pulling back the HN stratification of Bun_n along π_Q induces a stratification $\mathcal{S}_Q = \cup_{P \leq Q} \mathcal{S}_Q^P$ by locally closed sub-v-sheaves such that every stratum is nonempty. We observe that $|\mathcal{S}_Q^Q|$ consists of a single point $s_Q \in |\mathcal{S}_Q|$, and in fact $\mathcal{S}_Q^Q \simeq \mathrm{Spd}\,\overline{\mathbf{F}}_p$, since the Q-filtration splits and rigidifies the HN-filtration on this stratum. The key observation is that s_Q is contained in the closure of any stratum:

Theorem 1.3. *Any open neighborhood of s_Q in $|\mathcal{S}_Q|$ meets every stratum $|\mathcal{S}_Q^P|$, $P \leq Q$. Equivalently, the closure of $|\mathcal{S}_Q^P|$ in $|\mathcal{S}_Q|$ contains s_Q for every $P \leq Q$.*

From here, the proof of Theorem 1.1 is immediate: if P and $Q \geq P$ are fixed, then either $\mathrm{Bun}_n^Q \subset \overline{\mathrm{Bun}_n^P}$ or[1] $\mathrm{Bun}_n^Q \cap \overline{\mathrm{Bun}_n^P} = \emptyset$; but if the latter holds, we can find some open subset $U \subset |\mathrm{Bun}_n|$ containing $|\mathrm{Bun}_n^Q|$ and disjoint from $|\mathrm{Bun}_n^P|$, in which case $|\pi_Q|^{-1}(U) \subset |\mathcal{S}_Q|$ is a nonempty open neighborhood of s_Q disjoint from $|\mathcal{S}_Q^P|$, contradicting Theorem 1.3.

Let us sketch the argument for Theorem 1.3. Consider the locally profinite group

$$J_Q \overset{\mathrm{def}}{=} \prod_{1 \leq i \leq k} \mathrm{GL}_{m_i}(D_{\lambda_i}).$$

Any element $j = (j_i)_{1 \leq i \leq j} \in J_Q$ defines an automorphism of $\mathcal{S}_Q$ by sending an object $(\mathcal{E}, F_\bullet\mathcal{E}, r_\bullet)$ as before to the altered object $(\mathcal{E}, F_\bullet\mathcal{E}, r_\bullet \cdot j)$ where we abbreviate

$$r_\bullet \cdot j = \{r_i \circ j_i : \mathcal{O}(\lambda_i)^{m_i} \xrightarrow{\sim} F_i\mathcal{E}/F_{i-1}\mathcal{E}\}_{1 \leq i \leq k}.$$

This formula defines a right J_Q-action on $\mathcal{S}_Q$; note that the strata $\mathcal{S}_Q^P$ are stable under this action. The intuitive idea now is that $\mathcal{S}_Q$ is something like an iterated tower of H^1's, and the action of J_Q should move a point of $\mathcal{S}_Q$ "all around" inside these $\mathbf{Q}_p$-vector spaces. In particular, since s_Q is roughly the point corresponding to the product of the zero classes in these H^1's, one might hope that s_Q lies in the closure of the J_Q-orbit of any $x \in |\mathcal{S}_Q|$, which is a strictly stronger statement than Theorem 1.3. For example, take $n = 2$ and $Q = \mathrm{HN}(\mathcal{O} \oplus \mathcal{O}(1))$; then $\mathcal{S}_Q$ is just the sheafification of the presheaf sending $S \in \mathrm{Perf}_{\overline{\mathbf{F}_p}}$ to the $\mathbf{Q}_p$-vector space $H^1(\mathcal{X}_S, \mathcal{O}(-1))$, and an element (a, b) of $J_Q = \mathbf{Q}_p^\times \times \mathbf{Q}_p^\times$ acts by sending $f \in H^1(\mathcal{X}_S, \mathcal{O}(-1))$ to $b^{-1}a \cdot f$.

This intuition turns out to be correct in general:

Proposition 1.4. *For any point $x \in |\mathcal{S}_Q|$, the closure of the orbit $xJ_Q \subset |\mathcal{S}_Q|$ contains s_Q.*

[1] This dichotomy follows easily from the definition of stack-theoretic closure in our setting, together with the fact that each stratum Bun_n^Q is a gerbe.

Note that this is equivalent to the statement that the only J_Q-stable open neighborhood of s_Q is the entirety of $|\mathcal{S}_Q|$, cf. Lemma 3.2.

The proof of Proposition 1.4 runs by an induction on the number of slopes of Q. Note that when Q has a single slope, $\mathcal{S}_Q \cong \operatorname{Spd} \overline{\mathbf{F}_p}$ is a single point, and Proposition 1.4 is trivial. To explain the induction step, fix a general $Q = \mathrm{HN}(\oplus_{1 \le i \le k} \mathcal{O}(\lambda_i)^{m_i})$ as above, and let $Q' = \mathrm{HN}(\oplus_{1 \le i \le k-1} \mathcal{O}(\lambda_i)^{m_i})$ be the truncated polygon obtained by removing the side of largest slope from Q. There is a natural map

$$q : \mathcal{S}_Q \to \mathcal{S}_{Q'}$$

defined by sending an object $(\mathcal{E}, F_\bullet \mathcal{E}, r_\bullet)$ as before to $F_{k-1}\mathcal{E}$ equipped with the obvious truncated filtration and rigidification. We will see (in the proof of Proposition 3.1) that q is representable in locally spatial diamonds and is partially proper. Moreover, this map admits a canonical section

$$\sigma : \mathcal{S}_{Q'} \to \mathcal{S}_Q$$

sending $(\mathcal{E}', F_\bullet \mathcal{E}', r_\bullet)$ to $\mathcal{E}' \oplus \mathcal{O}(\lambda_k)^{m_k}$ equipped with the obvious k-step filtration and rigidification. Writing

$$J_Q \cong J_{Q'} \times \mathrm{GL}_{m_k}(D_{\lambda_k}),$$

the map q is then J_Q-equivariant for the evident actions on its source and target; in particular, the fibers of q are stable under the $\mathrm{GL}_{m_k}(D_{\lambda_k})$-action.

We now argue as follows. By induction, we may assume that Proposition 1.4 is known for $\mathcal{S}_{Q'}$; this implies that the J_Q-orbit closure of any point lying in the subset $|\mathcal{S}_{Q'}| \subset |\mathcal{S}_Q|$ has the desired property. The key observation is that it now suffices to check that for any point $x \in |\mathcal{S}_Q|$, *the orbit closure* $\overline{x\mathrm{GL}_{m_k}(D_{\lambda_k})}$ *meets* $|\mathcal{S}_{Q'}|$. This is a much more tractable problem, since the orbits in question lie in individual fibers of the map q, and the fibers of this map are closely related to Banach-Colmez spaces. Intuitively, the fibration structure of $\mathcal{S}_Q \to \mathcal{S}_{Q'}$ together with the product structure of the group J_Q allow us to prove the desired property of J_Q-orbit closures by a two-step procedure: first we use the $\mathrm{GL}_{m_k}(D_{\lambda_k})$-action to show that orbit closures meet $|\mathcal{S}_{Q'}|$, and then we use the induction hypothesis to show that the closure of the $J_{Q'}$-orbit of any point in $|\mathcal{S}_{Q'}|$ contains s_Q. We note that it seems hard to directly prove Theorem 1.3 by an inductive argument like this, since the fibration $\mathcal{S}_Q \to \mathcal{S}_{Q'}$ interacts quite poorly with the Harder-Narasimhan stratifications of its source and target.

1.2 Acknowledgments

Many of the ideas in this paper were also known to Laurent Fargues and Peter Scholze, and it's a pleasure to thank them both for some very interesting conversations over the years. I would also like to sincerely thank my coauthors on [BFH+17] for their enthusiastic participation in that project. Finally, I'm grateful to Johan de Jong for some helpful conversations, and Michael Thaddeus for several interesting conversations and for drawing my attention to the paper [FM02].

2 Preliminaries

2.1 Small v-stacks

Let Perf denote the site of characteristic p perfectoid spaces with its v-topology. In [Sch17, §12], Scholze defines the extremely general notion of a *small v-stack* on Perf. By definition, a small v-stack $\mathcal{X}$ is a stack in groupoids on Perf admitting some surjective map $U \to \mathcal{X}$ from a small v-sheaf such that $R = U \times_{\mathcal{X}} U$ is also a small v-sheaf. Equivalently, a small v-stack is a v-stack on Perf which can be presented as the quotient stack $[U/R]$ associated with some groupoid in small v-sheaves (U, R, s, t, c). Small v-stacks are presumably the most general class of v-stacks on Perf with some reasonable geometric meaning.

If $\mathcal{X}$ is a small v-stack, a *point* of $\mathcal{X}$ is an equivalence class of maps $\mathrm{Spd}(K, K^+) \to \mathcal{X}$ for some perfectoid field K with an open bounded valuation subring K^+; here two maps $\mathrm{Spd}(K_i, K_i^+) \to \mathcal{X}$ $(i = 1, 2)$ are equivalent if there exist surjective maps $\mathrm{Spd}(K_3, K_3^+) \to \mathrm{Spd}(K_i, K_i^+)$ for $i = 1, 2$ such that the diagram

$$\begin{array}{ccc} \mathrm{Spd}(K_3, K_3^+) & \longrightarrow & \mathrm{Spd}(K_2, K_2^+) \\ \downarrow & & \downarrow \\ \mathrm{Spd}(K_1, K_1^+) & \longrightarrow & \mathcal{X} \end{array}$$

is 2-commutative (as in [Sta17, Tag 04XF], one checks that this defines an equivalence relation). We write $|\mathcal{X}|$ for the set of points of $\mathcal{X}$. The set of points admits a canonical topology:

Proposition 2.1 ([Sch17, Prop. 12.7])**.** *Let $\mathcal{X}$ be a small v-stack with presentation $\mathcal{X} \simeq [U/R]$. Then $|\mathcal{X}| \cong |U|/|R|$, and the quotient topology on $|\mathcal{X}|$ induced by the surjection $|U| \to |\mathcal{X}|$ is independent of the choice of presentation. For any map $\mathcal{X} \to \mathcal{Y}$ of small v-stacks, the associated map $|\mathcal{X}| \to |\mathcal{Y}|$ is continuous.*

In [Sch17, Def. 10.7], Scholze defines open and closed immersions of small v-sheaves. We extend this notion to small v-stacks as follows.

Definition 2.2. Given a small v-stack $\mathcal{X}$, an *open* (resp. *closed*) *substack* of $\mathcal{X}$ is a strictly full subcategory $\mathcal{Z} \subset \mathcal{X}$ such that $\mathcal{Z} \times_{\mathcal{X}} W \to W$ is an open (resp. closed) immersion of small v-sheaves for any small v-sheaf W with a map $W \to \mathcal{X}$.

One easily checks that any open or closed substack of a small v-stack $\mathcal{X}$ is itself a small v-stack. Moreover, there is an equivalence between open substacks of $\mathcal{X}$ and open subsets of $|\mathcal{X}|$, cf. [Sch17, Prop. 12.9]. For closed substacks, a weaker result holds.

Proposition 2.3. *Let $\mathcal{X}$ be a small v-stack, and let $\mathcal{Z} \subset \mathcal{X}$ be a closed substack.*

i. The natural map $|\mathcal{Z}| \to |\mathcal{X}|$ is a closed embedding.

ii. There is a natural identification $\mathcal{Z} = \mathcal{X} \times_{|\mathcal{X}|} |\mathcal{Z}|$, in the sense that an arbitrary map of small v-stacks $\mathcal{Y} \to \mathcal{X}$ factors over the inclusion $\mathcal{Z} \subset \mathcal{X}$ if and only if $|\mathcal{Y}| \to |\mathcal{X}|$ factors through $|\mathcal{Z}| \to |\mathcal{X}|$.

Proof. 1. By the strict fullness of $\mathcal{Z}$ and the definition of points, one easily checks that $|\mathcal{Z}| \to |\mathcal{X}|$ is an injection. Moreover, for any small v-sheaf T with a map $T \to \mathcal{X}$, we have $|\mathcal{Z} \times_{\mathcal{X}} T| \cong |\mathcal{Z}| \times_{|\mathcal{X}|} |T|$ as subsets of $|T|$: one the one hand,

$$|\mathcal{Z} \times_{\mathcal{X}} T| \to |\mathcal{Z}| \times_{|\mathcal{X}|} |T|$$

is surjective by [Sch17, Prop. 12.10], while on the other hand the composite map

$$|\mathcal{Z} \times_{\mathcal{X}} T| \to |\mathcal{Z}| \times_{|\mathcal{X}|} |T| \to |T|$$

is a closed embedding.

Now, let $U \to \mathcal{X}$ be surjective map from a small v-sheaf. Then $|\mathcal{Z} \times_{\mathcal{X}} U| \to |U|$ is a closed embedding, since $\mathcal{Z} \times_{\mathcal{X}} U \to U$ is a closed immersion of small v-sheaves. But the map $|\mathcal{Z} \times_{\mathcal{X}} U| \to |U|$ identifies with the pullback of $|\mathcal{Z}| \to |\mathcal{X}|$ along the quotient map $|U| \to |\mathcal{X}|$, so we get the claim.

2. "Only if" is easy. For "if", consider a perfectoid space S with a map $S \to \mathcal{Y}$, corresponding to some $y \in \mathrm{Ob}(\mathcal{Y}_S)$. We need to check that the induced object of $\mathcal{X}_S$ is an object of the full subcategory $\mathcal{Z}_S$. Choose a surjective map $U \to \mathcal{X}$ from a small v-sheaf as before and set $V = U \times_{\mathcal{X}} \mathcal{Z}$ and $T = S \times_{\mathcal{X}} U$, so we get a commutative diagram

$$\begin{array}{ccccc} T & \longrightarrow & U & \longleftarrow & V \\ \downarrow & & \downarrow & & \downarrow \\ S & \longrightarrow & \mathcal{X} & \longleftarrow & \mathcal{Z} \end{array}$$

of small v-stacks. Now, since $|S| \to |\mathcal{X}|$ factors over $|\mathcal{Z}|$, the induced map $|T| \to |U|$ factors over $|V| \cong |U| \times_{|\mathcal{X}|} |\mathcal{Z}| \subset |U|$, so $T \to U$ factors over a map $\psi : T \to V$. But $T \to S$ is a surjective map of small v-sheaves, so after passing to some v-cover $\{S_i \to S\}$ we can choose sections fitting into a diagram

$$\begin{array}{ccccccc} & & & & \overset{\psi}{\frown} & & \\ & & T & \longrightarrow & U & \longleftarrow & V \\ & \overset{s_i}{\nearrow} & \downarrow & & \downarrow & & \downarrow \\ S_i & \longrightarrow & S & \longrightarrow & \mathcal{X} & \longleftarrow & \mathcal{Z} \end{array}$$

Going around the diagram via s_i and ψ, we see that $y|_{S_i}$ induces an object of $\mathcal{Z}_{S_i}$ for each i. Since $\mathcal{Z}$ is a stack, we conclude that y induces an object of $\mathcal{Z}_S$, as desired. $\square$

For a general small v-stack, not every closed subset of $|\mathcal{X}|$ arises as the topological space of a closed substack. For example, if X is a locally spatial diamond, the subsets of $|X|$ associated with closed sub-diamonds of X are exactly those subsets of $|X|$ which are closed and generalizing. This makes the notion of "stack-theoretic closure" slightly delicate. In particular, the existence of $\overline{\mathcal{Z}}$ in the following definition is not automatic.

Definition 2.4. Let $\mathcal{X}$ be a small v-stack, and let $\mathcal{Z} \subset \mathcal{X}$ be a small sub-v-stack. Suppose there exists a closed sub-v-stack $\overline{\mathcal{Z}} \subset \mathcal{X}$ such that the inclusion $\mathcal{Z} \to \mathcal{X}$ factors via $\mathcal{Z} \to \overline{\mathcal{Z}} \to \mathcal{X}$, such that $\overline{\mathcal{Z}}$ is initial among closed sub-v-stacks with this property. Then $\overline{\mathcal{Z}}$ (which is unique if it exists) is the *closure of* $\mathcal{Z}$ *in* $\mathcal{X}$.

Proposition 2.5. *Let $\mathcal{X}$ be a small v-stack, and let $\mathcal{Z} \subset \mathcal{X}$ be a small sub-v-stack such that $\mathcal{X} \times_{|\mathcal{X}|} \overline{|\mathcal{Z}|}$ is a closed substack of $\mathcal{X}$. Then $\mathcal{X} \times_{|\mathcal{X}|} \overline{|\mathcal{Z}|}$ is the closure of $\mathcal{Z}$ in $\mathcal{X}$.*

Proof. Let $\mathcal{Y} \subset \mathcal{X}$ be any closed substack such that the inclusion $\mathcal{Z} \to \mathcal{X}$ factors over $\mathcal{Y}$. Then $\overline{|\mathcal{Z}|} \to |\mathcal{X}|$ factors over an inclusion $\overline{|\mathcal{Z}|} \to |\mathcal{Y}|$. Since $\left|\mathcal{X} \times_{|\mathcal{X}|} \overline{|\mathcal{Z}|}\right| \cong \overline{|\mathcal{Z}|}$, Proposition 2.3.ii implies that the inclusion $\mathcal{X} \times_{|\mathcal{X}|} \overline{|\mathcal{Z}|} \to |\mathcal{X}|$ factors over a map $\mathcal{X} \times_{|\mathcal{X}|} \overline{|\mathcal{Z}|} \to \mathcal{Y}$. This shows that $\mathcal{X} \times_{|\mathcal{X}|} \overline{|\mathcal{Z}|}$ has the required universal property. $\square$

2.2 Relative Banach-Colmez spaces as diamonds

Given any perfectoid space $S/\mathbf{F}_p$, we have the (adic) relative Fargues-Fontaine curve $\mathcal{X}_S$. In this section we make a detailed study of the cohomology groups $H^i(\mathcal{X}_S, \mathcal{E})$ for $\mathcal{E}$ a vector bundle on S, in the language of diamonds. When $S = \operatorname{Spa} C^\flat$ is a tilted geometric point for some $C/\mathbf{Q}_p$, these are usually known as Banach-Colmez spaces.

A word on terminology: Suppose given S together with a vector bundle $\mathcal{E}/\mathcal{X}_S$ as above. By the *slopes* of $\mathcal{E}$, we mean the set

$$\{\lambda \in \mathbf{Q} \mid \lambda \,\text{is a slope of}\, \mathrm{HN}(\mathcal{E}_x) \,\text{for some}\, x \in S\}.$$

When S is quasicompact, this is a finite set by [KL15, Prop. 7.4.6].

Definition 2.6. Given a perfectoid space $S \in \mathrm{Perf}$ and a vector bundle $\mathcal{E}/\mathcal{X}_S$, we define functors $\mathcal{H}^i(\mathcal{E}) \to S$ for $i = 0, 1$ as the pro-étale sheafifications of the presheaves

$$\begin{aligned} \mathrm{Perf}_{/S} &\to \mathrm{Sets} \\ (T \to S) &\mapsto H^i(\mathcal{X}_T, \mathcal{E}_T), \end{aligned}$$

where $\mathcal{E}_T$ is the pullback of $\mathcal{E}$ along the canonical map $\mathcal{X}_T \to \mathcal{X}_S$.

We will sometimes write $\mathcal{H}^i_S(\mathcal{E})$ if we need to emphasize the base space S. These are sheaves of $\mathbf{Q}_p$-vector spaces over S, so the zero vector corresponds to a section $s : S \to \mathcal{H}^i(\mathcal{E})$ of the structure morphism. Note that the sheafification of $T \mapsto H^i(\mathcal{X}_T, \mathcal{E}_T)$ vanishes for any $i \geq 2$ by [KL15, Theorem 8.7.13]. In particular, applying the $\mathcal{H}^i$'s to a short exact sequence of vector bundles on $\mathcal{X}_S$ induces a six-term long exact sequence of sheaves of $\mathbf{Q}_p$-vector spaces over S in the obvious manner. Note also that $\mathcal{H}^i(\mathcal{E}_1 \oplus \mathcal{E}_2) \cong \mathcal{H}^i(\mathcal{E}_1) \times_S \mathcal{H}^i(\mathcal{E}_2)$.

Proposition 2.7. i. *If $\mathcal{E}$ has only negative slopes, then $\mathcal{H}^0(\mathcal{E}) = S$ via the zero section.*

ii. *If $\mathcal{E}$ has only nonnegative slopes, then $\mathcal{H}^1(\mathcal{E}) = S$ via the zero section.*

Proof. Part i. is immediate from Corollary 7.4.11 and Theorem 8.7.13 in [KL15].

Part ii. is local on S, so we may assume S is affinoid and that $\mathcal{E}$ has constant rank and degree. After passing to a further rational covering of S, if necessary, Lemma 8.8.13 and Corollary 8.8.14 of [KL15] guarantee the existence of a short exact sequence of vector bundles

$$0 \to \mathcal{G} \to \mathcal{F} \to \mathcal{E} \to 0$$

over $\mathcal{X}_S$ such that $\mathcal{G} \simeq \mathcal{O}(-1)^n$ and $\mathcal{F} \simeq \mathcal{O}^m$ after pullback along any geometric point $x \to S$. In particular, $\mathcal{F}$ and $\mathcal{G}(1)$ are pointwise-étale at all points of S. By the sheaf-theoretic surjectivity of $\mathcal{H}^1(\mathcal{F}) \to \mathcal{H}^1(\mathcal{E})$, it suffices to prove that $\mathcal{H}^1(\mathcal{F}) = S$ via the zero section. This can be checked pro-étale-locally on S. After passing to an affinoid pro-étale cover $S' \to S$, we can choose isomorphisms $\mathcal{F}_{S'} \simeq \mathcal{O}^m$ and $\mathcal{G}_{S'} \simeq \mathcal{O}(-1)^n$. By [Sch17, Lemma 7.18], we may assume, after passing to a further affinoid pro-étale cover of S', that any surjective étale map $V \to S'$ admits a section.

By Theorems 8.7.13 and 9.4.5 in [KL15],

$$H^1(\mathcal{X}_{S'}, \mathcal{F}_{S'}) \simeq H^1_{\mathrm{proet}}(S', \underline{\mathbf{Q}_p^m}),$$

so we're reduced to the claim that $H^1_{\mathrm{proet}}(S', \underline{\mathbf{Q}_p}) = 0$ for S' chosen as above. Since

$$H^1_{\mathrm{proet}}(S', \underline{\mathbf{Q}_p}) \cong \left(\varprojlim_n H^1_{\mathrm{proet}}(S', \underline{\mathbf{Z}/p^n\mathbf{Z}})\right)[\tfrac{1}{p}],$$

this reduces further to the vanishing of $H^1_{\mathrm{proet}}(S', \underline{\mathbf{Z}/p^n\mathbf{Z}})$. The sheaf $\underline{\mathbf{Z}/p^n\mathbf{Z}}$ on S'_{proet} is pulled back from S'_{et}, so [Sch17, Prop. 14.8] gives an isomorphism

$$H^1_{\mathrm{proet}}(S', \underline{\mathbf{Z}/p^n\mathbf{Z}}) \simeq H^1_{\mathrm{et}}(S', \underline{\mathbf{Z}/p^n\mathbf{Z}}).$$

But $H^1_{\mathrm{et}}(S', \underline{\mathbf{Z}/p^n\mathbf{Z}}) = 0$, since any étale cover of S' splits, and the result follows. ∎

It turns out that $\mathcal{H}^0(\mathcal{E})$ is well-behaved in all generality.

Proposition 2.8. *The functor $\mathcal{H}^0(\mathcal{E})$ is a locally spatial diamond, and the structure map $\mathcal{H}^0(\mathcal{E}) \to S$ is partially proper.*

Proof. This is local on S, so we can assume S is affinoid. Applying [KL15, Theorem 8.8.15], we can choose (locally on some rational covering of S) an exact sequence

$$0 \to \mathcal{E} \to \mathcal{O}(m_1)^{N_1} \xrightarrow{i} \mathcal{O}(m_2)^{N_2}$$

for some $N_1, N_2 \geq 0$ and $0 \ll m_1 \ll m_2$ (we learned this device from [Far16]). Applying $\mathcal{H}^0$ then presents $\mathcal{H}^0(\mathcal{E})$ as the fiber product

$$\mathcal{H}^0(\mathcal{E}) \cong \mathcal{H}^0\left(\mathcal{O}(m_1)^{N_1}\right) \times_{i,\mathcal{H}^0\left(\mathcal{O}(m_2)^{N_2}\right),s} S,$$

so it suffices to prove the result in the case where $\mathcal{E} = \mathcal{O}(m)^N$. This reduces further to $\mathcal{E} = \mathcal{O}(m)$, which can be proved as in e.g. [BFH$^+$17, Prop. 3.3.2].

For partial properness, the valuative criterion is obvious, so we need to check that the relative diagonal is closed. Writing it as the pullback of the zero section $S \to \mathcal{H}^0(\mathcal{E})$ along

$$\mathcal{H}^0(\mathcal{E}) \times_S \mathcal{H}^0(\mathcal{E}) \overset{(f,g)\mapsto f-g}{\longrightarrow} \mathcal{H}^0(\mathcal{E}),$$

it suffices to check that $S \to \mathcal{H}^0(\mathcal{E})$ is closed. Again, choose an injection $\mathcal{E} \to \mathcal{O}(m)^N$ for some large m and N, so we get an injective map $\mathcal{H}^0(\mathcal{E}) \to \mathcal{H}^0(\mathcal{O}(m)^N)$ compatible with the zero sections of the source and target. This reduces us to the case where $\mathcal{E} = \mathcal{O}(m)^N$, which again reduces to the case $\mathcal{E} = \mathcal{O}(m)$, in which case the result follows from [Far17, Lemme 2.10]. □

Proposition 2.9. *If $\mathcal{E}$ has only negative slopes, the functor $\mathcal{H}^1(\mathcal{E})$ is a diamond, and the structure map $\mathcal{H}^1(\mathcal{E}) \to S$ is partially proper.*

Proof. To check that $\mathcal{H}^1(\mathcal{E})$ is a diamond, we can (by [Sch17, Prop. 11.6]) work pro-étale-locally on S. Arguing as in the proof of Proposition 2.7.ii, we can find a short exact sequence

$$0 \to \mathcal{O}(-1)^n \to \mathcal{O}^m \to \mathcal{E}^\vee \to 0$$

of vector bundles locally on some pro-étale cover of S. Dualizing this sequence, passing to the associated long exact sequence of $\mathcal{H}^i$'s, and applying Proposition 2.7, we get a short exact sequence

$$0 \to \mathcal{H}^0(\mathcal{O}^m) \simeq \underline{\mathbf{Q}_p^m} \to \mathcal{H}^0(\mathcal{O}(1)^n) \to \mathcal{H}^1(\mathcal{E}) \to 0$$

of $\mathbf{Q}_p$-vector diamonds over S. Thus we get an isomorphism

$$\mathcal{H}^1(\mathcal{E}) \simeq \mathcal{H}^0(\mathcal{O}(1)^n)/\underline{\mathbf{Q}_p^m},$$

which presents $\mathcal{H}^1(\mathcal{E})$ as the quotient of a diamond by a quasi-pro-étale equivalence relation. Therefore $\mathcal{H}^1(\mathcal{E})$ is a diamond by [Sch17, Prop. 11.8].

For partial properness, the valuative criterion is obvious, so we again need to check that the diagonal is closed. Writing it as the pullback of the zero section $S \to \mathcal{H}^1(\mathcal{E})$ along

$$\mathcal{H}^1(\mathcal{E}) \times_S \mathcal{H}^1(\mathcal{E}) \overset{(f,g)\mapsto f-g}{\longrightarrow} \mathcal{H}^1(\mathcal{E})$$

as before, it suffices to check that the zero section $s : S \to \mathcal{H}^1(\mathcal{E})$ is closed. For this we first argue on presheaves. More precisely, suppose given a perfectoid space $T \to S$ and an element $c \in H^1(\mathcal{X}_T, \mathcal{E}_T)$, with associated extension bundle $\mathcal{F}/\mathcal{X}_T$. Using [KL15, Corollary 7.4.11], one checks that the presheaf $H^1(\mathcal{E}) : T \mapsto H^1(\mathcal{X}_T, \mathcal{E}_T)$ is separated in the sense of [Sta17, Tag 00WA]. Moreover, if $x = \mathrm{Spa}(K, K^+) \to T$ is any point, then the pullback of c to $H^1(\mathcal{X}_x, \mathcal{E}_x)$ vanishes if and only if the point $(1, 0)$ lies on or below the HN polygon of $\mathcal{F}_x$. By semicontinuity, the locus of such points is closed and generalizing in $|T|$; it therefore corresponds to a closed immersion of diamonds $X \to T$. It's then easy to see (using separatedness) that $X \to T$ satisfies the correct universal property: if $g : T' \to T$ is any map of perfectoid spaces, the pullback of c to $H^1(\mathcal{X}_{T'}, \mathcal{E}_{T'})$ vanishes if and only if g factors through a map $T' \to X$. Therefore $T \times_{c, H^1(\mathcal{E}), s} S$ is representable by a closed subdiamond of T, and in particular is already a sheaf. Thus

$$X = T \times_{c, H^1(\mathcal{E}), s} S \cong T \times_{c, \mathcal{H}^1(\mathcal{E}), s} S \to T$$

is a closed immersion of diamonds. (Here we use that sheafification commutes with finite limits.) In other words, we've shown that zero section $s : S \to \mathcal{H}^1(\mathcal{E})$ pulls back to a closed immersion of diamonds along any map $T \to \mathcal{H}^1(\mathcal{E})$ which factors through the canonical map $H^1(\mathcal{E}) \to \mathcal{H}^1(\mathcal{E})$.

To conclude the general case, choose any perfectoid space T with a map $f : T \to \mathcal{H}^1(\mathcal{E})$. Since $H^1(\mathcal{E})$ is separated, we can find (by [Sta17, Tags 00W9 & 00WB]) a pro-étale cover $\tilde{T} \to T$ such that the composite map $\tilde{f} : \tilde{T} \to \mathcal{H}^1(\mathcal{E})$ factors (uniquely) as

$$\tilde{T} \to H^1(\mathcal{E}) \to \mathcal{H}^1(\mathcal{E}),$$

in which case the base change

$$\left(T \times_{f, \mathcal{H}^1(\mathcal{E}), s} S \to T\right) \times_T \tilde{T} = \left(\tilde{T} \times_{\tilde{f}, \mathcal{H}^1(\mathcal{E}), s} S \to \tilde{T}\right)$$

is a closed immersion by our arguments so far. Since $\tilde{T} \to T$ is surjective as a map of v-sheaves, we deduce that $T \times_{\mathcal{H}^1(\mathcal{E})} S \to T$ is a closed immersion by [Sch17, Prop. 10.11.i], so the result follows. □

Our next goal is the local spatiality of $\mathcal{H}^1(\mathcal{E})$ for any $\mathcal{E}$ with only negative slopes. Although one can likely deduce this directly from the presentation of $\mathcal{H}^1(\mathcal{E})$ used in the proof of Proposition 2.9, it seems a little bit tricky to write this deduction out transparently. Our strategy instead is to show that $\mathcal{H}^1(\mathcal{E})$ admits a separated quasi-pro-étale map to a diamond whose local spatiality can be checked by hand, which implies the desired result by [Sch17, Corollary 11.28]. This argument relies on some auxiliary results which turn out to be very useful in their own right.

Theorem 2.10. *Suppose that $\mathcal{E}$ has only nonnegative slopes, and let $X \subset \mathcal{H}^0(\mathcal{E})$ be any open subdiamond such that $|X| \to |S|$ is surjective. Then $X \to S$ is surjective as a map of pro-étale sheaves.*

This result is extremely useful in practice: the surjectivity of $|X| \to |S|$ more or less amounts to the non-emptiness of $X \times_S \overline{s}$ for all geometric points $\overline{s} = \mathrm{Spa}(C, C^+) \to S$, and many natural X's of interest are much more accessible when the base is a geometric point.

Proof. Passing to an open-closed decomposition of S, we may assume $\mathcal{E}$ has constant rank and degree. The claim is clearly pro-étale-local on S, so replacing S by a suitable pro-étale cover and arguing as in the proof of Proposition 2.7.ii, we can then find a short exact sequence

$$0 \to \mathcal{O}(-1)^m \to \mathcal{O}^n \to \mathcal{E} \to 0.$$

Taking the associated long exact sequence of $\mathcal{H}^i$'s and applying Proposition 2.7, we get a short exact sequence

$$0 \to \mathcal{H}^0(\mathcal{O}^n) \simeq \underline{\mathbf{Q}_p^n} \xrightarrow{i} \mathcal{H}^0(\mathcal{E}) \xrightarrow{q} \mathcal{H}^1\left(\mathcal{O}(-1)^m\right) \to 0$$

of $\mathbf{Q}_p$-vector diamonds over S. Clearly

$$\mathcal{H}^0(\mathcal{E}) \times \underline{\mathbf{Q}_p^n} \overset{(f,a)\mapsto(f,f+i(a))}{\longrightarrow} \mathcal{H}^0(\mathcal{E}) \times_{\mathcal{H}^1(\mathcal{O}(-1)^m)} \mathcal{H}^0\left(\mathcal{E}\right)$$

is an isomorphism, so we get a cartesian diagram

$$\begin{array}{ccc} \mathcal{H}^0(\mathcal{E}) \times \underline{\mathbf{Q}_p^n} & \xrightarrow{(f,a)\mapsto f} & \mathcal{H}^0(\mathcal{E}) \\ {\scriptstyle (f,a)\mapsto f+i(a)}\downarrow & & \downarrow{\scriptstyle q} \\ \mathcal{H}^0(\mathcal{E}) & \xrightarrow{q} & \mathcal{H}^1\left(\mathcal{O}(-1)^m\right) \end{array}$$

of diamonds; note that the upper horizontal arrow is clearly a $\underline{\mathbf{Q}_p^n}$-torsor in the sense of [Sch17, Def. 10.12], and the vertical copy of q is surjective as a map of pro-étale sheaves. Thus the map q is a $\underline{\mathbf{Q}_p^n}$-torsor by [Sch17, Lemma 10.13]; in particular, q is universally open and quasi-pro-étale. Let $Y \subset \mathcal{H}^1(\mathcal{O}(-1)^m)$ be the open subdiamond associated with the open subset

$$\mathrm{im}\left(|\mathcal{H}^0(\mathcal{E})| \to |\mathcal{H}^1(\mathcal{O}(-1)^m)|\right)(|X|) \subset |\mathcal{H}^1(\mathcal{O}(-1)^m)|.$$

The map $\mathcal{H}^0(\mathcal{E}) \to \mathcal{H}^1(\mathcal{O}(-1)^m)$ then induces a map $X \to Y$ which is universally open and quasi-pro-étale, and such that $|X| \to |Y|$ is surjective. It's easy to check that any such map of locally spatial diamonds is surjective as a map of pro-étale sheaves.

It thus remains to check that the induced map $Y \to S$ is surjective as a map of pro-étale sheaves. In fact, we claim this map admits sections étale-locally on S. This is local on S, so we may assume that S is affinoid, in which case it admits a map $a : S \to \mathrm{Spd}\,\mathbf{Q}_p^{\mathrm{cyc}}$. The sheafification of the presheaf

$$\begin{array}{rcl} \mathrm{Perf}_{/\mathrm{Spd}\,\mathbf{Q}_p^{\mathrm{cyc}}} & \to & \mathrm{Sets} \\ T & \mapsto & H^1(\mathcal{X}_T, \mathcal{O}(-1)) \end{array}$$

is then representable by the sheaf quotient $\mathbf{A}^{1,\diamondsuit}_{\mathbf{Q}_p^{\mathrm{cyc}}}/\underline{\mathbf{Q}_p}$, cf. the proof of Proposition 2.13 below. Base changing back to S along a gives

$$\mathcal{H}^1_S(\mathcal{O}(-1)) \cong \mathbf{A}^{1,\diamondsuit}_{\mathbf{Q}_p^{\mathrm{cyc}}}/\underline{\mathbf{Q}_p} \times_{\mathrm{Spd}\,\mathbf{Q}_p^{\mathrm{cyc}}} S \cong \mathbf{A}^{1,\diamondsuit}_{S^\sharp}/\underline{\mathbf{Q}_p}$$

where $S^\sharp$ is the untilt of S specified by a. Pulling back the inclusion $Y \subset \mathcal{H}^1_S(\mathcal{O}(-1))$ along $\mathbf{A}^{1,\diamondsuit}_{S^\sharp} \to \mathbf{A}^{1,\diamondsuit}_{S^\sharp}/\underline{\mathbf{Q}_p}$ gives an open subdiamond of $\mathbf{A}^{1,\diamondsuit}_{S^\sharp}$, corresponding to an open adic subspace $W \subset \mathbf{A}^1_{S^\sharp}$ such that $|W| \to |S^\sharp|$ is surjective. We're now reduced to the existence of a section $\tilde{S}^\sharp \to W$ over some étale cover $\tilde{S}^\sharp \to S^\sharp$. This is local on $S^\sharp$, so we can assume that $S^\sharp = \mathrm{Spa}(A, A^+)$ is affinoid perfectoid.

Assume for the moment that $f : \mathbf{A}^1_{S^\sharp} \to S^\sharp$ is open. Writing W as a union of its quasicompact open subspaces W_i, the quasicompact open subsets $f(W_i)$ cover $S^\sharp$, and by quasicompacity of $S^\sharp$ this covering can be refined to a cover by $f(W_i)$ for some finite set of i's. Replacing W with the union of these finitely many W_i's reduces us to the case where W is quasicompact. Rescaling $W \subset \mathbf{A}^1_{S^\sharp}$ by a large power of p, we can assume that W is contained in

$$\mathbf{B}^1_{S^\sharp} \overset{\text{def}}{=} \mathrm{Spa}(A\langle T\rangle, A^+\langle T\rangle) \subset \mathbf{A}^1_{S^\sharp}.$$

As in [KL15, §2.6], we can write A as the completed direct limit of a directed system of strongly Noetherian Tate $\mathbf{Q}_p$-algebras A_i along submetric transition maps. Setting $X_i = \mathrm{Spa}(A_i, A_i^+)$ and $\mathbf{B}^1_{X_i} = \mathrm{Spa}(A_i\langle T\rangle, A_i^+\langle T\rangle)$, we have natural compatible homeomorphisms $|S^\sharp| \cong \lim_{\leftarrow} |X_i|$ and $|\mathbf{B}^1_{S^\sharp}| \cong \lim_{\leftarrow i} |\mathbf{B}^1_{X_i}|$. By spectrality of all spaces and maps, the subset $|W| \subset |\mathbf{B}^1_{S^\sharp}|$ is the preimage of a quasicompact open $|W_i| \subset |\mathbf{B}^1_{X_i}|$ for some large i; by taking i sufficiently large, we can also assume that $|W_i| \to |X_i|$ is surjective. Then $W_i \to X_i$ is a smooth surjective map of locally Noetherian adic spaces, so there exists an étale cover $\tilde{X}_i \to X_i$ and a section $s : \tilde{X}_i \to W_i$. Base changing the diagram

$$\begin{array}{ccc} & & W_i \\ & \overset{s}{\nearrow} & \downarrow \\ \tilde{X}_i & \longrightarrow & X_i \end{array}$$

along the map $S^\sharp \to X_i$ then gives the desired section.

It still remains to check that $\mathbf{A}^1_{S^\sharp} \to S^\sharp$ is open. Since $\mathbf{A}^1_{S^\sharp} = \cup_{n\geq 1}\mathrm{Spa}(A\langle p^nT\rangle, A^+\langle p^nT\rangle)$, it suffices to check that $\mathbf{B}^1_{S^\sharp} \to S^\sharp$ is open. By the limit argument above, this reduces to openness of $\mathbf{B}^1_{X_i} \to X_i$, which is a special case of [Hub96, Prop. 1.7.8]. □

As a first application, we have the following result.

Theorem 2.11. *Suppose S is an affinoid perfectoid space over $\mathbf{F}_p$, and let $\mathcal{E}/\mathcal{X}_S$ be a vector bundle of constant rank r and degree d whose slopes are all positive. Let N be any positive integer such that $\frac{1}{N}$ is less than the smallest slope of $\mathcal{E}$. Then, after pullback along some affinoid pro-étale cover $S' \to S$, we can find a short exact sequence*

$$0 \to \mathcal{O}^m \to \mathcal{O}(\tfrac{1}{N})^d \to \mathcal{E}_{S'} \to 0.$$

Note that necessarily $m = Nd - r$.

Proof. Fix N as in the theorem. Regarding $\mathcal{H}^0\left((\mathcal{O}(\frac{1}{N})^d)^\vee \otimes \mathcal{E}\right)$ as parametrizing bundle maps $\mathcal{O}(\frac{1}{N})^d \to \mathcal{E}$, let

$$\mathcal{S}\mathrm{urj}\left(\mathcal{O}(\tfrac{1}{N})^d, \mathcal{E}\right) \subset \mathcal{H}^0\left((\mathcal{O}(\tfrac{1}{N})^d)^\vee \otimes \mathcal{E}\right)$$

be the subfunctor whose T-points parametrize surjective bundle maps $\mathcal{O}(\frac{1}{N})^d \to \mathcal{E}_T$ on $\mathcal{X}_T$, and let $\mathcal{S}\mathrm{urj}\left(\mathcal{O}(\frac{1}{N})^d, \mathcal{E}\right)^{\mathrm{ss}}$ be the subfunctor of $\mathcal{S}\mathrm{urj}\left(\mathcal{O}(\frac{1}{N})^d, \mathcal{E}\right)$ whose T-points parametrize surjective bundle maps $\mathcal{O}(\frac{1}{N})^d \to \mathcal{E}_T$ whose kernel is pointwise-semistable at all points of T. These are clearly pro-étale sheaves. We claim that it suffices to prove that the structure map

$$\mathcal{S}\mathrm{urj}\left(\mathcal{O}(\tfrac{1}{N})^d, \mathcal{E}\right)^{\mathrm{ss}} \to S$$

is surjective as a map of pro-étale sheaves. Indeed, if this map is surjective, then we can choose a pro-étale cover $S'' \to S$ together with an S''-point of $\mathcal{S}\mathrm{urj}\left(\mathcal{O}(\frac{1}{N})^d, \mathcal{E}\right)^{\mathrm{ss}}$ lying over $S'' \to S$ and corresponding to a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{O}(\tfrac{1}{N})^d \to \mathcal{E}_{S''} \to 0$$

of vector bundles on $\mathcal{X}_{S''}$ such that $\mathcal{F}$ is pointwise-semistable of slope zero and degree m. Passing to a further pro-étale cover $S' \to S''$ such that $\mathcal{F}_{S'} \simeq \mathcal{O}^m$ and $S'' \to S$ is affinoid pro-étale, the theorem follows.

By [BFH⁺17, Prop. 3.3.5], $\mathcal{S}\mathrm{urj}\left(\mathcal{O}(\frac{1}{N})^d, \mathcal{E}\right)$ is an open subfunctor of $\mathcal{H}^0\left((\mathcal{O}(\frac{1}{N})^d)^\vee \otimes \mathcal{E}\right)$ (strictly speaking, this result is only proved in loc. cit. when the base is a geometric point, but the proof works verbatim in our more general setup). By semicontinuity, $\mathcal{S}\mathrm{urj}\left(\mathcal{O}(\frac{1}{N})^d, \mathcal{E}\right)^{\mathrm{ss}}$ is an open subfunctor of $\mathcal{S}\mathrm{urj}\left(\mathcal{O}(\frac{1}{N})^d, \mathcal{E}\right)$. Thus $\mathcal{S}\mathrm{urj}\left(\mathcal{O}(\frac{1}{N})^d, \mathcal{E}\right)^{\mathrm{ss}}$ is an open subdiamond of $\mathcal{H}^0\left((\mathcal{O}(\frac{1}{N})^d)^\vee \otimes \mathcal{E}\right)$. By the previous theorem, it now suffices to check that $\left|\mathcal{S}\mathrm{urj}\left(\mathcal{O}(\frac{1}{N})^d, \mathcal{E}\right)^{\mathrm{ss}}\right| \to |S|$ is surjective.

We first prove this in the special case where $S = \mathrm{Spa}(C, C^+)$ is a geometric point. For this, choose a decomposition $\mathcal{E} \simeq \oplus_i \mathcal{E}_i$ where $\mathcal{E}_i$ is semistable of rank r_i and degree d_i. By [BFH⁺17, Theorem 1.1.2], we can choose short exact sequences

$$0 \to \mathcal{O}^{Nd_i - r_i} \to \mathcal{O}(\tfrac{1}{N})^{d_i} \to \mathcal{E}_i \to 0$$

for each i, and taking the termwise direct sum of these sequences over i gives a short exact sequence as in the theorem. In particular, this shows that $\mathcal{S}\mathrm{urj}\left(\mathcal{O}(\frac{1}{N})^d, \mathcal{E}\right)^{\mathrm{ss}} \to S$ admits a section, so $\left|\mathcal{S}\mathrm{urj}\left(\mathcal{O}(\frac{1}{N})^d, \mathcal{E}\right)^{\mathrm{ss}}\right| \to |S|$ is surjective.

Now let S be any affinoid perfectoid space. For any given point $s \in |S|$, we can find a map from a geometric point $\tilde{x} = \mathrm{Spa}(C, C^+) \to S$ such that the topological image of the closed point in $\tilde{x}$ is s. For any such s and $\tilde{x}$, we get a cartesian diagram

$$\begin{array}{ccc} \mathcal{S}\mathrm{urj}\left(\mathcal{O}(\frac{1}{N})^d, \mathcal{E}\right)^{\mathrm{ss}} \times_S \tilde{x} & \xrightarrow{t} & \tilde{x} \\ \downarrow & & \downarrow \\ \mathcal{S}\mathrm{urj}\left(\mathcal{O}(\frac{1}{N})^d, \mathcal{E}\right)^{\mathrm{ss}} & \longrightarrow & S \end{array}$$

of diamonds. By the argument in the previous paragraph, t admits a section r. Passing to topological spaces and going around the resulting diagram via $|r|$ implies that

$$s \in \mathrm{im}\left(\left|\mathcal{S}\mathrm{urj}\left(\mathcal{O}(\tfrac{1}{N})^d, \mathcal{E}\right)^{\mathrm{ss}}\right| \to |S|\right),$$

and so the claim follows upon varying s and $\tilde{x}$. □

Corollary 2.12. *Let S be any perfectoid space, and let $\mathcal{E}$ be any vector bundle on $\mathcal{X}_S$ with only positive slopes. Then the structure morphism $\mathcal{H}^0(\mathcal{E}) \to S$ is smooth.*

Proof. This is pro-étale-local on S, so we can assume that S is affinoid and that $\mathcal{E}$ has constant rank r and degree d and moreover fits into a short exact sequence

$$0 \to \mathcal{O}^m \to \mathcal{O}(\tfrac{1}{N})^d \to \mathcal{E} \to 0$$

as in the previous theorem. Passing to $\mathcal{H}^i$'s and applying Proposition 2.7, we get a short exact sequence

$$0 \to \mathcal{H}^0(\mathcal{O}^m) \simeq \underline{\mathbf{Q}_p^m} \to \mathcal{H}^0(\mathcal{O}(\tfrac{1}{N})^d) \xrightarrow{q} \mathcal{H}^0(\mathcal{E}) \to 0$$

of $\mathbf{Q}_p$-vector diamonds over S. Arguing as in the proof of Theorem 2.10 gives that q is a pro-étale $\underline{\mathbf{Q}_p^m}$-torsor. In particular, choosing any open compact subgroup $K \subset \mathbf{Q}_p^m$, we get (by [Sch17, Lemma 10.13]) a surjective separated étale map

$$\mathcal{H}^0(\mathcal{O}(\tfrac{1}{N})^d)/\underline{K} \xrightarrow{q'} \mathcal{H}^0(\mathcal{O}(\tfrac{1}{N})^d)/\underline{\mathbf{Q}_p^m} \simeq \mathcal{H}^0(\mathcal{E}).$$

The diamond $\mathcal{H}^0(\mathcal{O}(\frac{1}{N})^d)$ is smooth over S, since it's represented by an open d-variable polydisk over S. Since K is a pro-p group acting freely and continuously on $\mathcal{H}^0(\mathcal{O}(\frac{1}{N})^d)$, we deduce that $\mathcal{H}^0(\mathcal{O}(\frac{1}{N})^d)/\underline{K}$ is smooth over S, and thus the target of q' is smooth as well, since smoothness is étale-local on the source. □

Finally, we need the following "seed" result.

Proposition 2.13. *For any $N \geq 1$ and any S, the functor $\mathcal{H}^1(\mathcal{O}(\frac{-1}{N}))$ is a locally spatial diamond.*

Proof. Arguing locally on S, we can assume that S admits a map $a : S \to \mathrm{Spa}\, \mathbf{Q}_p^{\mathrm{cyc}}$, so it's enough to show that $\mathcal{H}^1_{\mathrm{Spa}\, \mathbf{Q}_p^{\mathrm{cyc}}}(\mathcal{O}(\frac{-1}{N}))$ is a locally spatial diamond. By [Sch17, Prop. 13.4.ii-iv], it suffices to show that $\mathcal{H}^1_{\mathrm{Spa}\, \mathbf{C}_p}(\mathcal{O}(\frac{-1}{N}))$ is a locally spatial diamond.

Next, observe that over $\mathcal{X}_{\mathrm{Spa}\, \mathbf{C}_p}$ we can choose a short exact sequence of coherent sheaves

$$0 \to \mathcal{O}(-\tfrac{1}{N}) \to \mathcal{O}^N \to i_* \mathbf{C}_p \to 0,$$

where $i : \mathrm{Spa} \mathbf{C}_p \to \mathcal{X}_{\mathrm{Spa}\, \mathbf{C}_p}$ is the inclusion of the point at infinity. Note that for a general S with a map $S \to \mathrm{Spd}\, \mathbf{C}_p$, the pullback of $i_* \mathbf{C}_p$ along $\mathcal{X}_S \to \mathcal{X}_{\mathrm{Spa}\, \mathbf{C}_p}$ is $i_* \mathcal{O}_{S^\sharp}$, where $i : S^\sharp \to \mathcal{X}_S$ is the closed immersion of the specified untilt of S into $\mathcal{X}_S$, cf. [Han16, §2.3]. In particular, the functor $\mathcal{H}^0(i_* \mathbf{C}_p)$ on perfectoid spaces over $\mathrm{Spd}\, \mathbf{C}_p$ is represented by the diamond $(\mathbf{A}^1_{\mathbf{C}_p})^\diamond$. Therefore, passing to $\mathcal{H}^i$'s and applying 2.7, we get a short exact sequence

$$0 \to \mathcal{H}^0(\mathcal{O}^N) \simeq \underline{\mathbf{Q}_p^N} \xrightarrow{\alpha} \mathcal{H}^0(i_* \mathbf{C}_p) \simeq (\mathbf{A}^1_{\mathbf{C}_p})^\diamond \xrightarrow{\beta} \mathcal{H}^1(\mathcal{O}(-\tfrac{1}{N})) \to 0.$$

Note that α corresponds to an embedding of $\mathbf{Q}_p^N$ as a closed subgroup of $|\mathbf{A}^1_{\mathbf{C}_p}|$ consisting of classical points; here of course we give $\mathbf{A}^1_{\mathbf{C}_p}$ the usual additive group structure. Now, writing

$$\mathbf{A}^1_{\mathbf{C}_p} = \cup_{n \geq 1} \mathrm{Spa}(\mathbf{C}_p \langle p^n T \rangle, \mathcal{O}_{\mathbf{C}_p} \langle p^n T \rangle) = \cup U_n$$

as a union of quasicompact open subgroups, we get an associated covering of $\mathcal{H}^1_{\mathrm{Spd}\, \mathbf{C}_p}(\mathcal{O}(-\frac{1}{N}))$ by open subdiamonds

$$\mathcal{H}^1_{\mathrm{Spd}\, \mathbf{C}_p}(\mathcal{O}(-\tfrac{1}{N})) \simeq \cup_{n \geq 1} U_n^\diamond / (\alpha(\underline{\mathbf{Q}_p^N}) \cap U_n^\diamond).$$

Since $\alpha(\underline{\mathbf{Q}_p^N}) \cap U_n \simeq \underline{\mathbf{Z}_p^N}$ is profinite and $U_n^\diamond$ is spatial, we're now reduced to the following lemma, whose proof we leave as an exercise for the reader. □

Lemma 2.14. *Let G be a profinite group, and let X be a spatial diamond with a $\underline{G}$-action. Then $X/\underline{G}$ is a spatial diamond.*

Theorem 2.15. *Suppose that $\mathcal{E}$ has only negative slopes. Then $\mathcal{H}^1(\mathcal{E})$ is a locally spatial diamond.*

Proof. This is local on S, so we can assume that S is affinoid and that $\mathcal{E}$ has constant rank r and degree $-d$. We already know that $\mathcal{H}^1(\mathcal{E})$ is a diamond with separated structure map $\mathcal{H}^1(\mathcal{E}) \to S$, so by [Sch17, Prop. 13.4], the local spatiality of $\mathcal{H}^1(\mathcal{E})$ can be checked pro-étale-locally on S. Applying Theorem 2.11, we may choose (after replacing S by some affinoid pro-étale cover) a short exact sequence

$$0 \to \mathcal{O}^m \to \mathcal{O}(\tfrac{1}{N})^d \to \mathcal{E}^\vee \to 0$$

of vector bundles on S for some fixed large N. Dualizing this sequence, passing to the associated long exact sequence of $\mathcal{H}^i$'s, and applying Proposition 2.7, we get a short exact sequence

$$0 \to \mathcal{H}^0(\mathcal{O}^m) \simeq \underline{\mathbf{Q}_p^m} \xrightarrow{i} \mathcal{H}^1(\mathcal{E}) \xrightarrow{q} \mathcal{H}^1\left(\mathcal{O}(\tfrac{-1}{N})^d\right) \to 0$$

of $\mathbf{Q}_p$-vector diamonds over S. Clearly

$$\mathcal{H}^1(\mathcal{E}) \times \underline{\mathbf{Q}_p^m} \overset{(f,a)\mapsto(f,f+i(a))}{\longrightarrow} \mathcal{H}^1(\mathcal{E}) \times_{\mathcal{H}^1\left(\mathcal{O}(\frac{-1}{N})^d\right)} \mathcal{H}^1(\mathcal{E})$$

is an isomorphism, so we get a cartesian diagram

$$\begin{array}{ccc} \mathcal{H}^1(\mathcal{E}) \times \underline{\mathbf{Q}_p^m} & \xrightarrow{(f,a)\mapsto f} & \mathcal{H}^1(\mathcal{E}) \\ {\scriptstyle (f,a)\mapsto f+i(a)} \downarrow & & \downarrow {\scriptstyle q} \\ \mathcal{H}^1(\mathcal{E}) & \xrightarrow{q} & \mathcal{H}^1\left(\mathcal{O}(\frac{-1}{N})^d\right) \end{array}$$

of diamonds; note that the upper horizontal arrow is clearly separated and quasi-pro-étale. In particular, (the lower horizontal copy of) q becomes separated and quasi-pro-étale after base changing along the v-cover given by (the right vertical copy of) q, so q is separated and quasi-pro-étale by [Sch17, Prop. 10.11.v]. Since the target of q is locally spatial, we can now conclude the local spatiality of its source by applying [Sch17, Corollary 11.28]. □

2.3 The stack Bun_n

In this section we check that the stack Bun_n is a small v-stack.

Proposition 2.16. *The fibered category* Bun_n *is a v-stack.*

Proof. This follows from Proposition 20.2.1 and Lemma 20.2.2 of [SW15] □

Proposition 2.17. *The diagonal map* $\Delta : \mathrm{Bun}_n \to \mathrm{Bun}_n \times \mathrm{Bun}_n$ *is representable in locally spatial diamonds.*

Proof. Let S be a perfectoid space with a map $S \to \mathrm{Bun}_n \times \mathrm{Bun}_n$, corresponding to a pair of vector bundles $\mathcal{E}_1$ and $\mathcal{E}_2$ over $\mathcal{X}_S$. We need to check that $\mathrm{Bun}_n \times_{\mathrm{Bun}_n \times \mathrm{Bun}_n} S$ is a locally spatial diamond. By definition, this fiber product is the sheaf

$$\begin{aligned} \mathcal{I}\mathrm{som}_S(\mathcal{E}_1, \mathcal{E}_2) : \mathrm{Perf}_{/S} &\to \mathrm{Sets} \\ T \to S &\mapsto \mathcal{O}_{\mathcal{X}_T}\text{-module isomorphisms } \mathcal{E}_{1,T} \xrightarrow{\sim} \mathcal{E}_{2,T}. \end{aligned}$$

For any two bundles $\mathcal{E}, \mathcal{F}$ on $\mathcal{X}_S$, let $\mathcal{H}\mathrm{om}_S(\mathcal{E}, \mathcal{F}) \cong \mathcal{H}^0_S(\mathcal{E}^\vee \otimes \mathcal{F})$ be the functor on $\mathrm{Perf}_{/S}$ sending $T \to S$ to the set of $\mathcal{O}_{\mathcal{X}_T}$-module maps $\mathcal{E}_T \to \mathcal{F}_T$. By Proposition 2.8, this is a locally spatial diamond. Note that the identity map $\mathcal{E} \to \mathcal{E}$ defines a distinguished section $\mathrm{id} : S \to \mathcal{H}\mathrm{om}_S(\mathcal{E}, \mathcal{E})$ of the structure morphism to S. We then conclude by observing the isomorphism

$$\mathcal{I}\mathrm{som}_S(\mathcal{E}_1, \mathcal{E}_2) \cong (\mathcal{H}\mathrm{om}_S(\mathcal{E}_1, \mathcal{E}_2) \times_S \mathcal{H}\mathrm{om}_S(\mathcal{E}_2, \mathcal{E}_1)) \times_{\gamma, \mathcal{H}\mathrm{om}_S(\mathcal{E}_1,\mathcal{E}_1) \times_S \mathcal{H}\mathrm{om}_S(\mathcal{E}_2,\mathcal{E}_2), \mathrm{id}^2} S,$$

where γ is the map sending $(f, g) \in \mathcal{H}\mathrm{om}_S(\mathcal{E}_1, \mathcal{E}_2) \times_S \mathcal{H}\mathrm{om}_S(\mathcal{E}_2, \mathcal{E}_1)$ to $(g \circ f, f \circ g) \in \mathcal{H}\mathrm{om}_S(\mathcal{E}_1, \mathcal{E}_1) \times_S \mathcal{H}\mathrm{om}_S(\mathcal{E}_2, \mathcal{E}_2)$. □

It remains to construct reasonable charts for Bun_n. To facilitate this, note that Bun_n decomposes as the disjoint union of open and closed substacks $\mathrm{Bun}_n^d \subset \mathrm{Bun}_n$ parametrizing rank n vector bundles of constant degree d. It thus suffices to find small v-sheaves X_d together with surjective maps $X_d \to \mathrm{Bun}_n^d$ for each d. There are several options for how to do this; in particular, one can build suitable X_d's from affine Grassmannians and prove a Beauville-Laszlo type uniformization, or one can build X_d's inspired by the theory of Quot schemes. We take the latter approach, following an idea of Fargues.

For any fixed $m \gg 0$, consider the functor

$$X_{d,m} = \mathcal{S}\mathrm{urj}\left(\mathcal{O}(m)^{mn+n-d}, \mathcal{O}(m+1)^{mn-d}\right)$$

on perfectoid spaces over $\breve{\mathbf{Q}}_p$. By the proof of Proposition 2.11, this is a locally spatial diamond. An easy calculation shows that for any complete algebraically closed field $C/\mathbf{F}_p$ and any surjection $q : \mathcal{O}(m)^{mn+n-d} \twoheadrightarrow \mathcal{O}(m+1)^{mn-d}$ of vector bundles over $\mathcal{X}_{\mathrm{Spa}\, C}$, the bundle $\ker q$ has rank n, degree d, and maximal HN slope at most m. Moreover, *every* vector bundle $\mathcal{E}$ satisfying these three numerical conditions arises as the kernel of such a surjection: after replacing $\mathcal{E}$ by $\mathcal{E}^\vee(m)$, this becomes the statement that any vector bundle of rank n with (positive) degree e and with all HN slopes non-negative can be realized as the cokernel of an injection $\mathcal{O}(-1)^e \to \mathcal{O}^{e+n}$, which again follows from Lemma 8.8.13 and Corollary 8.8.14 of [KL15]. In particular, the natural map

$$\begin{array}{rcl} \pi_m : X_{d,m} & \to & \mathrm{Bun}_n^d \\ (q : \mathcal{O}(m)^{mn+n-d} \twoheadrightarrow \mathcal{O}(m+1)^{mn-d}) & \mapsto & \ker q \end{array}$$

factors through the inclusion of the open substack $\mathrm{Bun}_n^{d,\mathrm{max.slope} \le m}$ parametrizing bundles with maximal slope $\le m$. Let S be a perfectoid space with a map $f : S \to \mathrm{Bun}_n^{d,\mathrm{max.slope} \le m}$, corresponding to a bundle $\mathcal{E}/\mathcal{X}_S$. Replacing $\mathcal{E}$ with $\mathcal{E}^\vee(m)$ and arguing as in the proof of Proposition 2.7.ii, we can find a pro-étale cover $S' \to S$ such that the composite map $S' \to \mathrm{Bun}_n^{d,\mathrm{max.slope} \le m}$ lifts to an S'-point of $X_{d,m}$. In particular, the map

$$\pi_m : X_{d,m} \to \mathrm{Bun}_n^{d,\mathrm{max.slope} \le m}$$

is surjective as a map of v-stacks. Setting $X = \coprod_{m > |d|} X_{d,m}$, the evident map $X \to \mathrm{Bun}_n^d$ is then surjective as a map of v-stacks, and the source is a locally spatial diamond, so we conclude.

3 Dynamics on Banach-Colmez spaces

3.1 The space $\mathcal{S}_Q$

Proposition 3.1. *The map $\mathcal{S}_Q \to \mathrm{Spd}\, \overline{\mathbf{F}_p}$ is representable in locally spatial diamonds and is partially proper.*

Proof. We argue by induction on the number of slopes of Q. When Q has one slope, $\mathcal{S}_Q \cong \mathrm{Spd}\, \overline{\mathbf{F}_p}$, so we may assume Q has two or more slopes. Write $Q = \mathrm{HN}(\oplus_{1 \le i \le k} \mathcal{O}(\lambda_i)^{m_i})$ as in the introduction. Notation as in the introduction, it then suffices to show that

$$q : \mathcal{S}_Q \to \mathcal{S}_{Q'}$$

is representable in locally spatial diamonds and partially proper. Let T be a perfectoid space with a map $f : T \to \mathcal{S}_{Q'}$, corresponding to a bundle $\mathcal{E}'/\mathcal{X}_T$ with filtration and rigidification. One then checks directly from the definitions that the sheaf of sets

$$\mathcal{S}_Q \times_{\mathcal{S}_{Q'}} T$$

on $\mathrm{Perf}_{/T}$ is represented by the functor

$$\mathcal{H}^1\left((\mathcal{O}(\lambda_k)^{m_k})^\vee \otimes \mathcal{E}'\right).$$

By [BFH+17, Corollary 2.2.13], the maximal slope of $\mathcal{E}'_x$ at any point $x \in T$ is at most λ_{k-1}, so $(\mathcal{O}(\lambda_k)^{m_k})^\vee \otimes \mathcal{E}'$ has only negative slopes. We then conclude by Proposition 2.9 and Theorem 2.15. □

3.2 Orbit closures

In this section we fill in the details of the proof of Proposition 1.4. We begin with some easy lemmas.

Lemma 3.2. *Let X be a topological space with an action of a group G, and let $x \in X$ be a G-fixed point. Then $x \in \overline{yG}$ for all $y \in X$ if and only if X is the unique G-stable open neighborhood of x.*

Proof. The existence of a G-stable open neighborhood U of x with $U \subsetneq X$ is clearly equivalent to the existence of a non-empty G-stable closed subset $V \subset X$ with $x \notin V$. But the existence of such a V is clearly equivalent to the existence of a G-orbit yG with $x \notin \overline{yG}$ (one direction is obvious; for the other direction, write $V = \cup_{y\in V}\overline{yG}$). □

Lemma 3.3. *Let X and Y be topological spaces with actions of a group G, and let $f : Y \to X$ be a continuous G-equivariant map. Then for any G-fixed point $y \in Y$ and any $y' \in Y$ such that $y \in \overline{y'G}$, we have $f(y) \in \overline{f(y')G}$.*

Proof. Observe that

$$f(y) \in f(\overline{y'G}) \subseteq \overline{f(y'G)} = \overline{f(y')G},$$

where the middle containment follows from continuity. □

Lemma 3.4. *Let G be a group with a product decomposition $G = H \times K$, and let X be a topological space with a G-action. Let $x \in X$ be a G-fixed point, and let $S \subset X$ be a K-stable subspace containing x. Suppose that every H-orbit closure in X meets S and that every K-orbit closure of a point of S contains x. Then every G-orbit closure in X contains x.*

Proof. Let $x' \in X$ be any point. By assumption, we may choose some $s \in S$ with $s \in \overline{x'H}$. Then $sK \subseteq \overline{x'H}K$, so

$$\overline{sK} \subseteq \overline{\overline{x'H}K} = \overline{x'HK} = \overline{x'G},$$

where the middle equality follows from the general identity

$$\overline{\cup_{i\in I}\overline{V_i}} = \overline{\cup_{i\in I}V_i}$$

for any collection of subsets V_i of any topological space X. Since $x \in \overline{sK}$ by assumption, the result follows. □

We now return to the problem at hand.

Proof of Proposition 1.4. Let $x \subset |\mathcal{S}_Q|$ be any point. We need to prove that $s_Q \in \overline{xJ_Q}$. As in the introduction, we have the fibration $q : \mathcal{S}_Q \to \mathcal{S}_{Q'}$ with its canonical section $\sigma : \mathcal{S}_{Q'} \to \mathcal{S}_Q$, so we can regard $|\mathcal{S}_{Q'}|$ as a closed subspace of $|\mathcal{S}_Q|$ via σ; note also that $\sigma(s_{Q'}) = s_Q$. By induction, we can assume that the the $J_{Q'}$-orbit closure of any point in $|\mathcal{S}_{Q'}| \subset |\mathcal{S}_Q|$ contains s_Q. By Lemma 3.4, it then suffices to check that

$$q(x) \in \overline{x\mathrm{GL}_{m_k}(D_{\lambda_k})}.$$

To verify this, choose a complete algebraically closed extension $C/\mathbf{Q}_p$ and some open bounded valuation subring $C^+ \subset C$ together with a map

$$\mathrm{Spd}(C, C^+) \to \mathcal{S}_{Q'}$$

such that the topological image of the unique closed point of $|\mathrm{Spd}(C, C^+)|$ is $q(x)$. Let $\mathcal{E}'/\mathcal{X}_{\mathrm{Spa}(C^\flat, C^{+\flat})}$ be the vector bundle (with $k-1$-step filtration and rigidication) defined by this map. Set $\mathcal{S} = \mathcal{S}_Q \times_{q, \tilde{\mathcal{S}}_{Q'}} \mathrm{Spd}(C, C^+)$, so

$$\mathcal{S} \cong \mathcal{H}^1(\mathcal{E}' \otimes \mathcal{O}(-\lambda_k)^{m_k})$$

is a locally spatial $\mathbf{Q}_p$-vector diamond over $\mathrm{Spd}(C, C^+)$ by the arguments in the previous sections. Let $0 \in |\mathcal{S}|$ be the topological image of the unique closed point in $\mathrm{Spd}(C, C^+)$ along the zero section, so we get a natural $\mathrm{GL}_{m_k}(D_{\lambda_k})$-equivariant commutative diagram

$$\begin{array}{ccc} |\mathcal{S}| & \xrightarrow{\pi} & |\mathcal{S}_Q| \\ \uparrow & & \uparrow \\ \{0\} & \longrightarrow & \{q(x)\} \end{array}$$

such that the image of π contains x. By Lemma 3.3, it suffices to check that any $\mathrm{GL}_{m_k}(D_{\lambda_k})$-orbit closure in $|\mathcal{S}|$ contains 0.

To proceed, note it suffices to prove that for any given $\mathcal{F}/\mathcal{X}_{\mathrm{Spa}(C^\flat, C^{+\flat})}$ with only negative slopes, the $p^{\mathbf{Z}}$-orbit (for the scaling action of $p^{\mathbf{Z}} \subset \mathbf{Q}_p^\times$) of any point $x \in |\mathcal{H}^1(\mathcal{F})|$ has the point 0 in its closure. Indeed, for the particular $\mathcal{F}$ of interest to us, the scaling action of $a \in \mathbf{Q}_p^\times$ corresponds to the action of the element $\mathrm{diag}(a, a, \ldots, a) \in \mathrm{GL}_{m_k}(D_{\lambda_k})$, and so the $p^{\mathbf{Z}}$-orbit of any $x \in |\mathcal{H}^1(\mathcal{F})|$ is contained in the $\mathrm{GL}_{m_k}(D_{\lambda_k})$-orbit of x.

To check this claim about $p^{\mathbf{Z}}$-orbit closures, we observe that $\mathcal{F}$ can be written as the kernel of a surjection $\mathcal{O}^m \to \mathcal{O}(1)^n$ for some m, n. Taking cohomology of the associated short exact sequence

$$0 \to \mathcal{F} \to \mathcal{O}^m \to \mathcal{O}(1)^n \to 0,$$

we get a surjection of vector diamonds $\mathcal{H}^0(\mathcal{O}(1)^n) \to \mathcal{H}^1(\mathcal{F})$ over $\mathrm{Spd}(C, C^+)$. Applying Lemma 3.3 again reduces us to checking that the closure of any $p^{\mathbf{Z}}$-orbit in $|\mathcal{H}^0(\mathcal{O}(1)^n)|$ contains 0. This statement, finally, can be checked by hand. Indeed, there is a natural identification of $\mathcal{H}^0(\mathcal{O}(1)^n)$ with the n-variable open perfectoid unit disk

$$\tilde{\mathbf{D}}^n = \mathrm{Spa}\left(C^+[[T_1^{1/p^\infty}, \ldots, T_n^{1/p^\infty}]], C^+[[T_1^{1/p^\infty}, \ldots, T_n^{1/p^\infty}]]\right)_\eta$$

over $\mathrm{Spa}(C, C^+)$, matching the scaling action of p with the Frobenius operator $\varphi : T_i \mapsto T_i^p$. Moreover, the point 0 identifies with the (unique) point x_0 lying over the closed point of $\mathrm{Spa}(C, C^+)$ and whose associated valuation sends each T_i to 0.

By Lemma 3.2, we're now reduced to checking that the only φ-stable open neighborhood of x_0 in $|\tilde{\mathbf{D}}^n|$ is the entirety of $|\tilde{\mathbf{D}}^n|$, which is easy. Indeed, it suffices to check that if $U \subset |\tilde{\mathbf{D}}^n|$ is an open neighborhood of x_0, then $\cup_{j\gg 0}\varphi^{-j}(U) = |\tilde{\mathbf{D}}^n|$; but the subsets

$$V_m = \left\{x \in |\tilde{\mathbf{D}}^n| \mid |T_i|_x \leq |p|_x^m \,\forall 1 \leq i \leq n\right\}$$

are cofinal among open neighborhoods of x_0, and clearly $\cup_{j\gg 0}\varphi^{-j}(V_m) = |\tilde{\mathbf{D}}^n|$ for any m. □

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