Artin vanishing in rigid analytic geometry

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Abstract

We prove a rigid analytic analogue of the Artin vanishing theorem. Precisely, we prove (under mild hypotheses) that the geometric étale cohomology of any Zariski-constructible sheaf on any affinoid rigid space X vanishes in all degrees above the dimension of X. Along the way, we show that branched covers of normal rigid spaces can often be extended across closed analytic subsets, in analogy with a classical result for complex analytic spaces. We also prove a general comparison theorem relating the algebraic and analytic étale cohomologies of any affinoid rigid space.

Contents

1 Introduction

Let $X \subset \mathbf{C}^m$ be a smooth affine variety over $\mathbf{C}$, or more generally any complex Stein manifold. According to a classical theorem of Andreotti and Frankel [AF59], X has the homotopy type of a CW complex of real dimension $\leq \dim X$. In particular, the cohomology groups $H^i(X, A)$ vanish for any abelian group A and any $i > \dim X$. This vanishing theorem was significantly generalized by Artin, who proved the following striking result.

Theorem 1.1 (Artin, Corollaire XIV.3.2 in [SGA73])**.** *Let X be an affine variety over a separably closed field k, and let $\mathscr{F}$ be any torsion abelian sheaf on the étale site of X. Then*

$$H^i_{\text{ét}}(X, \mathscr{F}) = 0$$

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for all $i > \dim X$.

We remind the reader that for a general k-variety X, the groups $H^i_{ét}(X, \mathscr{F})$ vanish in degrees $i > 2\dim X$, and this bound is sharp.

It's natural to wonder whether there is a rigid analytic analogue of the Artin vanishing theorem. Again, we have a general sharp vanishing theorem due to Berkovich and Huber (cf. [Ber93, Corollary 4.2.6], [Hub96, Corollary 2.8.3]): for any quasicompact and quasiseparated[1] rigid space X over a complete algebraically closed nonarchimedean field C, and any torsion abelian sheaf $\mathscr{F}$ on $X_{ét}$, the cohomology group $H^i_{ét}(X, \mathscr{F})$ vanishes for all $i > 2\dim X$. Now in rigid geometry the *affinoid spaces* play the role of basic affine objects, and the most naive guess for an analogue of Artin vanishing would be that $H^i_{ét}(X, \mathscr{F})$ vanishes for all affinoids X/C, all torsion abelian sheaves $\mathscr{F}$ on $X_{ét}$ and all $i > \dim X$. Unfortunately, after some experimentation, one discovers that this fails miserably: there are plenty of torsion abelian sheaves on the étale site of any d-dimensional affinoid with nonzero cohomology in all degrees $i \in [0, 2d]$. However, the following conjecture seems to be a reasonable salvage.

Conjecture 1.2. *Let X be an affinoid rigid space over a complete algebraically closed nonarchimedean field C, and let $\mathscr{G}$ be any* Zariski-constructible *sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules on $X_{ét}$ for some n prime to the residue characteristic of C. Then*

$$H^i_{ét}(X, \mathscr{G}) = 0$$

for all $i > \dim X$.

Here for a given rigid space X and Noetherian coefficient ring Λ, we say a sheaf $\mathscr{G}$ of Λ-modules on $X_{ét}$ is Zariski-constructible if X admits a locally finite stratification into subspaces $Z_i \subset X$, each locally closed for the Zariski topology on X, such that $\mathscr{G}|_{Z_{i,ét}}$ is a locally constant sheaf of Λ-modules of finite type for each i. Note that Zariski-constructible sheaves are overconvergent, and so it is immaterial whether one interprets their étale cohomology in the framework of Berkovich spaces or adic spaces (cf. [Hub96, Theorem 8.3.5]).

The main result of this paper confirms Conjecture 1.2 in the case where C has characteristic zero and the pair $(X, \mathscr{G})$ arises via base extension from a discretely valued nonarchimedean field.

Theorem 1.3. *Let X be an affinoid rigid space over a complete discretely valued nonarchimedean field K of characteristic zero, and let $\mathscr{F}$ be any Zariski-constructible sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules on $X_{ét}$ for some n prime to the residue characteristic of K. Then the cohomology groups $H^i_{ét}(X_{\widehat{\overline{K}}}, \mathscr{F})$ are finite for all i, and*

$$H^i_{ét}(X_{\widehat{\overline{K}}}, \mathscr{F}) = 0$$

for all $i > \dim X$.

For a slightly more general result, see Corollary 3.7. As far as we know, this is the first progress on Conjecture 1.2 since Berkovich [Ber96] treated some cases where $\mathscr{F} = \mathbf{Z}/n\mathbf{Z}$ is constant and X is assumed algebraizable in a certain sense. In particular, using a deep algebraization theorem of Elkik [Elk73, Théorème 7], Berkovich proved Conjecture 1.2 when $\mathscr{F}$ is constant and X is *smooth,* which might give one some confidence in the general conjecture.

Our proof of Theorem 1.3 doesn't explicitly use any algebraization techniques. Instead, we reduce to the special case where $\mathscr{F}$ is constant. In this situation, it turns out we can argue directly, with fewer assumptions on K:

[1](More generally, one can allow any quasiseparated rigid space admitting a covering by countably many quasi-compact open subsets.)

Theorem 1.4. *Let $X = \mathrm{Spa}\,A$ be an affinoid rigid space over a complete discretely valued nonarchimedean field K. Then*

$$H^i_{\text{ét}}(X_{\widehat{\overline{K}}}, \mathbf{Z}/n\mathbf{Z}) = 0$$

for all $i > \dim X$ and all n prime to the residue characteristic of K.

The proof of this theorem uses a number of ingredients, including some theorems of Greco and Valabrega on excellent rings, a remarkable formula of Huber for the stalks of the nearby cycle sheaves $R^q\lambda_*(\mathbf{Z}/n\mathbf{Z})$ on $\mathrm{Spec}(A^\circ/\varpi)_{\text{et}}$, a special case of Gabber's delicate "affine Lefschetz theorem" for quasi-excellent schemes, and the classical Artin vanishing theorem.

The reduction step involves an ingredient which seems interesting in its own right. To explain this, we make the following definition.

Definition 1.5. Let X be a normal rigid space. A *cover* of X is a finite surjective map $\pi : Y \to X$ from a normal rigid space Y, such that there exists some closed nowhere-dense analytic subset $Z \subset X$ with $\pi^{-1}(Z)$ nowhere-dense and such that $Y \smallsetminus \pi^{-1}(Z) \to X \smallsetminus Z$ is finite étale.

We then have the following result, which seems to be new.

Theorem 1.6. *Let X be a normal rigid space over a complete nonarchimedean field K, and let $Z \subset X$ be any closed nowhere-dense analytic subset. Then the restriction functor*

$$\begin{array}{ccc} \left\{ \begin{array}{c} \text{covers of } X \\ \text{étale over } X \smallsetminus Z \end{array} \right\} & \to & \left\{ \begin{array}{c} \text{finite étale covers} \\ \text{of } X \smallsetminus Z \end{array} \right\} \\ Y & \mapsto & Y \times_X (X \smallsetminus Z) \end{array}$$

is fully faithful. Moreover, if K has characteristic zero, it is an equivalence of categories; in other words, any finite étale cover of $X \smallsetminus Z$ extends uniquely to a cover of X.

We remind the reader that in the schemes setting, the analogue of the equivalence in Theorem 1.6 is an easy exercise in taking normalizations, and holds essentially whenever the base scheme X is Nagata, while for complex analytic spaces the problem was solved by Stein and Grauert-Remmert in the 50's, cf. [DG94].

Let us say something about the proof. Full faithfulness is an easy consequence of Bartenwerfer's rigid analytic version of Riemann's Hebbarkeitssatz, which says that bounded functions on normal rigid spaces extend uniquely across nowhere-dense closed analytic subsets. Essential surjectivity in characteristic zero is more subtle; indeed, it provably fails in positive characteristic. When X is smooth and Z is a strict normal crossings divisor, however, essential surjectivity was proved by Lütkebohmert in his work [Lüt93] on Riemann's existence problem.[2] We reduce the general case to Lütkebohmert's result using recent work of Temkin on embedded resolution of singularities for quasi-excellent schemes in characteristic zero

As another application of this circle of ideas, we prove the following very general comparison result.

Theorem 1.7. *Let $X = \mathrm{Spa}\,A$ be any affinoid rigid space, and set $\mathcal{X} = \mathrm{Spec}\,A$, so there is a natural morphism of sites $\mu_X : X_{\text{ét}} \to \mathcal{X}_{\text{ét}}$. Then for any torsion abelian sheaf $\mathscr{F}$ on $\mathcal{X}_{\text{ét}}$, the natural comparison map*

$$H^n_{\text{ét}}(\mathcal{X}, \mathscr{F}) \to H^n_{\text{ét}}(X, \mu_X^*\mathscr{F})$$

is an isomorphism for all $n \geq 0$.

[2] Although curiously, Lütkebohmert doesn't explicitly state the result in his paper, nor does he discuss full faithfulness.

When $\mathscr{F}$ is a *constant* sheaf, this is proved in Huber's book; we reduce the general case of Theorem 1.7 to Huber's result, using the schemes version of Theorem 1.6 together with a cohomological descent argument. In particular, we use the fact that any surjective qcqs morphism of rigid analytic spaces is universally of cohomological descent relative to the étale topology, which seems to be a new observation.

We end the introduction with a conjecture, which would give some further justification for the definition of Zariski-constructible sheaves.

Conjecture 1.8. *In the notation of the previous theorem, the pullback functor*

$$\mu_X^* : \mathrm{Sh}(\mathcal{X}_{\text{ét}}, \Lambda) \to \mathrm{Sh}(X_{\text{ét}}, \Lambda)$$

induces an equivalence of categories from constructible étale sheaves on $\mathcal{X}$ to Zariski-constructible étale sheaves on X.

With Theorem 1.6 in hand, one can reduce this conjecture in the characteristic zero case to a comparison of sheaf Exts on $\mathcal{X}_{\text{ét}}$ and $X_{\text{ét}}$. The essential point in this reduction is that while open subsets $\mathcal{U} \subset \mathcal{X}$ are in tautological bijection with Zariski-open subsets $U \subset X$, it's *a priori* difficult to understand the essential image of the analytification functor $\mathcal{U}_{\text{fét}} \mapsto U_{\text{fét}}$ for a given U, since U is essentially never affinoid; however, as an easy consequence of Theorem 1.6, one gets an equivalence of categories $\mathcal{U}_{\text{fét}} \cong U_{\text{fét}}$ (at least when X is normal). Using this, it's easy to deduce that if $\mathscr{F}$ is a Zariski-constructible sheaf on X and $X = \coprod Z_i$ is a suitably nice partition such that $\mathscr{F}|_{Z_i}$ is locally constant, than each $\mathscr{F}|_{Z_i}$ is the pullback of a locally constant sheaf on the corresponding subset $\mathcal{Z}_i \subset \mathcal{X}$.

Finally, let us note that our argument for the reduction of Theorem 1.3 to Theorem 1.4 reduces Conjecture 1.2 to the special case where $\mathscr{F}$ is constant, at least for C of characteristic zero. With Theorem 1.4 in hand, this might put the general characteristic zero case of Conjecture 1.2 within reach of some approximation argument.

Remarks on terminology and conventions.

Our convention is that a "nonarchimedean field" is a topological field whose topology is defined by a *nontrivial* nonarchimedean valuation of rank one. If K is any nonarchimedean field, we regard rigid analytic spaces over K as a full subcategory of the category of adic spaces locally of topologically finite type over $\mathrm{Spa}(K, K^\circ)$. If A is any topological ring, we write A° for the subset of power-bounded elements; if A is a Huber ring, we write $\mathrm{Spa}\, A$ for $\mathrm{Spa}(A, A^\circ)$.

We use the terms "Zariski-closed subset" and "closed analytic subset" interchangeably, and we always regard Zariski-closed subsets of rigid spaces as rigid spaces via the induced reduced structure. Finally, we remind the reader that in rigid geometry the phrases "dense Zariski-open subset" and "Zariski-dense open subset" have very different meanings.

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2 Preliminaries

2.1 Zariski-constructible sheaves on rigid spaces

In this section we discuss some basics on Zariski-constructible étale sheaves on rigid spaces. For simplicity we fix a nonachimedean field K and a Noetherian coefficient ring Λ; until further notice, roman letters $X, Y, \ldots$ denote rigid spaces over K, and "Zariski-constructible" means a Zariski-constructible sheaf of Λ-modules, as defined in the introduction, on the étale site of some rigid space X over K. We'll also say that an object $\mathscr{F} \in D^b(X_{\text{ét}}, \Lambda)$ is Zariski-constructible if it has Zariski-constructible cohomology sheaves.

Proposition 2.1. *Let $\mathscr{F}$ be a Zariski-constructible sheaf on a rigid space X.*
i. *If $f : Y \to X$ is any morphism of rigid spaces, then $f^*\mathscr{F}$ is Zariski-constructible.*
ii. *If $i : X \to W$ is a closed immersion, then $i_*\mathscr{F}$ is Zariski-constructible.*
iii. *If $j : X \to V$ is a Zariski-open immersion and $\mathscr{F}$ is locally constant, then $j_!\mathscr{F}$ is Zariski-constructible.*

Proof. Trivial. □

We will often verify Zariski-constructibility via the following *dévissage*, which is a trivial consequence of the previous proposition.

Proposition 2.2. *Let X be any rigid space, and let $\mathscr{F}$ be a sheaf of Λ-modules on $X_{\text{ét}}$. The following are equivalent:*
i. *$\mathscr{F}$ is Zariski-constructible.*
ii. *There is some dense Zariski-open subset $j : U \to X$ with closed complement $i : Z \to X$ such that $i^*\mathscr{F}$ is Zariski-constructible and $j^*\mathscr{F}$ is locally constant of finite type.*

Note that one probably cannot weaken the hypotheses in ii. here to the condition that $j^*\mathscr{F}$ is Zariski-constructible; this is related to the fact that the Zariski topology in the rigid analytic world is not transitive.

Proposition 2.3. *If $f : X' \to X$ is a finite morphism and $\mathscr{F}$ is a Zariski-constructible sheaf on X', then $f_*\mathscr{F}$ is Zariski-constructible.*

We note in passing that if $f : X \to Y$ is a finite morphism, or more generally any quasi-compact separated morphism with finite fibers, then $f_* : \mathrm{Sh}(X_{\text{ét}}, \Lambda) \to \mathrm{Sh}(Y_{\text{ét}}, \Lambda)$ is an exact functor, cf. Proposition 2.6.4 and Lemma 1.5.2 in [Hub96].

Proof. We treat the case where $X' = \mathrm{Spa}\, A'$ and $X = \mathrm{Spa}\, A$ are affinoid, which is all we'll need later. We can assume they are reduced and that f is surjective. If $i : Z \to X$ is Zariski-closed and nowhere dense, then $\dim Z < \dim X$; setting $Z' = Z \times_X X'$ and writing $f' : Z' \to Z$ and $i' : Z' \to X'$ for the evident morphisms, we can assume that $i^* f_*\mathscr{F} \cong f'_* i'^* \mathscr{F}$ is Zariski-constructible by induction on $\dim X$. By dévissage, it now suffices to find a dense Zariski-open subset $j : U \to X$ such that $j^* f_*\mathscr{F}$ is locally constant. To do this, choose a dense Zariski-open subset $V \subset X'$ such that $\mathscr{F}|_V$ is locally constant. Then $W = X \smallsetminus f(X' \smallsetminus V)$ is a dense Zariski-open subset of X, and $\mathscr{F}$ is locally constant after pullback along the open immersion $W' = W \times_X X' \to X'$. If $\mathrm{char}(K) = 0$, we now conclude by taking U to be any dense Zariski-open subset contained in W such that $U' = U \times_X X' \to U$ is finite étale; if $\mathrm{char}(K) = p$, we instead choose U so that $U' \to U$ factors as the composition of a universal homeomorphism followed by a finite étale map. (For the existence of such a U, look at the map of schemes $\mathrm{Spec}\, A' \to \mathrm{Spec}\, A$; this morphism has the desired structure over all generic points of the target, and these structures then spread out over a dense Zariski-open subset of $\mathrm{Spec}\, A$. One then concludes by analytifying.) □

Next we check the two-out-of-three property:

Proposition 2.4. *Let X is a rigid space and let $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ be a short exact sequence of étale sheaves of Λ-modules on X. If two of the three sheaves $\{\mathscr{F}, \mathscr{G}, \mathscr{H}\}$ are Zariski-constructible, then so is the third.*

Using this result, it's straightforward to check that Zariski-constructible sheaves form an abelian subcategory of $\mathrm{Sh}(X_{\text{ét}}, \Lambda)$; since we never need this result, we leave it as an exercise for the interested reader.

Proof. By induction on $\dim X$ and dévissage, it suffices to find some dense Zariski-open subset $j : U \to X$ such that all three sheaves are locally constant after restriction to U. By assumption, we can choose U such that two of the three sheaves have this property. Looking at the exact sequence $0 \to \mathscr{F}|_U \to \mathscr{G}|_U \to \mathscr{H}|_U \to 0$, [Hub96, Lemma 2.7.3] implies that all three sheaves are constructible (in the sense of [Hub96]). By [Hub96, Lemma 2.7.11], it now suffices to check that all three sheaves are overconvergent. But if $\overline{x} \rightsquigarrow \overline{y}$ is any specialization of geometric points, this follows immediately by applying the snake lemma to the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{F}_{\overline{y}} \arrow[r] \arrow[d] & \mathscr{G}_{\overline{y}} \arrow[r] \arrow[d] & \mathscr{H}_{\overline{y}} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathscr{F}_{\overline{x}} \arrow[r] & \mathscr{G}_{\overline{x}} \arrow[r] & \mathscr{H}_{\overline{x}} \arrow[r] & 0 \end{tikzcd}$$

since by assumption two of the three vertical arrows are isomorphisms. □

It seems interesting to develop some more theory around Zariski-constructible sheaves on rigid spaces. In particular, we expect this notion to have some non-trivial stabilities under certain of the six functors:

Conjecture 2.5. *Let X be a finite-dimensional rigid space over a complete algebraically closed field C. For simplicity, assume that $\Lambda = \mathbf{Z}/n\mathbf{Z}$ for some n prime to the residue characteristic, and suppose that X admits a dualizing complex $\omega_X \in D^b(X_{\text{ét}}, \Lambda)$.*[3] *Let $\mathscr{F} \in D^b(X_{\text{ét}}, \Lambda)$ be any Zariski-constructible object. Then:*

i. *The Verdier dual $\mathbf{D}_X \mathscr{F} \overset{\mathrm{def}}{=} R\mathscr{H}\mathrm{om}(\mathscr{F}, \omega_X)$ is Zariski-constructible.*

ii. *If $f : Y \to X$ is any morphism which is finite-dimensional and compactifiable in the sense of [Hub96, Definition 5.1.1], so $Rf_!$ and $Rf^!$ are defined, then $Rf^! \mathscr{F}$ is Zariski-constructible.*

[3] It seems plausible that ω_X can be defined for essentially any X by constructing it locally on an étale hypercover by disjoint unions of affinoids and then applying some version of the "BBD gluing lemma"; the only nonformal input one should need is that $\mathrm{Ext}^i_{D^b(U_{\text{ét}}, \Lambda)}(\omega_U, \omega_U) = 0$ for any affinoid U and any $i < 0$, which is immediate from [Han17, Proposition B.3.1].

iii. *If* $j : X \to W$ *is a Zariski-open immersion and* $\mathscr{F}$ *is locally constant, then* $Rj_*\mathscr{F}$ *is Zariski-constructible.*

iv. *If* $f : X \to S$ *is any proper morphism of rigid spaces, then* $Rf_*\mathscr{F}$ *is Zariski-constructible.*

It is true, but not obvious, that we have implications iii. $\Leftrightarrow$ i. $\Rightarrow$ ii. among these statements, and that iv. is true when $\dim S = 1$; we omit the proofs, since these results aren't needed in the present paper. We hope to return to this conjecture in a future article.

2.2 Extending covers across closed subsets

In this section we prove a slight strengthening of Theorem 1.6. We'll freely use basic facts about irreducible components of rigid spaces, as developed in [Con99], without any comment. The following result of Bartenwerfer [Bar76, §3] is also crucial for our purposes.

Theorem 2.6 (Bartenwerfer)**.** *Let* X *be a normal rigid space, and let* $Z \subset X$ *be a nowhere-dense closed analytic subset, with* $j : X \smallsetminus Z \to X$ *the inclusion of the open complement. Then* $\mathcal{O}_X^+ \xrightarrow{\sim} j_*\mathcal{O}_{X\smallsetminus Z}^+$ *and* $\mathcal{O}_X \xrightarrow{\sim} \left(j_*\mathcal{O}_{X\smallsetminus Z}^+\right)[\frac{1}{\varpi}]$*. In particular, if* X *is affinoid and* $f \in \mathcal{O}_X(X \smallsetminus Z)$ *is bounded, then* f *extends uniquely to an element of* $\mathcal{O}_X(X)$*, so* $\mathcal{O}_X(X) \cong \mathcal{O}_X^+(X \smallsetminus Z)[\frac{1}{\varpi}]$*.*

Corollary 2.7. *If* X *is a connected normal rigid space and* $Z \subset X$ *is a nowhere-dense closed analytic subset, then* $X \smallsetminus Z$ *is connected.*

Proof. Any idempotent in $\mathcal{O}_X(X \smallsetminus Z)$ is power-bounded, so this is immediate from the previous theorem. □

Proposition 2.8. *Let* X *be a normal rigid space, and let* $\pi : Y \to X$ *be a cover of* X*. Then each irreducible component of* Y *maps surjectively onto some irreducible component of* X*. Moreover, if* $V \subset X$ *is any closed nowhere-dense analytic subset, then* $\pi^{-1}(V)$ *is nowhere-dense.*

Proof. We immediately reduce to the case where X is connected. Let $Z \subset X$ be as in the definition of a cover, and let Y_i be any connected component of Y, so then $Y_i \cap \pi^{-1}(Z)$ is closed and nowhere-dense in Y_i and $Y_i \smallsetminus Y_i \cap \pi^{-1}(Z) \to X \smallsetminus Z$ is finite étale. Then

$$\mathrm{im}\left(Y_i \smallsetminus Y_i \cap \pi^{-1}(Z) \to X \smallsetminus Z\right)$$

is a nonempty open and closed subset of $X \smallsetminus Z$, so it coincides with $X \smallsetminus Z$ by the previous corollary. In particular, $\pi(Y_i)$ contains a dense subset of X. On the other hand, $\pi(Y_i)$ is a closed analytic subset of X since π is finite. Therefore $\pi(Y_i) = X$.

For the second claim, note that if V is a closed analytic subset of a connected normal space X, then $V \subsetneq X$ if and only if V is nowhere-dense if and only if $\dim V < \dim X$. Since

$$\dim \pi^{-1}(V) \cap Y_i = \dim V < \dim X = \dim Y_i$$

for any irreducible component Y_i of Y, this gives the claim. □

Proposition 2.9. *Let* X *be a normal rigid space, and let* $Z \subset X$ *be any closed nowhere-dense analytic subset. Then the restriction functor*

$$\begin{array}{ccc} \left\{ \text{ covers of } X \right\} & \to & \left\{ \text{ covers of } X \smallsetminus Z \right\} \\ Y & \mapsto & Y \times_X (X \smallsetminus Z) \end{array}$$

is fully faithful.

Proof. If $\pi : Y \to X$ is any cover and $U \subset X$ is any open affinoid, then $\pi^{-1}(U)$ is affinoid as well, and $\pi^{-1}(Z \cap U)$ is nowhere-dense in U by the previous proposition. But then $\mathcal{O}_Y(\pi^{-1}(U)) \cong \mathcal{O}_Y^+(\pi^{-1}(U \smallsetminus U \cap Z))[\frac{1}{\varpi}]$ by Theorem 2.6, so $\mathcal{O}_Y(U)$ only depends on $Y \times_X (X \smallsetminus Z)$. This immediately gives the result. $\square$

It remains to prove the following result.

Theorem 2.10. *Let X be a normal rigid space over a characteristic zero complete nonarchimedean field K, and let $Z \subset X$ be any closed nowhere-dense analytic subset. Then the restriction functor*

$$\left\{ \begin{array}{c} \text{covers of } X \\ \text{étale over } X \smallsetminus Z \end{array} \right\} \to \left\{ \begin{array}{c} \text{finite étale covers} \\ \text{of } X \smallsetminus Z \end{array} \right\}$$
$$Y \mapsto Y \times_X (X \smallsetminus Z)$$

is essentially surjective.

In other words, given a (surjective) finite étale cover $\pi : Y \to X \smallsetminus Z$, we need to find a cover $\tilde{\pi} : \tilde{Y} \to X$ and an open immersion $Y \to \tilde{Y}$ such that the diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "\pi"'] & \tilde{Y} \arrow[d, "\tilde{\pi}"] \\ X \smallsetminus Z \arrow[r] & X \end{tikzcd}$$

is cartesian. We refer to this as the problem of *extending* Y to a cover of X. Note that by the full faithfulness proved above, we're always free to work locally on X when extending a given cover of $X \smallsetminus Z$.

Until further notice, fix K of characteristic zero. The key special case is the following result.

Theorem 2.11 (Lütkebohmert)**.** *If X is a smooth rigid space and $D \subset X$ is a strict normal crossings divisor, then any finite étale cover of $X \smallsetminus D$ extends to a cover of X.*

This is more or less an immediate consequence of the arguments in §3 in Lütkebohmert's paper [Lüt93] (and is implicit in the proof of Theorem 3.1 of loc. cit.). For the convenience of the reader, we explain the deduction in detail. Let $\mathbf{B}^r = \operatorname{Spa} K\langle X_1, \dots, X_r \rangle$ denotes the r-dimensional closed affinoid ball.

Lemma 2.12 (Lemma 3.3 in [Lüt93])**.** *Let S be a smooth K-affinoid space, and let $r \geq 1$ be any integer. If Y_0 is a cover of $S \times (\mathbf{B}^r \smallsetminus V(X_1, \dots, X_r))$ which is étale over $S \times (\mathbf{B}^r \smallsetminus V(X_1 \cdots X_r))$, then Y_0 extends to a cover $\tilde{Y}$ of $S \times \mathbf{B}^r$.*

We also need a result of Kiehl on the existence of "tubular neighborhoods" of strict normal crossings divisors in smooth rigid spaces.

Lemma 2.13. *If $D \subset X$ is a strict normal crossings divisor in a smooth rigid space, then for any (adic) point x in X contained in exactly r irreducible components $D_1, \dots, D_r$ of D, we can find some small open affinoid $U \subset X$ containing x together with a smooth affinoid S and an isomorphism $U \simeq S \times \mathbf{B}^r$, under which the individual components $D_i \cap U$ containing x identify with the zero loci of the coordinate functions $X_i \in \mathcal{O}(\mathbf{B}^r)$.*

Proof. This follows from a careful reading of Theorem 1.18 in [Kie67b] (cf. also [Mit, Theorem 2.11]). $\square$

Granted these results, we deduce Theorem 2.11 as follows. By full faithfulness we can assume that X is quasicompact, or even affinoid. We now argue by induction on the maximal number $\iota(D)$ of irreducible components of D passing through any individual point of X. If $\iota(D) = 1$, then D is smooth, so arguing locally around any point in D, Lemma 2.13 puts us exactly in the situation covered by the case $r = 1$ of Lemma 2.12. If $\iota(D) = n$, then locally on X we can assume that D has (at most) n smooth components $D_1, D_2, \ldots, D_n$. By the induction hypothesis, any finite étale cover Y of $X \smallsetminus D$ extends to a cover Y_i of $X \smallsetminus D_i$ for each $1 \leq i \leq n$, since $\iota(D \smallsetminus D_i) \leq n-1$ for $D \smallsetminus D_i$ viewed as a strict normal crossings divisor in $X \smallsetminus D_i$. By full faithfulness the Y_i's glue to a cover Y_0 of $X \smallsetminus \cap_{1 \leq i \leq n} D_i$, and locally around any point in $\cap_{1 \leq i \leq n} D_i$ Lemma 2.13 again puts us in the situation handled by Lemma 2.12, so Y_0 extends to a cover $\tilde{Y}$ of X, as desired.

Proof of Theorem 2.10. We can assume that $X = \operatorname{Spa} A$ is an affinoid rigid space, so $Z = \operatorname{Spa} B$ is also affinoid, and we get a corresponding closed immersion of schemes $\mathcal{Z} = \operatorname{Spec} B \to \mathcal{X} = \operatorname{Spec} A$. These are quasi-excellent schemes over $\mathbf{Q}$, so according to Theorem 1.11 in [Tem17], we can find a projective birational morphism $f : \mathcal{X}' \to \mathcal{X}$ such that $\mathcal{X}'$ is regular and $f^{-1}(\mathcal{Z})^{\mathrm{red}}$ is a strict normal crossings divisor, and such that f is an isomorphism away from $\mathcal{Z} \cup \mathcal{X}^{\mathrm{sing}}$. Analytifying, we get a proper morphism of rigid spaces $g : X' \to X$ with X' smooth such that $g^{-1}(Z)^{\mathrm{red}}$ is a strict normal crossings divisor.

Suppose now that we're given a finite étale cover $Y \to X \smallsetminus Z$. Base changing along g, we get a finite étale cover of $X' \smallsetminus g^{-1}(Z)$, which then extends to a cover $h : Y' \to X'$ by Theorem 2.11. Now, since $g \circ h$ is proper, the sheaf $(g \circ h)_* \mathcal{O}_{Y'}$ defines a sheaf of coherent $\mathcal{O}_X$-algebras by [Kie67a]. Taking the normalization of the affinoid space associated with the global sections of this sheaf, we get a normal affinoid Y'' together with a finite map $Y'' \to X$ and a canonical isomorphism $Y''|_{(X \smallsetminus Z)^{\mathrm{sm}}} \cong Y|_{(X \smallsetminus Z)^{\mathrm{sm}}}$. The cover $\tilde{Y} \to X$ we seek can then be defined as the Zariski closure of $Y''|_{(X \smallsetminus Z)^{\mathrm{sm}}}$ in Y''; note that this is just a union of irreducible components of Y'', so it's still normal, and it's easy to check that $\tilde{Y}$ is a cover of X. Finally, since $\tilde{Y}$ and Y are canonically isomorphic after restriction to $(X \smallsetminus Z)^{\mathrm{sm}}$, the full faithfulness argument shows that this isomorphism extends to an isomorphism $\tilde{Y}|_{X \smallsetminus Z} \cong Y$, since $(X \smallsetminus Z)^{\mathrm{sm}}$ is a dense Zariski-open subset of $X \smallsetminus Z$. This concludes the proof. $\square$

For completeness, we state the following mild generalization of Theorem 2.10.

Theorem 2.14. *Let X be a normal rigid space over a characteristic zero complete nonarchimedean field K, and let $V \subset X$ be any closed nowhere-dense analytic subset. Suppose that $Y \to X \smallsetminus V$ is a cover, and that there is some closed nowhere-dense analytic set $W \subset X \smallsetminus V$ such that $V \cup W$ is an analytic set in X and such that*

$$Y \times_{(X \smallsetminus V)} (X \smallsetminus V \cup W) \to X \smallsetminus V \cup W$$

is finite étale. Then Y extends to a cover $\tilde{Y} \to X$.

Proof. Apply Theorem 2.10 with $Z = V \cup W$ to construct $\tilde{Y} \to X$ extending

$$Y \times_{(X \smallsetminus V)} (X \smallsetminus V \cup W) \to X \smallsetminus V \cup W,$$

and then use full faithfulness to deduce that $\tilde{Y}|_{X \smallsetminus V} \cong Y$. $\square$

Combining this extension theorem with classical Zariski-Nagata purity, we get a purity theorem for rigid analytic spaces.

Corollary 2.15. *Let X be a smooth rigid analytic space over a characteristic zero complete nonarchimedean field, and let $Z \subset X$ be any closed analytic subset which is everywhere of codimension ≥ 2. Then finite étale covers of X are equivalent to finite étale covers of $X \smallsetminus Z$.*

3 Vanishing and comparison theorems

3.1 The affinoid comparison theorem

In this section we prove Theorem 1.7. Note that when $\mathscr{F}$ is a constant sheaf of torsion abelian groups, this theorem is exactly Corollary 3.2.3 in [Hub96], and we'll eventually reduce to this case. The crucial input is the following lemma.

Lemma 3.1. *Let A be a normal K-affinoid over a complete nonarchimedean field K; set $\mathcal{X} = \operatorname{Spec} A$ and $X = \operatorname{Spa} A$. Let $j : \mathcal{U} \to \mathcal{X}$ be the inclusion of any Zariski-open subset, with analytification $j^{\mathrm{an}} : U \to X$, and let $\mathscr{F}$ be a locally constant constructible sheaf of $\mathbf{Z}/m\mathbf{Z}$-modules on $\mathcal{U}_{\text{ét}}$ for some m. Then writing $\mu_X : X_{\text{ét}} \to \mathcal{X}_{\text{ét}}$ as before, the natural map*

$$H^n_{\text{ét}}(\mathcal{X}, j_!\mathscr{F}) \to H^n_{\text{ét}}(X, \mu_X^* j_!\mathscr{F})$$

is an isomorphism for all $n \geq 0$.

In what follows we usually write $\mathscr{F}^{\mathrm{an}} = \mu_X^*\mathscr{F}$ when context is clear.

Before continuing, note that if $j : \mathcal{U} \to \mathcal{X}$ is any open immersion with closed complement $i : \mathcal{Z} \to \mathcal{X}$, the four functors $j_!, j^*, i_*, i^*$ and their analytifications can be canonically and functorially commuted with the appropriate μ^*'s in the evident sense. Indeed, for j^* and i^* this is obvious (by taking adjoints of the obvious equivalences $j_*\mu_{U*} \cong \mu_{X*}j_*^{\mathrm{an}}$ and $i_*\mu_{Z*} \cong \mu_{X*}i_*^{\mathrm{an}}$), for $j_!$ it is a special case of [Ber93, Corollary 7.1.4], and for i_* it's a very special case of [Hub96, Theorem 3.7.2]. Moreover, if $f : \mathcal{Y} = \operatorname{Spec} \mathcal{B} \to \mathcal{X}$ is any finite morphism with analytification $f^{\mathrm{an}} : Y = \operatorname{Spa} B \to X$, then $\mu_X^* f_* \cong f_*^{\mathrm{an}}\mu_Y^*$ (by [Hub96, Theorem 3.7.2] and [Hub96, Proposition 2.6.4] again) and $\mu_Y^* f^* \cong f^{\mathrm{an}*}\mu_X^*$ (by taking adjoints to the obvious equivalence $\mu_{X*}f_*^{\mathrm{an}} \cong f_*\mu_{Y*}$). We'll use all of these compatibilities without further comment.

Proof of Theorem 1.7. First, observe that all functors involved in the statement of the theorem commute with filtered colimits: for $H^n_{\text{ét}}(\mathcal{X}, -)$ this is standard, for $H^n_{\text{ét}}(X, -)$ this follows from [Hub96, Lemma 2.3.13], and for μ_X^* it is trivial (because μ_X^* is a left adjoint). Writing $\mathscr{F}$ as the filtered colimit of its m-torsion subsheaves, we therefore reduce to the case where $\mathscr{F}$ is killed by some integer $m \geq 1$. Since $\mathcal{X}$ is qcqs, we can write any sheaf of $\mathbf{Z}/m\mathbf{Z}$-modules on $\mathcal{X}_{\text{ét}}$ as a filtered colimit of constructible sheaves of $\mathbf{Z}/m\mathbf{Z}$-modules, cf. [Sta17, Tag 03SA], which reduces us further to the case where $\mathscr{F}$ is a constructible sheaf of $\mathbf{Z}/m\mathbf{Z}$-modules.

We now argue by induction on $\dim \mathcal{X}$ ($= \dim X$). By Noether normalization for affinoids and the aforementioned compatibility of μ^* with pushforward along finite maps, we can assume that $A \simeq K\langle X_1, \ldots, K_{\dim X}\rangle$, so in particular that A is normal. Choose some dense Zariski-open $j : \mathcal{U} \to \mathcal{X}$ such that $j^*\mathscr{F}$ is locally constant. Writing $\mathscr{F}^{\mathrm{an}} = \mu_X^*\mathscr{F}$ for brevity and taking the cohomology of the sequence

$$0 \to j_!j^*\mathscr{F} \to \mathscr{F} \to i_*i^*\mathscr{F} \to 0$$

before and after applying μ^*, we get a pair of long exact sequence sitting in a commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & H^n_{\text{ét}}(\mathcal{X}, j_! j^* \mathscr{F}) \arrow[r] \arrow[d, "\psi_{1,n}"] & H^n_{\text{ét}}(\mathcal{X}, \mathscr{F}) \arrow[r] \arrow[d, "\psi_{2,n}"] & H^n_{\text{ét}}(\mathcal{Z}, i^* \mathscr{F}) \arrow[r] \arrow[d, "\psi_{3,n}"] & \cdots \\ \cdots \arrow[r] & H^n_{\text{ét}}(X, j_!^{\text{an}} j^{\text{an}*} \mathscr{F}^{\text{an}}) \arrow[r] & H^n_{\text{ét}}(X, \mathscr{F}^{\text{an}}) \arrow[r] & H^n_{\text{ét}}(Z, i^{\text{an}*} \mathscr{F}^{\text{an}}) \arrow[r] & \cdots \end{tikzcd}$$

(here we've freely used the various compatibilities of the four functors and analytification). But then $\psi_{1,n}$ is an isomorphism for all n by Lemma 3.1, and $\psi_{3,n}$ is an isomorphism for all n by the induction hypothesis, so $\psi_{2,n}$ is an isomorphism for all n by the five lemma. $\square$

It remains to prove Lemma 3.1. For this we make mild use of cohomological descent; in particular, we need the following observation, which might be useful in other contexts.

Proposition 3.2. *Let $f : Y \to X$ be a surjective qcqs map of rigid analytic spaces. Then f is universally of cohomological descent relative to the étale topology.*

Proof. Using Huber's general qcqs base change theorem [Hub96, Theorem 4.1.1.(c)'] and arguing as in the proof of [Con03, Theorem 7.7], one reduces via [Con03, Theorem 7.2] to the fact that if Y is a qcqs adic space which is topologically of finite type over a (possibly higher rank) geometric point $\mathrm{Spa}(C, C^+)$, then the structure map $Y \to \mathrm{Spa}(C, C^+)$ admits a section, which follows e.g. as in [Sch17, Lemma 9.5]. $\square$

In particular, if $\mathcal{X} = \mathrm{Spec}\, A$ is as above and $f : \mathcal{Y} \to \mathcal{X}$ is a surjective finite morphism (or more generally, a surjective proper morphism) with analytification $f^{\text{an}} : Y \to X = \mathrm{Spa}\, A$, then on the one hand f is surjective and proper and therefore universally of cohomological descent in the usual sense, while on the other hand f^{an} is surjective and qcqs and thus universally of cohomological descent by the previous proposition. Considering the 0-coskeleta $\mathcal{Y}^\bullet_{/\mathcal{X}} = \left(\mathcal{Y}^{(p)}_{/\mathcal{X}} = \underbrace{\mathcal{Y} \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} \mathcal{Y}}_{p} \overset{\varepsilon_p}{\to} \mathcal{X} \right)_{p \geq 0}$ and $Y^\bullet_{/X} = (\cdots)_{p\geq 0}$ of the maps f and f^{an}, we then get a pair of cohomological descent spectral sequences

$$E_1^{p,q} = H^q_{\text{ét}}(\mathcal{Y}^{(p)}_{/\mathcal{X}}, \varepsilon_p^* \mathscr{F}) \Rightarrow H^{p+q}_{\text{ét}}(\mathcal{X}, \mathscr{F})$$

and

$$E_{1,\text{an}}^{p,q} = H^q_{\text{ét}}(Y^{(p)}_{/X}, \varepsilon_p^{\text{an}*} \mathscr{F}^{\text{an}}) \Rightarrow H^{p+q}_{\text{ét}}(X, \mathscr{F}^{\text{an}})$$

for any torsion abelian sheaf $\mathscr{F}$ on $\mathcal{X}_{\text{ét}}$, as in [Con03, Theorem 6.11]. By general nonsense, there is a morphism from the first spectral sequence to the second compatible with the evident maps between the individual terms in the E_1-pages and in the abutments.

Proof of Lemma 3.1. We can assume that $\mathcal{U}$ is dense in $\mathcal{X}$. By assumption, we can choose some surjective finite étale map $\pi : \mathcal{Y} \to \mathcal{U}$ such that $\pi^* \mathscr{F}$ is constant. Since A is normal and Nagata, we can apply the schemes version of Theorem 1.6 to obtain a cartesian diagram

$$\begin{tikzcd} \mathcal{Y} \arrow[r, "j_0"] \arrow[d, "\pi"'] & \tilde{\mathcal{Y}} \arrow[d, "\tilde{\pi}"] \\ \mathcal{U} \arrow[r, "j"] & \mathcal{X} \end{tikzcd}$$

where $\tilde{\pi}$ is finite and surjective and $\tilde{\mathcal{Y}} = \operatorname{Spec} \tilde{B}$ is normal (to be clear, $\tilde{\mathcal{Y}}$ is simply the normalization of $\mathcal{X}$ in $\mathcal{Y}$, cf. Lemma 3.3 below). Let $\tilde{\mathcal{Y}}^{\bullet}_{/\mathcal{X}}$ and $\mathcal{Y}^{\bullet}_{/\mathcal{U}}$ be the evident 0-coskeleta, so we get a cartesian diagram

$$\begin{tikzcd} \mathcal{Y}^{\bullet}_{/\mathcal{U}} \arrow[r, "j_\bullet"] \arrow[d, "\varepsilon_\bullet"'] & \tilde{\mathcal{Y}}^{\bullet}_{/\mathcal{X}} \arrow[d, "\tilde{\varepsilon}_\bullet"] \\ \mathcal{U} \arrow[r, "j"] & \mathcal{X} \end{tikzcd}$$

where the vertical maps are the evident augmentations. Denote the analogous rigid analytic objects and diagrams by roman letters and $(-)^{\mathrm{an}}$'s in the obvious way. Since $\tilde{\mathcal{Y}} \to \mathcal{X}$ and its analytification are universally of cohomological descent, the discussion above gives a morphism of spectral sequences

$$\begin{array}{ccc} H^q_{\text{ét}}(\tilde{\mathcal{Y}}^{(p)}_{/\mathcal{X}}, \tilde{\varepsilon}^*_p j_! \mathscr{F}) & \Rightarrow & H^{p+q}_{\text{ét}}(\mathcal{X}, j_! \mathscr{F}) \\ \downarrow & & \downarrow \\ H^q_{\text{ét}}(\tilde{Y}^{(p)}_{/X}, \tilde{\varepsilon}^{\mathrm{an}*}_p j^{\mathrm{an}}_! \mathscr{F}) & \Rightarrow & H^{p+q}_{\text{ét}}(X, j^{\mathrm{an}}_! \mathscr{F}^{\mathrm{an}}) \end{array}$$

and it now suffices to check that the individual maps on the E_1-page are isomorphisms. Each of these maps sits as the lefthand vertical arrow in a commutative squares

$$\begin{tikzcd} H^q_{\text{ét}}(\tilde{\mathcal{Y}}^{(p)}_{/\mathcal{X}}, \tilde{\varepsilon}^*_p j_! \mathscr{F}) \arrow[r] \arrow[d] & H^q_{\text{ét}}(\tilde{\mathcal{Y}}^{(p)}_{/\mathcal{X}}, j_{p!} \varepsilon^*_p \mathscr{F}) \arrow[d] \\ H^q_{\text{ét}}(\tilde{Y}^{(p)}_{/X}, \tilde{\varepsilon}^{\mathrm{an}*}_p j^{\mathrm{an}}_! \mathscr{F}) \arrow[r] & H^q_{\text{ét}}(\tilde{Y}^{(p)}_{/X}, j^{\mathrm{an}}_{p!} \varepsilon^{\mathrm{an}*}_p \mathscr{F}) \end{tikzcd}$$

where the horizontal arrows (exist and) are isomorphisms by proper base change. It's now enough to show that the righthand vertical arrow is an isomorphism; since each $\varepsilon^*_p \mathscr{F}$ is a constant sheaf, this reduces us to the special case of the lemma where $\mathscr{F}$ is *constant.*

Returning to the original notation of the lemma, we can now assume that $\mathscr{F}$ is the constant sheaf associated with some finite abelian group G. Again, we get a pair of long exact sequences sitting in a commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & H^n_{\text{ét}}(\mathcal{X}, j_! G) \arrow[r] \arrow[d, "\psi_{1,n}"] & H^n_{\text{ét}}(\mathcal{X}, G) \arrow[r] \arrow[d, "\psi_{2,n}"] & H^n_{\text{ét}}(\mathcal{Z}, G) \arrow[r] \arrow[d, "\psi_{3,n}"] & \cdots \\ \cdots \arrow[r] & H^n_{\text{ét}}(X, j^{\mathrm{an}}_! G) \arrow[r] & H^n_{\text{ét}}(X, G) \arrow[r] & H^n_{\text{ét}}(Z, G) \arrow[r] & \cdots \end{tikzcd}$$

as before. But now $\psi_{2,n}$ and $\psi_{3,n}$ are isomorphisms for all n by [Hub96, Corollary 3.2.3], so applying the five lemma again we conclude that $\psi_{1,n}$ is an isomorphism for all n, as desired. □

In this argument, we used the following schemes version of Theorem 1.6.

Lemma 3.3. *Let X be any scheme which is normal and Nagata, and let $Z \subset X$ be any closed nowhere-dense subset. Then the restriction functor*

$$\begin{array}{ccc} \left\{ \begin{array}{c} \text{covers of } X \\ \text{étale over } X \smallsetminus Z \end{array} \right\} & \to & \left\{ \begin{array}{c} \text{finite étale covers} \\ \text{of } X \smallsetminus Z \end{array} \right\} \\ Y & \mapsto & Y \times_X (X \smallsetminus Z) \end{array}$$

is an equivalence of categories, with essential inverse given by sending a finite étale cover $U \to X \smallsetminus Z$ to the normalization of X in U.

We remind the reader that a scheme is Nagata if it is locally Noetherian and admits an open covering by spectra of universally Japanese rings, cf. [Sta17, Tag 033R].

Proof sketch. Let $U \to X \smallsetminus Z$ be a finite étale cover, and let $\tilde{U} \to X$ be the normalization of X in U as in [Sta17, Tag 035G]. By [Sta17, Tag 03GR], $\tilde{U} \to X$ is a finite morphism. By [Sta17, Tags 035K and 03GP], the diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & \tilde{U} \arrow[d] \\ X \smallsetminus Z \arrow[r] & X \end{tikzcd}$$

is cartesian. By [Sta17, Tag 035L], the scheme $\tilde{U}$ is normal. The remaining verifications are left as an exercise for the interested reader. $\square$

3.2 The reduction step

In this section we deduce Theorem 1.3 from Theorem 1.4. For clarity we focus on the vanishing statement in the theorem; it's easy to see that the following argument also reduces the finiteness of the groups $H^i_{\text{ét}}(X_{\widehat{\overline{X}}}, \mathscr{F})$ to finiteness in the special case where $\mathscr{F} = \mathbf{Z}/n\mathbf{Z}$ is constant, and finiteness in the latter case follows from [Ber15, Theorem 1.1.1].

Proof of Theorem 1.3. Fix a nonarchimedean field K and a coefficient ring $\Lambda = \mathbf{Z}/n\mathbf{Z}$ as in the theorem. In what follows, "sheaf" is shorthand for "étale sheaf of Λ-modules". For nonnegative integers d, i, consider the following statement.

Statement $\mathcal{T}_{d,i}$: "For all K-affinoids X of dimension $\leq d$, all Zariski-constructible sheaves $\mathscr{F}$ on X, and all integers $j > i$, we have $H^j_{\text{ét}}(X_{\widehat{\overline{K}}}, \mathscr{F}) = 0$."

We are trying to prove that $\mathcal{T}_{d,d}$ is true for all $d \geq 0$. The idea is to argue by ascending induction on d and descending induction on i. More precisely, it clearly suffices to assume the truth of $\mathcal{T}_{d-1,d-1}$ and then show that $\mathcal{T}_{d,i+1}$ implies $\mathcal{T}_{d,i}$ for any $i \geq d$; as noted in the introduction, $\mathcal{T}_{d,2d}$ is true for any $d \geq 0$, which gives a starting place for the descending induction.

We break the details into several steps.

Step One. *Suppose that $\mathcal{T}_{d-1,d-1}$ holds. Then for any d-dimensional affinoid X, any Zariski-constructible sheaf $\mathscr{F}$ on X, and any dense Zariski-open subset $j : U \to X$, the natural map $H^i_{\text{ét}}(X_{\widehat{\overline{K}}}, j_! j^* \mathscr{F}) \to H^i_{\text{ét}}(X_{\widehat{\overline{K}}}, \mathscr{F})$ is surjective for $i = d$ and bijective for $i > d$.*

Letting $i : Z \to X$ denote the closed complement, this is immediate by looking at the long exact sequence

$$\cdots \to H^{i-1}_{\text{ét}}(Z_{\widehat{\overline{K}}}, i^* \mathscr{F}) \to H^i_{\text{ét}}(X_{\widehat{\overline{K}}}, j_! j^* \mathscr{F}) \to H^i_{\text{ét}}(X_{\widehat{\overline{K}}}, \mathscr{F}) \to H^i_{\text{ét}}(Z_{\widehat{\overline{K}}}, i^* \mathscr{F}) \to \cdots$$

associated with the short exact sequence

$$0 \to j_! j^* \mathscr{F} \to \mathscr{F} \to i_* i^* \mathscr{F} \to 0$$

and then applying $\mathcal{T}_{d-1,d-1}$ to control the outer terms.

Step Two. *For any d, i, $\mathcal{T}_{d,i}$ holds if and only if it holds for all* normal *affinoids.*

One direction is trivial. For the other direction, note that by Noether normalization for affinoids [BGR84, Corollary 6.1.2/2], any d-dimensional affinoid X admits a finite map

$$\tau : X \to \mathbf{B}^d = \operatorname{Spa} K \langle T_1, \ldots, T_d \rangle ,$$

and $\tau_* = R\tau_*$ preserves Zariski-constructibility by Proposition 2.3.

Step Three.[4] *Suppose that $\mathcal{T}_{d-1,d-1}$ holds. Then for any d-dimensional normal affinoid X, any dense Zariski-open subset $j : U \to X$, and any locally constant constructible sheaf $\mathscr{H}$ on U, we can find a Zariski-constrictible sheaf $\mathscr{G}$ on X together with a surjection $s : \mathscr{G} \to j_!\mathscr{H}$, such that moreover $H^i_{\text{ét}}(X_{\widehat{\overline{K}}}, \mathscr{G}) = 0$ for all $i > d$.*

To prove this, suppose we are given X, U, and $\mathscr{H}$ as in the statement. By definition, we can find a finite étale cover $\pi : Y \to U$ such that $\pi^*\mathscr{H}$ is constant, i.e. such that there exists a surjection $\Lambda^n_Y \to \pi^*\mathscr{H}$ for some n; fix such a surjection. This is adjoint to a surjection $\pi_!(\Lambda^n_Y) \to \mathscr{H}$,[5] which extends by zero to a surjection $s : j_!\pi_!(\Lambda^n_Y) \to j_!\mathscr{H}$. We claim that the sheaf $\mathscr{G} = j_!\pi_!(\Lambda^n_Y)$ has the required properties. Zariski-constructibility is clear from the identification $\pi_! = \pi_*$ and Proposition 2.3. For the vanishing statement, we apply Theorem 1.6 to extend $Y \to U$ to a cover $\tilde{\pi} : \tilde{Y} \to X$ sitting in a cartesian diagram

$$\begin{tikzcd} Y \arrow[r, "h"] \arrow[d, "\pi"'] & \tilde{Y} \arrow[d, "\tilde{\pi}"] \\ U \arrow[r, "j"] & X \end{tikzcd}$$

where $\tilde{Y}$ is a normal affinoid and h is a dense Zariski-open immersion. By proper base change and the finiteness of $\tilde{\pi}$, we get isomorphisms

$$\mathscr{G} = j_!\pi_!(\Lambda^n_Y) \cong \tilde{\pi}_! h_!(\Lambda^n_Y) \cong \tilde{\pi}_* h_!(\Lambda^n_Y),$$

so $H^i_{\text{ét}}(X_{\widehat{\overline{K}}}, \mathscr{G}) \cong H^i_{\text{ét}}(\tilde{Y}_{\widehat{\overline{K}}}, h_!\Lambda_Y)^{\oplus n}$. Now, writing $i : V \to \tilde{Y}$ for the closed complement of Y, we get exact sequences

$$H^{i-1}_{\text{ét}}(V_{\widehat{\overline{K}}}, \Lambda_V) \to H^i_{\text{ét}}(\tilde{Y}_{\widehat{\overline{K}}}, h_!\Lambda_Y) \to H^i_{\text{ét}}(\tilde{Y}_{\widehat{\overline{K}}}, \Lambda_{\tilde{Y}})$$

for all i. Examining this sequence for any fixed $i > d$, we see that the rightmost term vanishes by Theorem 1.4, while the leftmost term vanishes by the assumption that $\mathcal{T}_{d-1,d-1}$ holds.[6] Therefore $H^i_{\text{ét}}(\tilde{Y}_{\widehat{\overline{K}}}, h_!\Lambda_Y) = 0$ for $i > d$, as desired.

Step Four. *Suppose that $\mathcal{T}_{d-1,d-1}$ holds. Then $\mathcal{T}_{d,i+1}$ implies $\mathcal{T}_{d,i}$ for any $i \geq d$.*

Fix d and $i \geq d$ as in the statement, and assume $\mathcal{T}_{d,i+1}$ is true. Let X be a d-dimensional affinoid, and let $\mathscr{F}$ be a Zariski-constructible sheaf on X. We need to show that $H^{i+1}_{\text{ét}}(X_{\widehat{\overline{K}}}, \mathscr{F}) = 0$. By Step Two, we can assume X is normal (or even that X is the d-dimensional affinoid ball). By Step One, it suffices to show that $H^{i+1}_{\text{ét}}(X_{\widehat{\overline{K}}}, j_!j^*\mathscr{F}) = 0$ where $j : U \to X$ is the inclusion of any dense Zariski-open subset. Fix a choice of such a U with the property that $j^*\mathscr{F}$ is locally constant. By Step Three, we can choose a Zariski-constructible sheaf $\mathscr{G}$ on X and a surjection $s : \mathscr{G} \to j_!j^*\mathscr{F}$ such that $H^n_{\text{ét}}(X_{\widehat{\overline{K}}}, \mathscr{G}) = 0$ for all $n > d$. By Proposition 2.4, the sheaf $\mathscr{K} = \ker s$ is Zariski-constructible. Now, looking at the exact sequence

$$H^{i+1}_{\text{ét}}(X_{\widehat{\overline{K}}}, \mathscr{G}) \to H^{i+1}_{\text{ét}}(X_{\widehat{\overline{K}}}, j_!j^*\mathscr{F}) \to H^{i+2}_{\text{ét}}(X_{\widehat{\overline{K}}}, \mathscr{K}),$$

[4] This step was inspired by some constructions in Nori's beautiful paper [Nor02].

[5] Surjectivity here can be checked either by a direct calculation or by "pure thought" ($\pi_!$ is left adjoint to π^*, and left adjoints preserve epimorphisms).

[6] One really needs the induction hypothesis to control the leftmost term here, since V may not be normal.

we see that the leftmost term vanishes by the construction of $\mathscr{G}$, while the rightmost term vanishes by the induction hypothesis. Therefore

$$H^{i+1}_{\text{ét}}(X_{\widehat{\overline{K}}}, j_! j^* \mathscr{F}) = 0,$$

as desired. $\square$

3.3 Constant coefficients

In this section we prove Theorem 1.4. The following technical lemma plays an important role in the argument.

Lemma 3.4. *Let K be a complete discretely valued nonarchimedean field, and let A be a reduced K-affinoid algebra. Then A° is an excellent Noetherian ring. Moreover, the strict Henselization of any localization of A° is excellent as well.*

Proof. By Noether normalization for affinoids and [BGR84, Corollary 6.4.1/6], A° can be realized as a module-finite integral extension of $\mathcal{O}_K \langle T_1, \dots, T_n \rangle$ with $n = \dim A$. By a result of Valabrega (cf. [Val75, Proposition 7] and [Val76, Theorem 9]), the convergent power series ring $\mathcal{O}_K \langle T_1, \dots, T_n \rangle$ is excellent for any complete discrete valuation ring $\mathcal{O}_K$. Since excellence propagates along finite type ring maps and localizations, cf. [Sta17, Tag 07QU], we see that A° and any localization thereof is excellent. Now, by a result of Greco [Gre76, Corollary 5.6.iii], the strict Henselization of any excellent local ring is excellent, which gives what we want. $\square$

We also need the following extremely powerful theorem of Gabber.

Theorem 3.5 (Gabber)**.** *Let B be a quasi-excellent strictly Henselian local ring, and let $U \subset \operatorname{Spec} B$ be an affine open subscheme. Then $H^i_{\text{ét}}(U, \mathbf{Z}/n\mathbf{Z}) = 0$ for any $i > \dim B$ and any integer n invertible in B.*

Proof. This is a special case of Gabber's affine Lefschetz theorem for quasi-excellent schemes, cf. Corollaire XV.1.2.4 in [PS14]. $\square$

Finally, we recall the following strong form of Artin's vanishing theorem [SGA73, §XIV.3].

Theorem 3.6 (Artin)**.** *Let X be an affine variety over a separably closed field k, and let $\mathscr{F}$ be a torsion abelian sheaf on $X_{\text{ét}}$. Set*

$$\delta(\mathscr{F}) = \sup \{ \operatorname{tr.deg} k(x)/k \mid \mathscr{F}_{\overline{x}} \neq 0 \}.$$

Then $H^i_{\text{ét}}(X, \mathscr{F}) = 0$ for all $i > \delta(\mathscr{F})$.

Proof of Theorem 1.4. Let $X = \operatorname{Spa} A$ be a K-affinoid as in the theorem. After replacing K by $\widehat{K^{\text{nr}}}$ and X by $X^{\text{red}}_{\widehat{K^{\text{nr}}}}$, we can assume that A is reduced and that K has separably closed residue field k. By an easy induction we can also assume that $n = l$ is prime. For notational simplicity we give the remainder of the proof in the case where $\operatorname{char}(k) = p > 0$; the equal characteristic zero case is only easier.

By e.g. Corollary 2.4.6 in [Ber93], $\operatorname{Gal}_{\overline{K}/K}$ sits in a short exact sequence

$$1 \to P \to \operatorname{Gal}_{\overline{K}/K} \to T \simeq \prod_{q \neq p} \mathbf{Z}_q \to 1$$

where P is pro-p. In particular, if $L \subset \overline{K}$ is any finite extension of K, then

$$H^i(\mathrm{Gal}_{\overline{K}/L}, \mathbf{Z}/l\mathbf{Z}) \simeq \begin{cases} \mathbf{Z}/l\mathbf{Z} \text{ if } i = 0,1 \\ 0 \text{ if } i > 1 \end{cases}.$$

For any such L, look at the Cartan-Leray spectral sequence

$$E_2^{i,j} = H^i(\mathrm{Gal}_{\overline{K}/L}, H^j_{\text{ét}}(X_{\widehat{\overline{K}}}, \mathbf{Z}/l\mathbf{Z})) \Rightarrow H^{i+j}_{\text{ét}}(X_L, \mathbf{Z}/l\mathbf{Z}).$$

The group $\oplus_{j\geq 0} H^j_{\text{ét}}(X_{\widehat{\overline{K}}}, \mathbf{Z}/l\mathbf{Z})$ is finite by vanishing in degrees $> 2\dim X$ together with [Ber15, Theorem 1.1.1], so all the Galois actions in the E_2-page are trivial for any large enough L,[7] in which case we can rewrite the spectral sequence as

$$E_2^{i,j} = H^i(\mathrm{Gal}_{\overline{K}/L}, \mathbf{Z}/l\mathbf{Z}) \otimes H^j_{\text{ét}}(X_{\widehat{\overline{K}}}, \mathbf{Z}/l\mathbf{Z}) \Rightarrow H^{i+j}_{\text{ét}}(X_L, \mathbf{Z}/l\mathbf{Z}).$$

Now if d is the largest integer such that $H^d_{\text{ét}}(X_{\widehat{\overline{K}}}, \mathbf{Z}/l\mathbf{Z}) \neq 0$, then the $E_2^{1,d}$ term survives the spectral sequence, so $H^{d+1}_{\text{ét}}(X_L, \mathbf{Z}/l\mathbf{Z}) \neq 0$; moreover, this latter group coincides with $H^{d+1}_{\text{ét}}(X_{L^{\mathrm{t}}}, \mathbf{Z}/l\mathbf{Z})$ where L^{t} denotes the maximal subfield of L tamely ramified over K (this is immediate from Cartan-Leray, since $\mathrm{Gal}_{L/L^{\mathrm{t}}}$ is a p-group). By Proposition 2.4.7 in [Ber93] the residue field of L^{t} is still separably closed. It thus suffices to prove the following statement:

(†) For any reduced affinoid $X = \mathrm{Spa}\, A$ over a complete discretely valued nonarchimedean field K with separably closed residue field k of characteristic $p > 0$, we have $H^i_{\text{ét}}(X, \mathbf{Z}/l\mathbf{Z}) = 0$ for all $i > 1 + \dim X$ and all primes $l \neq p$.

Fix a uniformizer $\varpi \in \mathcal{O}_K$. Set $\mathcal{X} = \mathrm{Spec}\, A^\circ$ and $\mathcal{X}_s = \mathrm{Spec}\, A^\circ/\varpi$, so $\mathcal{X}_s$ is an affine variety over k. As in [Hub96, §3.5] or [Ber94], there is a natural map of sites $\lambda : X_{\text{ét}} \to \mathcal{X}_{s,\text{ét}}$, corresponding to the natural functor

$$\begin{array}{ccc} \mathcal{X}_{s,\text{ét}} & \to & X_{\text{ét}} \\ \mathcal{U}/\mathcal{X}_s & \mapsto & \eta(\mathcal{U})/X \end{array}$$

given by (uniquely) deforming an étale map $\mathcal{U} \to \mathcal{X}_s$ to a ϖ-adic formal scheme étale over $\mathrm{Spf}\, A^\circ$ and then passing to rigid generic fibers. (We follow Huber's notation in writing λ - Berkovich denotes this map by Θ.) For any abelian étale sheaf $\mathscr{F}$ on X, derived pushforward along λ gives rise to the so-called nearby cycle sheaves $R^j\lambda_*\mathscr{F}$ on $\mathcal{X}_{s,\text{ét}}$, which can be calculated as the sheafifications of the presheaves $\mathcal{U} \mapsto H^j_{\text{ét}}(\eta(\mathcal{U}), \mathscr{F})$, and there is a spectral sequence

$$H^i_{\text{ét}}(\mathcal{X}_s, R^j\lambda_*\mathscr{F}) \Rightarrow H^{i+j}_{\text{ét}}(X, \mathscr{F}),$$

cf. Proposition 4.1 and Corollary 4.2.(iii) in [Ber94]. Taking $\mathscr{F} = \mathbf{Z}/l\mathbf{Z}$, we see that to prove (†) it's enough to show that $H^i_{\text{ét}}(\mathcal{X}_s, R^j\lambda_*\mathbf{Z}/l\mathbf{Z}) = 0$ for any $j \geq 0$ and any $i > 1 + \dim X - j$. By the strong form of the Artin vanishing theorem recalled above, we're reduced to proving that if $x \in \mathcal{X}_s$ is any point such that

$$(R^j\lambda_*\mathbf{Z}/l\mathbf{Z})_{\overline{x}} \neq 0,$$

then $\mathrm{tr.deg}\, k(x)/k \leq 1 + \dim X - j$.

[7] This trick was inspired by a discussion of Poincaré dualities in an IHES lecture by Peter Scholze, cf. https://www.youtube.com/watch?v=E3zAEqkd9cQ.

We check this by a direct computation. So, let $x \in \mathcal{X}_s$ be any point, and let $\mathfrak{p}_x \subset A^\circ$ be the associated prime ideal. Crucially, we have a "purely algebraic" description of the stalk $(R^j\lambda_*\mathbf{Z}/l\mathbf{Z})_{\overline{x}}$: letting $\mathcal{O}_{\mathcal{X},\overline{x}}$ denote the strict Henselization of $\mathcal{O}_{\mathcal{X},x} = (A^\circ)_{\mathfrak{p}_x}$ as usual, then

$$(R^j\lambda_*\mathbf{Z}/l\mathbf{Z})_{\overline{x}} \cong H^j_{\text{ét}}\left(\operatorname{Spec}\mathcal{O}_{\mathcal{X},\overline{x}}[\tfrac{1}{\varpi}], \mathbf{Z}/l\mathbf{Z}\right). \qquad (*)$$

This is a special case of [Hub96, Theorem 3.5.10], and it's remarkable that we have a description like this which doesn't involve taking some completion. By Lemma 3.4, $\mathcal{O}_{\mathcal{X},\overline{x}}$ is excellent, so Theorem 3.5 implies that $H^j_{\text{ét}}(U, \mathbf{Z}/l\mathbf{Z}) = 0$ for any open affine subscheme $U \subset \operatorname{Spec}\mathcal{O}_{\mathcal{X},\overline{x}}$ and any $j > \dim\mathcal{O}_{\mathcal{X},\overline{x}}$. In particular, taking $U = \operatorname{Spec}\mathcal{O}_{\mathcal{X},\overline{x}}[\frac{1}{\varpi}]$ and applying Huber's formula $(*)$ above, we see that if j is an integer such that $(R^j\lambda_*\mathbf{Z}/l\mathbf{Z})_{\overline{x}} \neq 0$, then necessarily

$$j \leq \dim\mathcal{O}_{\mathcal{X},\overline{x}} = \dim\mathcal{O}_{\mathcal{X},x} = \operatorname{ht}\mathfrak{p}_x,$$

where the first equality follows from e.g. [Sta17, Tag 06LK]. Writing $R = A^\circ/\varpi$ and $\overline{\mathfrak{p}_x} = \mathfrak{p}_x/\varpi \subset R$, we then have

$$\begin{aligned} j + \operatorname{tr.deg} k(x)/k &\leq \operatorname{ht}\mathfrak{p}_x + \operatorname{tr.deg} k(x)/k \\ &= 1 + \operatorname{ht}\overline{\mathfrak{p}_x} + \operatorname{tr.deg} k(x)/k \\ &= 1 + \dim R_{\overline{\mathfrak{p}_x}} + \dim R/\overline{\mathfrak{p}_x} \\ &\leq 1 + \dim R \\ &\leq 1 + \dim X. \end{aligned}$$

Here the second line follows from the fact that $\varpi \in \mathfrak{p}_x$ is part of a system of parameters of $(A^\circ)_{\mathfrak{p}_x}$, so $\operatorname{ht}\overline{\mathfrak{p}_x} = \dim R_{\overline{\mathfrak{p}_x}} = \dim(A^\circ)_{\mathfrak{p}_x} - 1$; the third line is immediate from the equality $\operatorname{tr.deg} k(x)/k = \dim R/\overline{\mathfrak{p}_x}$, which is a standard fact about domains of finite type over a field; the fourth line is trivial; and the fifth line follows from the fact that R is module-finite over $k[T_1, \ldots, T_n]$ with $n = \dim X$. But then

$$\operatorname{tr.deg} k(x)/k \leq 1 + \dim X - j,$$

as desired. □

3.4 Stein spaces

We end with the following slight generalization of the main theorem.

Corollary 3.7. *Let X be a rigid space over a characteristic zero complete discretely valued nonarchimedean field K which is* weakly Stein *in the sense that it admts an admissible covering $X = \cup_{n\geq 1}U_n$ by a nested sequence of open affinoid subsets $U_1 \subset U_2 \subset U_3 \subset \cdots$. Let $\mathscr{F}$ be any Zariski-constructible sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules on $X_{\text{ét}}$ for some n prime to the residue characteristic of K. Then*

$$H^i_{\text{ét}}(X_{\widehat{\overline{K}}}, \mathscr{F}) = 0$$

for all $i > \dim X$.

Proof. By [Hub96, Lemma 3.9.2], we have a short exact sequence

$$0 \to \varprojlim_n{}^1 H^{i-1}_{\text{ét}}(U_{n,\widehat{\overline{K}}}, \mathscr{F}) \to H^i_{\text{ét}}(X_{\widehat{\overline{K}}}, \mathscr{F}) \to \varprojlim_n H^i_{\text{ét}}(U_{n,\widehat{\overline{K}}}, \mathscr{F}) \to 0.$$

But the groups $H^i_{\text{ét}}(U_{n,\widehat{\overline{K}}}, \mathscr{F})$ are finite, so the $\lim^1$ term vanishes, and the result now follows from Theorem 1.3. □

This argument also shows that Conjecture 1.2, for a fixed choice of C, is equivalent to the apparently more general conjecture that the cohomology of any Zariski-constructible sheaf on any weakly Stein space X over C vanishes in all degrees $> \dim X$.

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