

On the Kottwitz conjecture for local Shimura varieties

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Abstract

Kottwitz’s conjecture describes the contribution of a supercuspidal representation to the cohomology of a local Shimura variety in terms of the local Langlands correspondence. Using a Lefschetz-Verdier fixed-point formula, we prove a weakened generalized version of Kottwitz’s conjecture. The weakening comes from ignoring the action of the Weil group and only considering the actions of the groups G and J_b up to non-elliptic representations. The generalization is that we allow arbitrary connected reductive groups G and non-minuscule coweights μ .

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1 Introduction

Let F be a finite extension of the field \mathbf{Q}_p of p -adic numbers, and let \check{F} be the completion of the maximal unramified extension of F , relative to a fixed algebraic closure \bar{F} . Let $\sigma \in \text{Aut}(\check{F}/F)$ be the arithmetic Frobenius element. Let G be a connected reductive group defined over F , $[b] \in B(G)$ a σ -conjugacy class of elements of $G(\check{F})$, and $\{\mu\}$ a conjugacy class of cocharacters $\mathbf{G}_m \rightarrow G$ defined over \bar{F} . Assume that $\{\mu\}$ is minuscule and that $[b] \in B(G, \{\mu\})$. The triple $(G, [b], \{\mu\})$ is called a local Shimura datum [RV14, §5]. It is conjectured that there is an associated tower $\mathcal{M}_{G,b,\mu,K}$ of rigid analytic spaces over \check{E} , indexed by open compact subgroups $K \subset G(F)$. Here E is the field of definition of the conjugacy class $\{\mu\}$, a finite extension of F . The isomorphism class of the tower only depends on the classes $[b]$ and $\{\mu\}$. The theory of Rapoport-Zink spaces [RZ96] provides instances in which such a tower exists.

The *Kottwitz conjecture* [Rap95, Conjecture 5.1], [RV14, Conjecture 7.3] relates the cohomology of the $\mathcal{M}_{G,b,\mu,K}$ to the local Langlands correspondence, in the case that $[b]$ is basic. Let us review the precise statement. Let $B(G)_{\text{bas}} \subset B(G)$ be the set of basic σ -conjugacy classes. Assume that $[b] \in B(G)_{\text{bas}}$ and choose a representative $b \in [b]$. Let J_b be the associated inner form of G . Note that since $B(G, \{\mu\})$ contains a unique basic element, $[b]$ is uniquely determined by $\{\mu\}$.

The tower $\mathcal{M}_{G,b,\mu,K}$ receives commuting actions of $J_b(F)$ and $G(F)$. The action of $J_b(F)$ preserves each $\mathcal{M}_{G,b,\mu,K}$, while the action of $g \in G(F)$ sends $\mathcal{M}_{G,b,\mu,K}$ to $\mathcal{M}_{G,b,\mu,gKg^{-1}}$. There is furthermore a *Weil descent datum* on this tower from \check{E} down to E . It need not be effective.

We have the cohomology

$$R\Gamma_c(G, b, \mu, K) = R\Gamma_c(\mathcal{M}_{G,b,\mu,K} \times_{\check{E}} \widehat{\bar{E}}, \bar{\mathbf{Q}}_\ell),$$

an object in the derived category of $\bar{\mathbf{Q}}_\ell$ -vector spaces which is equipped with an action of $J_b(F)$. It also comes equipped with a natural action of I_E , which extends to an action of W_E due to the Weil descent datum. The actions of $J_b(F)$ and W_E commute.

Given an irreducible smooth admissible representation ρ of $J_b(F)$ we define

$$H^i(G, b, \mu)[\rho] = \varinjlim_K \text{Ext}_{J_b(F)}^i(R\Gamma_c(G, b, \mu, K), \rho)$$

This is a $\overline{\mathbf{Q}}_\ell$ -vector space with a smooth action of $G(F) \times W_E$. Finally we define the virtual $G(F) \times W_E$ -representation

$$H^*(G, b, \mu)[\rho] := \sum_{i \in \mathbf{Z}} (-1)^i H_c^i(\mathcal{M}_{G,b,\mu})[\rho](-d),$$

where $d = \dim \mathcal{M}_{G,b,\mu} = \langle 2\rho_G, \mu \rangle$ and ρ_G is half the sum of the positive roots of G . The isomorphism class of $H^*(G, b, \mu)[\rho]$ only depends on $(G, [b], \{\mu\})$ and ρ .

The Kottwitz conjecture describes $H^*(G, b, \mu)[\rho]$ in terms of the local Langlands correspondence. To state it, fix a quasi-split group G^* and a $G^*(\overline{F})$ -conjugacy class Ψ of inner twists $G^* \rightarrow G$. The choice of Ψ gives an identification of the (complex) Langlands dual groups of G^* , G , and J_b ; we shall denote them all by \widehat{G} . The group \widehat{G} carries an action of $\Gamma = \text{Gal}(\overline{F}/F)$; we denote by ${}^L G$ the corresponding L -group. The basic form of the local Langlands conjecture [Kala, Conjecture A] predicts that the set of isomorphism classes of essentially square-integrable representations of $G(F)$ (resp., $J_b(F)$) is partitioned into L -packets $\Pi_\phi(G)$ (resp., $\Pi_\phi(J_b)$), each such packet indexed by a discrete Langlands parameter $\phi : W_F \times \text{SL}_2(\mathbf{C}) \rightarrow {}^L G$. When ϕ is trivial on $\text{SL}_2(\mathbf{C})$ it is expected that the packets $\Pi_\phi(G)$ and $\Pi_\phi(J_b)$ consist entirely of supercuspidal representations.

Let $S_\phi = \text{Cent}(\phi, \widehat{G})$. For $\lambda \in X^*(Z(\widehat{G})^\Gamma)$, write $\text{Rep}(S_\phi, \lambda)$ for the set of isomorphism classes of algebraic representations of the algebraic group S_ϕ whose restriction to $Z(\widehat{G})^\Gamma$ is λ -isotypic, and write $\text{Irr}(S_\phi, \lambda)$ for the subset of irreducible such representations. The class of b corresponds to a character $\lambda_b : Z(\widehat{G})^\Gamma \rightarrow \mathbf{C}^\times$ via the isomorphism $B(G)_{\text{bas}} \rightarrow X^*(Z(\widehat{G})^\Gamma)$ of [Kot85, Proposition 5.6]. For any $\pi \in \Pi_\phi(G)$ and $\rho \in \Pi_\phi(J_b)$ there is an element $\delta_{\pi,\rho} \in \text{Rep}(S_\phi, \lambda_b)$, which can be thought of as measuring the relative position of π and ρ , and whose definition will be given in Section 2.

The conjugacy class $\{\mu\}$ of cocharacters gives dually a conjugacy class of weights of \widehat{G} and we denote by $r_{\{\mu\}}$ the irreducible representation of \widehat{G} of highest weight μ . There is a natural extension of $r_{\{\mu\}}$ to ${}^L G_E$, the L -group of the base change of G to E [Kot84a, Lemma 2.1.2].

Let $\text{Groth}(G(F) \times W_E)$ be the Grothendieck group of the category of $(G(F) \times W_E)$ -modules over $\overline{\mathbf{Q}}_\ell$ which are admissible as a $G(F)$ -module and smooth as a W_E -module.

Conjecture 1.0.1 (Kottwitz). *Let $\phi : W_F \rightarrow {}^L G$ be a discrete Langlands parameter. Write $r_{\{\mu\}} \circ \phi_E$ for the representation of $S_\phi \times W_E$ given by*

$$r_{\{\mu\}} \circ \phi_E(s, w) = r_{\{\mu\}}(s \cdot \phi(w)).$$

Given $\rho \in \Pi_\phi(J_b)$, each $H^i(G, b, \mu)[\rho]$ is admissible, and we have the following equality¹ in $\text{Groth}(G(F) \times W_E)$:

$$H^*(G, b, \mu)[\rho] = (-1)^d \sum_{\pi \in \Pi_\phi(G)} \pi \boxtimes \text{Hom}_{S_\phi}(\delta_{\pi, \rho}, r_{\{\mu\}} \circ \phi_E) \left(-\frac{d}{2}\right). \quad (1.0.1)$$

Remark 1.0.2. In [RV14], $H^*(G, b, \mu)[\rho]$ is defined as the alternating sum

$$\sum_{i, j \in \mathbf{Z}} (-1)^{i+j} H^{i,j}(G, b, \mu)[\rho],$$

where

$$H^{i,j}(G, b, \mu)[\rho] = \varinjlim_K \text{Ext}_{J_b(F)}^j(R^i \Gamma(G, b, \mu, K), \rho).$$

There is a spectral sequence $H^{i,j}(G, b, \mu)[\rho] \implies H^{i+j}(G, b, \mu)[\rho]$, so that if one knew that each $H^{i,j}(G, b, \mu)[\rho]$ were an admissible representation of $J_b(F)$ which is nonzero for only finitely many (i, j) , then the admissibility of $H^{i+j}(G, b, \mu)[\rho]$ would follow; in that case the two definitions of $H^*(G, b, \mu)[\rho]$ are consistent.

[RV14, Proposition 6.1] proves the admissibility of $H^{i,j}(G, b, \mu)[\rho]$ under an assumption (Properties 5.3(iii)) that $\mathcal{M}_{G, b, \mu, K}$ admits a covering by $J_b(F)$ -translates of a quasi-compact open subset. Such a covering exists in the Lubin-Tate case, but probably not in general. So we do not claim that $H^{i,j}(G, b, \mu)[\rho]$ is admissible for general (G, b, μ) .

In this article we prove a version of this conjecture that is both weaker and more general. The weakening comes from ignoring the Weil-group action and thus working in $\text{Groth}(G(F))$ instead of $\text{Groth}(G(F) \times W_E)$. Moreover, we only detect the behavior of representations on the set of elliptic conjugacy classes in $G(F)$. This means that, while we identify the right hand side as contributing to the left hand side, we are not able to exclude potential contributions to the left hand side of non-elliptic representations (meaning those whose distribution characters are supported away from the locus of regular elliptic elements in $G(F)$).

The generalization is that we remove two conditions that are present in the formulations of Kottwitz's conjecture in [Rap95] and [RV14]. One of them is that G is a B -inner form of its quasi-split inner form G^* . This condition, reviewed in Subsection 2.2, has the effect of making the definition of $\delta_{\pi, \rho}$ straightforward. To remove it, we use the formulation of the refined

¹[RV14, Conjecture 7.3] omits the sign $(-1)^d$.

local Langlands correspondence [Kala, Conjecture G] based on the cohomology sets $H^1(u \rightarrow W, Z \rightarrow G)$ of [Kalb]. The definition of $\delta_{\pi, \rho}$ in this setting is a bit more involved and is given in Subsection 2.3, see Definition 2.3.2.

The second condition that we remove is that the conjugacy class $\{\mu\}$ consists of minuscule cocharacters. When μ is not minuscule, then the $\mathcal{M}_{G, b, \mu, K}$ are not rigid spaces; rather they belong to Scholze’s category of diamonds.

The proof of our theorem is based on the following assumptions, each of which is currently being addressed by work in progress due to various authors.

Assumptions 1.0.3.

1. We assume the refined local Langlands correspondence for supercuspidal L -parameters, in the formulation of [Kala, Conjecture G]. Some of it is reviewed in Section 2.
2. We assume that the geometric Satake equivalence holds for Scholze’s mixed-characteristic affine Grassmannian, see Subsection 4.2.
3. The final assumption concerns the moduli stack Bun_G of G -bundles on the Fargues-Fontaine curve [Far]. We need to assume that certain ℓ -adic sheaves on Bun_G are *reflexive*, meaning that they are isomorphic to their double Verdier duals. This is part of forthcoming work of Scholze on the “automorphic to Galois” direction of the local Langlands correspondence, which in turn is modeled on V. Lafforgue’s work [Laf02] which accomplishes the same direction for function fields. Assumption 3 is stated in Subsection 4.10.

Theorem 1.0.4. *We work under assumptions (1)-(3). Let $\phi: W_F \rightarrow {}^L G$ be a discrete Langlands parameter. Let $\text{Groth}(G(F))^{\text{ell}}$ be the quotient of $\text{Groth}(G(F))$ by the subgroup generated by non-elliptic representations. Then each $H^i(G, b, \mu)[\rho]$ is an admissible representation of $G(F)$. Furthermore, (1.0.1) is true in $\text{Groth}(G(F))^{\text{ell}}$. That is, the following equation holds in $\text{Groth}(G(F))^{\text{ell}}$:*

$$H^*(G, b, \mu)[\rho] = (-1)^d \sum_{\pi \in \Pi_\phi(G)} [\dim \text{Hom}_{S_\phi}(\delta_{\pi, \rho}, r_{\{\mu\}})] \pi$$

1.1 Remarks on the proof, and relation with prior work

Theorem 1.0.4 is proved by an application of a Lefschetz-Verdier fixed-point formula. Let us illustrate the idea in the Lubin-Tate case, when $G = \text{GL}_n$,

$\mu = (1, 0, \dots, 0)$, and b is basic of slope $1/n$. In this case $J_b(F) = D^\times$, where D/F is the division algebra of invariant $1/n$, and the spaces $\mathcal{M}_{G,b,\mu,K}$ are known as the Lubin-Tate tower. Atop the tower sits the infinite-level Lubin-Tate space \mathcal{M}_∞ as described in [SW13]. \mathcal{M}_∞ is a pre-perfectoid space admitting an action of $\mathrm{GL}_n(F) \times D^\times$. The Hodge-Tate period map exhibits \mathcal{M}_∞ as a pro-étale cover of Drinfeld's upper half-space Ω^{n-1} (the complement in \mathbf{P}_F^{n-1} of all F -rational hyperplanes).

Now suppose $g \in \mathrm{GL}_n(F)$ is a regular elliptic element, and let C/F be a complete algebraically closed field. Then g has exactly n fixed points on Ω_C^{n-1} . For each such fixed point x , g acts on the fiber $\mathcal{M}_{\infty,x}$.

Key observation. The action of g on $\mathcal{M}_{\infty,x}$ agrees with the action of an element $g' \in D^\times$, where g and g' are related (meaning they become conjugate over \bar{F}).

Suppose that ρ is an admissible representation of D^\times . There is a corresponding ℓ -adic local system \mathcal{L}_ρ on $\Omega_{C,\acute{e}t}^{n-1}$.

A naïve form of the Lefschetz trace formula would predict:

$$\mathrm{tr}(g|H_c^*(\Omega_C^{n-1}, \mathcal{L}_\rho)) = \sum_{x \in (\Omega_C^{n-1})^g} \mathrm{tr}(g|\mathcal{L}_{\rho,x}),$$

where $\mathbf{P}^{n-1}(C)^g$ is the set of g -fixed points. For each such point x , the key observation above gives $\mathrm{tr}(g|\mathcal{L}_{\rho,x}) = \mathrm{tr}(\rho(g'))$, where g and g' are related. By the Jacquet-Langlands correspondence, there exists a discrete series representation π of $\mathrm{GL}_n(F)$ satisfying $\mathrm{tr}(\pi(g)) = (-1)^{n-1} \mathrm{tr}(\rho(g'))$ (here $\mathrm{tr}(\pi(g))$ is interpreted as a Harish-Chandra character). Therefore in $\mathrm{Groth}(\mathrm{GL}_n(F))^{\mathrm{ell}}$ we have an equality

$$H_c^*(\Omega_C^{n-1}, \mathcal{L}_\rho) = (-1)^{n-1} n\pi.$$

The virtual $\mathrm{GL}_n(F) \times W_F$ -representation $H(G, [b], \{\mu\})[\rho]$ is dual to the Euler characteristic $H_c^*(\Omega_C^{n-1}, \mathcal{L}_{\rho^\vee})$, where ρ^\vee is the smooth dual; thus the above is in accord with Theorem 1.0.4.

This argument goes back at least to work of Michael Harris in the 1990s. The difficulty lies in proving the validity of the Lefschetz formula. Prior work of Strauch and Mieda proved Theorem 1.0.4 in the case of the Lubin-Tate tower [Str08], [Mie12], [Mie14a] and also in the case of a basic Rapoport-Zink space for $\mathrm{GSp}(4)$ [Mie]. These results are unconditional. In each of these cases the local Langlands correspondence was already known in sufficient detail (no Assumption 1 was necessary), and also the cocharacter μ was

minuscule, so the relevant period space (generally a mixed-characteristic affine Grassmannian) is simply a flag variety (and thus no Assumption 2 was necessary).

In applying a Lefschetz formula to a non-proper rigid space, care must be taken to treat the boundary. For instance, if D is the closed unit disc $\{|T| \leq 1\}$ in the adic space \mathbf{A}^1 , then the automorphism $T \mapsto T + 1$ has Euler characteristic 1 on D , despite having no fixed points. The culprit is that this automorphism fixes the single boundary point in $\overline{D} \setminus D$. Mieda [Mie14b] proves a Lefschetz formula for an operator on a rigid space, under an assumption that the operator has no topological fixed points on a compactification. Now, in all of the above cases, $\mathcal{M}_{G,b,\mu,K}$ admits a *cellular decomposition*. This means (approximately) that $\mathcal{M}_{G,b,\mu,K}$ contains a compact open subset, whose translates by Hecke operators cover all of $\mathcal{M}_{G,b,\mu,K}$. This is enough to establish the “topological fixed point” hypothesis necessary to apply Mieda’s Lefschetz formula. Shen [She14] constructs a cellular decomposition for a basic Rapoport-Zink space attached to the group $U(1, n - 1)$, which paves the way for an unconditional proof of Theorem 1.0.4 in this case as well.

For general (G, b, μ) , the $\mathcal{M}_{G,b,\mu,K}$ do not admit a cellular decomposition, and so there is probably no hope of applying [Mie14b]. This is where Assumption 3 comes in: it is precisely the input necessary to prove the Lefschetz formula we need.

1.2 Overview of the article

In Section 2, we review the refined local Langlands conjectures of [Kala] and then give the construction of $\delta_{\pi,\rho}$ without assuming that G is a B -inner form of G^* .

In Section 3, we review Scholze’s theory of diamonds [SW13], and prove a Lefschetz-Verdier fixed-point formula for their cohomology, along the lines of [Var07]. This formula applies to those sheaves which are *reflexive*, meaning that they are isomorphic to their double Verdier dual.

In Section 4, we review Scholze’s mixed-characteristic Grassmannian Gr_G (also called the B_{dR}^+ -Grassmannian). It is expected that the geometric Satake equivalence, linking representations of \widehat{G} and equivariant perverse sheaves on Gr_G , holds as it does in the classical case [MV07]. This is our Assumption 2. We also define the diamonds $\mathcal{M}_{G,b,\mu,K}$ and the representations $H^i(G, b, \mu)[\pi]$. We introduce Assumption 3, which implies that each $H^i(G, b, \mu)[\pi]$ is an admissible representation of $J_b(F)$.

In Section 5, we apply our Lefschetz-Verdier fixed point formula to a bounded Grassmannian $\mathrm{Gr}_{G, \leq \mu}$ to prove Theorem 1.0.4.

1.3 Acknowledgements

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2 Review of the local Langlands correspondence

2.1 Basic notions

Recall that we have fixed a quasi-split group G^* and a $G^*(\check{F})$ -conjugacy Ψ class of inner twists $\psi : G^* \rightarrow G$. Given an element $b \in G(\check{F})$, there is an associated inner form J_b of a Levi subgroup of G^* as described in [Kot97, §3.3, §3.4]. Its group of F -points is given by

$$J_b(F) \cong \left\{ g \in G(\check{F}) \mid \mathrm{Ad}(b)\sigma(g) = g \right\}.$$

Up to isomorphism the group J_b depends only on the σ -conjugacy class $[b]$. It will be convenient to choose b to be decent [RZ96, Definition 1.8]. Then there exists a finite unramified extension F'/F such that $b \in G(F')$. This allows us to replace \check{F} by F' in the above formula. The slope morphism $\nu : \mathbf{D} \rightarrow G_{\check{F}}$ of b , [Kot85, §4], is also defined over F' . The centralizer $G_{F', \nu}$ of ν in $G_{F'}$ is a Levi subgroup of $G_{F'}$. The $G(F')$ -conjugacy class of ν is defined over F , and then so is the $G(F')$ -conjugacy class of $G_{F', \nu}$. There is a Levi subgroup M^* of G^* defined over F and $\psi \in \Psi$ that restricts to an inner twist $\psi : M^* \rightarrow J_b$, see [Kot97, §4.3].

From now on assume that b is basic. This is equivalent to $M^* = G^*$, so that J_b is in fact an inner form of G^* and of G .

2.2 The case that G is a B -inner form of G^*

The assumption that G is a B -inner form of G^* means that some $\psi \in \Psi$ can be equipped with a decent basic $b^* \in G^*(F^{\mathrm{nr}})$ such that ψ is an isomorphism $G_{F^{\mathrm{nr}}}^* \rightarrow G_{F^{\mathrm{nr}}}$ satisfying $\psi^{-1}\sigma(\psi) = \mathrm{Ad}(b^*)$. In other words, ψ becomes an isomorphism over F from the group J_{b^*} , now constructed relative to G^* and b^* , and G . Under this assumption, and after choosing a Whittaker

datum \mathfrak{w} for G^* , the isocrystal formulation of the refined local Langlands correspondence [Kala, Conjecture F] predicts the existence of bijections

$$\begin{aligned}\Pi_\phi(G) &\cong \text{Irr}(S_\phi, \lambda_{b^*}) \\ \Pi_\phi(J_b) &\cong \text{Irr}(S_\phi, \lambda_{b^*} + \lambda_b)\end{aligned}$$

where we have used the isomorphisms $B(G)_{\text{bas}} \cong X^*(Z(\widehat{G})^\Gamma) \cong B(G^*)_{\text{bas}}$ of [Kot85, Proposition 5.6] to obtain from $[b] \in B(G)_{\text{bas}}$ and $[b^*] \in B(G^*)_{\text{bas}}$ characters λ_b and λ_{b^*} of $Z(\widehat{G})^\Gamma$.

These bijections are uniquely characterized by the endoscopic character identities which are part of [Kala, Conjecture F]. Write $\pi \mapsto \tau_{b^*, \mathfrak{w}, \pi}$, $\rho \mapsto \tau_{b^*, \mathfrak{w}, \rho}$ for these bijections, and $\tau \mapsto \pi_{b^*, \mathfrak{w}, \tau}$, $\tau \mapsto \rho_{b^*, \mathfrak{w}, \tau}$ for their inverses and define

$$\delta_{\pi, \rho} := \check{\tau}_{b^*, \mathfrak{w}, \pi} \otimes \tau_{b^*, \mathfrak{w}, \rho}.$$

While all of these bijections depend on the choice of Whittaker datum \mathfrak{w} and the choice of b^* , we will argue in Subsection 2.3 that for any pair π and ρ the representation $\delta_{\pi, \rho}$ is independent of these choices. Of course it does depend on b , but this we take as part of the given data.

2.3 The general case

We now drop the assumption that G is a B -inner form of G^* . Because of this, we no longer have the isocrystal formulation of the refined local Langlands correspondence. However, we do have the formulation based on rigid inner twists [Kala, Conjecture G]. What this means with regards to the Kottwitz conjecture is that neither π nor ρ correspond to representations of S_ϕ . Rather, they correspond to representations τ_π and τ_ρ of a different group $\pi_0(S_\phi^+)$. Nonetheless it will turn out that $\check{\tau}_\pi \otimes \tau_\rho$ provides in a natural way a representation $\delta_{\pi, \rho}$ of S_ϕ .

In order to make this precise we will need the material of [Kalb] and [Kalc], some of which is summarized in [Kala]. First, we will need the cohomology set $H^1(u \rightarrow W, Z \rightarrow G^*)$ defined in [Kalb, §3] for any finite central subgroup $Z \subset G^*$ defined over F . As in [Kalc, §3.2] it will be convenient to package these sets for varying Z into the single set

$$H^1(u \rightarrow W, Z(G^*) \rightarrow G^*) := \varinjlim H^1(u \rightarrow W, Z \rightarrow G^*).$$

The transition maps on the right are injective, so the colimit can be seen as an increasing union.

Next, we will need the reinterpretation, given in [Kot], of $B(G)$ as the set of cohomology classes of algebraic 1-cocycles of a certain Galois gerbe

$1 \rightarrow \mathbf{D}(\bar{F}) \rightarrow \mathcal{E} \rightarrow \Gamma \rightarrow 1$. This reinterpretation is also reviewed in [Kalc, §3.1]. Let us make it explicit for basic decent elements $b' \in G(F^{\text{nr}})$. There is a uniquely determined 1-cocycle $\mathbf{Z} \cong \langle \sigma \rangle \rightarrow G(F^{\text{nr}})$ whose value at σ is equal to b' . By inflation we obtain a 1-cocycle of W_F in $G(\bar{F})$. Since b' is basic and decent, for some finite Galois extension K/F splitting G the restriction of this 1-cocycle to W_K factors through K^\times and is a homomorphism $K^\times \rightarrow Z(G)(K)$. Moreover, this homomorphism is algebraic and is in fact given by a multiple of the slope morphism $\nu : \mathbf{D} \rightarrow G$. In this way we obtain a 1-cocycle valued in $G(\bar{F})$ of the extension $1 \rightarrow K^\times \rightarrow W_{K/F} \rightarrow \Gamma_{K/F} \rightarrow 1$, which we can pull-back along $\Gamma \rightarrow \Gamma_{K/F}$ and then combine with ν to obtain a 1-cocycle of \mathcal{E} valued in $G(\bar{F})$, that is algebraic in the sense that its restriction to $\mathbf{D}(\bar{F})$ is given by a morphism of algebraic groups, namely ν . The reader is referred to [Kot97, §8 and App B] for further details.

Finally, we will need the comparison map

$$B(G)_{\text{bas}} \rightarrow H^1(u \rightarrow W, Z(G) \rightarrow G)$$

of [Kalc, §3.3]. In fact, this comparison map is already defined on the level of cocycles, via pull-back along the diagram [Kalc, (3.13)], and takes the form

$$G(F^{\text{nr}})_{\text{d,bas}} \rightarrow Z^1(u \rightarrow W, Z(G) \rightarrow G) \quad (2.3.1)$$

where on the left we have the decent basic elements in $G(F^{\text{nr}})$.

After this short review we turn to the construction of $\delta_{\pi,\rho} \in \text{Rep}(S_\phi, \lambda_b)$. Choose any inner twist $\psi \in \Psi$ and let $\bar{z}_\sigma := \psi^{-1}\sigma(\psi) \in G_{\text{ad}}^*(\bar{F})$. Then $\bar{z} \in Z^1(F, G_{\text{ad}}^*)$ and the surjectivity of the natural map $H^1(u \rightarrow W, Z(G^*) \rightarrow G^*) \rightarrow H^1(F, G_{\text{ad}}^*)$ asserted in [Kalb, Corollary 3.8] allows us to choose $z \in Z^1(u \rightarrow W, Z(G^*) \rightarrow G^*)$ lifting \bar{z} . Then $(\psi, z) : G^* \rightarrow G$ is a rigid inner twist. Let $z_b \in Z^1(u \rightarrow W, Z(G) \rightarrow G)$ be the image of b under (2.3.1). For psychological reasons, let $\xi : G_{F'} \rightarrow J_{b,F'}$ denote the identity map. Then $(\xi \circ \psi, \psi^{-1}(z) \cdot z_b) : G^* \rightarrow J_b$ is also a rigid inner twist.

The L -packets $\Pi_\phi(G)$ and $\Pi_\phi(J_b)$ are now parameterized by representations of a certain cover S_ϕ^+ of S_ϕ . While [Kala, Conjecture G] is formulated in terms of a finite cover depending on an auxiliary choice of a finite central subgroup $Z \subset G^*$, we will adopt here the point of view of [Kalc] and work with a canonical infinite cover. Following [Kalc, §3.3] we let $Z_n \subset Z(G)$ be the subgroup of those elements whose image in $Z(G)/Z(G_{\text{der}})$ is n -torsion, and let $G_n = G/Z_n$. Then G_n has adjoint derived subgroup and connected center. More precisely, $G_n = G_{\text{ad}} \times C_n$, where $C_n = C_1/C_1[n]$ and $C_1 = Z(G)/Z(G_{\text{der}})$. It is convenient to identify $C_n = C_1$ as algebraic tori and take the m/n -power map $C_1 \rightarrow C_1$ as the transition map $C_n \rightarrow C_m$ for

$n|m$. The isogeny $G \rightarrow G_n$ dualizes to $\widehat{G}_n \rightarrow \widehat{G}$ and we have $\widehat{G}_n = \widehat{G}_{\text{sc}} \times \widehat{C}_1$. Note that $\widehat{C}_1 = Z(\widehat{G})^\circ$. The transition map $\widehat{G}_m \rightarrow \widehat{G}_n$ is then the identity on \widehat{G}_{sc} and the m/n -power map on \widehat{C}_1 . Set $\widehat{\widehat{G}} = \varprojlim \widehat{G}_n = \widehat{G}_{\text{sc}} \times \widehat{C}_\infty$, where $\widehat{C}_\infty = \varprojlim \widehat{C}_n$. Elements of $\widehat{\widehat{G}}$ can be written as $(a, (b_n)_n)$, where $a \in \widehat{G}_{\text{sc}}$ and $(b_n)_n$ is a sequence of elements $b_n \in \widehat{C}_1$ satisfying $b_n = b_m^{\frac{m}{n}}$ for $n|m$. In this presentation, the natural map $\widehat{\widehat{G}} \rightarrow \widehat{G}$ sends $(a, (b_n)_n)$ to $a_{\text{der}} \cdot b_1$, where $a_{\text{der}} \in \widehat{G}_{\text{der}}$ is the image of $a \in \widehat{G}_{\text{sc}}$ under the natural map $\widehat{G}_{\text{sc}} \rightarrow \widehat{G}_{\text{der}}$.

Definition 2.3.1. Let $Z(\widehat{\widehat{G}})^+ \subset S_\phi^+ \subset \widehat{\widehat{G}}$ be the preimages of $Z(\widehat{G})^F \subset S_\phi \subset \widehat{G}$ under $\widehat{\widehat{G}} \rightarrow \widehat{G}$.

Given a character $\lambda : \pi_0(Z(\widehat{\widehat{G}})^+) \rightarrow \mathbf{C}^\times$ (which we will always assume trivial on the kernel of $Z(\widehat{\widehat{G}})^+ \rightarrow \widehat{G}_n$ for some n) let $\text{Rep}(\pi_0(S_\phi^+), \lambda)$ denote the set of isomorphism classes of representations of $\pi_0(S_\phi^+)$ whose pull-back to $\pi_0(Z(\widehat{\widehat{G}})^+)$ is λ -isotypic, and let $\text{Irr}(\pi_0(S_\phi^+), \lambda)$ be the (finite) subset of irreducible representations. Let λ_z be the character corresponding to the class of z under the Tate-Nakayama isomorphism

$$H^1(u \rightarrow W, Z(G^*) \rightarrow G^*) \rightarrow \pi_0(Z(\widehat{\widehat{G}})^+)^*$$

of [Kalb, Corollary 5.4], and let λ_{z_b} be the character corresponding to the class of z_b in $H^1(u \rightarrow W, Z(G) \rightarrow G)$. Then according to [Kala, Conjecture G], upon fixing a Whittaker datum \mathfrak{w} for G^* there are bijections

$$\begin{aligned} \Pi_\phi(G) &\cong \text{Irr}(\pi_0(S_\phi^+), \lambda_z) \\ \Pi_\phi(J_b) &\cong \text{Irr}(\pi_0(S_\phi^+), \lambda_z + \lambda_{z_b}) \end{aligned}$$

again uniquely determined by the endoscopic character identities. We write $\pi \mapsto \tau_{z, \mathfrak{w}, \pi}$, $\rho \mapsto \tau_{z, \mathfrak{w}, \rho}$ for these bijections, and $\tau \mapsto \pi_{z, \mathfrak{w}, \tau}$, $\tau \mapsto \rho_{z, \mathfrak{w}, \tau}$ for their inverses. We form the representation $\check{\tau}_{z, \mathfrak{w}, \pi} \otimes \tau_{z, \mathfrak{w}, \rho} \in \text{Rep}(\pi_0(S_\phi^+), \lambda_{z_b})$.

Recall the map [Kalc, (4.7)]

$$S_\phi^+ \rightarrow S_\phi, \quad (a, (b_n)) \mapsto \frac{a_{\text{der}} \cdot b_1}{N_{E/F}(b_{[E:F]})}. \quad (2.3.2)$$

Here $a_{\text{der}} \in \widehat{G}_{\text{der}}$ is the image of $a \in \widehat{G}_{\text{sc}}$ under the natural map $\widehat{G}_{\text{sc}} \rightarrow \widehat{G}_{\text{der}}$ and E/F is a sufficiently large finite Galois extension. This map is independent of the choice of E/F . According to [Kalc, Lemma 4.1] pulling back along this map sets up a bijection $\text{Irr}(\pi_0(S_\phi^+), \lambda_{z_b}) \rightarrow \text{Irr}(S_\phi, \lambda_b)$. Note

that since ϕ is discrete the group S_ϕ^{\natural} defined in loc. cit. is equal to S_ϕ . The lemma remains valid, with the same proof, if we remove the requirement of the representations being irreducible, and we obtain the bijection $\text{Rep}(\pi_0(S_\phi^+), \lambda_{z_b}) \rightarrow \text{Rep}(S_\phi, \lambda_b)$.

Definition 2.3.2. Let $\delta_{\pi, \rho}$ be the image of $\check{\tau}_{z, \mathfrak{w}, \pi} \otimes \tau_{z, \mathfrak{w}, \rho}$ under the bijection $\text{Rep}(\pi_0(S_\phi^+), \lambda_{z_b}) \rightarrow \text{Rep}(S_\phi, \lambda_b)$.

In the situation when G is a B -inner form of G^* , this definition of $\delta_{\pi, \rho}$ agrees with the one of Subsection 2.2, because then we can take z to be the image of b^* under (2.3.1) and then $\tau_{z, \mathfrak{w}, \pi}$ and $\tau_{b^*, \mathfrak{w}, \pi}$ are related via (2.3.2), and so are $\tau_{z, \mathfrak{w}, \rho}$ and $\tau_{b^*, \mathfrak{w}, \rho}$, see [Kalc, §4.2].

Lemma 2.3.3. *The representation $\delta_{\pi, \rho}$ is independent of the choices of Whittaker datum \mathfrak{w} and of a rigidifying 1-cocycle $z \in Z^1(u \rightarrow W, Z(G^*) \rightarrow G^*)$.*

Proof. Both of these statements follow from [Kala, Conjecture G]. For the independence of Whittaker datum, one can prove that the validity of this conjecture implies that if \mathfrak{w} is replaced by another choice \mathfrak{w}' then there is an explicitly constructed character $(\mathfrak{w}, \mathfrak{w}')$ of $\pi_0(S_\phi/Z(\widehat{G})^\Gamma)$ whose inflation to $\pi_0(S_\phi^+)$ satisfies $\tau_{z, \mathfrak{w}, \sigma} = \tau_{z, \mathfrak{w}', \sigma} \otimes (\mathfrak{w}, \mathfrak{w}')$ for any $\sigma \in \Pi_\phi(G) \cup \Pi_\phi(J_b)$. See §4 and in particular Theorem 4.3 of [Kal13], the proof of which is valid for a general G that satisfies [Kala, Conjecture G], bearing in mind that the transfer factor we use here is related to the one used there by $s \mapsto s^{-1}$. The independence of z follows from the same type of argument, but now using [Kalc, Lemma 6.2]. \square

2.4 Endoscopic character relations

We recall here the endoscopic character identities, which are part of the refined local Langlands correspondence, following the formulation of [Kalb, §5.4], also recalled in [Kala, §4.2]. They will be an important ingredient in the proof of our main result.

We summarize the notation established so far.

- F/\mathbf{Q}_p is a finite extension.
- G is a connected reductive group defined over F .
- G^* is a quasi-split connected reductive group defined over F .
- Ψ is a G^* -conjugacy class of inner twists $\psi: G^* \rightarrow G$.

- $\bar{z}_\sigma = \psi^{-1}\sigma(\psi) \in G_{\text{ad}}^*$, so that $\bar{z} \in Z^1(F, G_{\text{ad}}^*)$.
- $z \in Z^1(u \rightarrow W, Z(G^*) \rightarrow G^*)$ is a lift of \bar{z} .
- $b \in G(F^{\text{nr}})$ is a decent basic element.
- J_b is the corresponding inner form of G .
- $\xi: G_{F^{\text{nr}}} \rightarrow J_{b, F^{\text{nr}}}$ is the identity map.
- $z_b \in Z^1(u \rightarrow W, Z(G) \rightarrow G)$ is the image of b under (2.3.1).
- \mathfrak{w} is a Whittaker datum for G^* .
- $\phi: W_F \rightarrow {}^L G$ is a discrete parameter.
- $S_\phi = \text{Cent}(\phi, \widehat{G})$.
- S_ϕ^+ is the group defined in Definition 2.3.1.

Associated to ψ are the L -packets $\Pi_\phi(G)$ and $\Pi_\phi(J_b)$ and the bijections

$$\Pi_\phi(G) \rightarrow \text{Irr}(\pi_0(S_\phi^+), \lambda_z), \quad \Pi_\phi(J_b) \rightarrow \text{Irr}(\pi_0(S_\phi^+), \lambda_z + \lambda_{z_b})$$

denoted by $\pi \mapsto \tau_{z, \mathfrak{w}, \pi}$ and $\rho \mapsto \tau_{z, \mathfrak{w}, \rho}$.

We now choose a semi-simple element $s \in S_\phi$ and an element $\dot{s} \in S_\phi^+$ which lifts s . Let $e(G)$ and $e(J_b)$ be the Kottwitz signs of the groups G and J_b , as defined in [Kot83]. Consider the virtual characters

$$e(G) \sum_{\pi \in \Pi_\phi(G)} \text{tr } \tau_{z, \mathfrak{w}, \pi}(\dot{s}) \cdot \Theta_\pi \quad \text{and} \quad e(J_b) \sum_{\rho \in \Pi_\phi(J_b)} \text{tr } \tau_{z, \mathfrak{w}, \rho}(\dot{s}) \cdot \Theta_\rho.$$

The endoscopic character identities are equations which relate these two virtual characters to virtual characters on an endoscopic group H_1 of G and J_b . From the pair (ϕ, \dot{s}) one obtains a refined elliptic endoscopic datum

$$\dot{\mathfrak{e}} = (H, \mathcal{H}, \dot{s}, \eta) \tag{2.4.1}$$

in the sense of [Kalb, §5.3] as follows. Let $\widehat{H} = \text{Cent}(s, \widehat{G})^\circ$. The image of ϕ is contained in $\text{Cent}(s, \widehat{G})$, which in turns acts by conjugation on its connected component \widehat{H} . This gives a homomorphism $W_F \rightarrow \text{Aut}(\widehat{H})$. Letting $\Psi_0(\widehat{H})$ be the based root datum of \widehat{H} [Kot84b, §1.1] and $\Psi_0^\vee(\widehat{H})$ its dual, we obtain the homomorphism

$$W_F \rightarrow \text{Aut}(\widehat{H}) \rightarrow \text{Out}(\widehat{H}) = \text{Aut}(\Psi_0(\widehat{H})) = \text{Aut}(\Psi_0^\vee(\widehat{H}))^\vee.$$

Since the target is finite, this homomorphism extends to Γ_F and we obtain a based root datum with Galois action, hence a quasi-split connected reductive group H defined over F . Its dual group is by construction equal to \widehat{H} . We let $\mathcal{H} = \widehat{H} \cdot \phi(W_F)$, noting that the right factor normalizes the left so their product \mathcal{H} is a subgroup of ${}^L G$. Finally, we let $\eta : \mathcal{H} \rightarrow {}^L G$ be the natural inclusion. Note that by construction ϕ takes image in \mathcal{H} , i.e. it factors through η .

We can realize the L -group of H as ${}^L H = \widehat{H} \rtimes W_F$, but we caution the reader that W_F does not act on \widehat{H} via the map $W_F \rightarrow \text{Aut}(\widehat{H})$ given by ϕ as above. Rather, we have to modify this action to ensure that it preserves a pinning of \widehat{H} . More precisely, after fixing an arbitrary pinning of \widehat{H} we obtain a splitting $\text{Out}(\widehat{H}) \rightarrow \text{Aut}(\widehat{H})$ and the action of W_F on \widehat{H} we use to form ${}^L H$ is given by composing the above map $W_F \rightarrow \text{Out}(\widehat{H})$ with this splitting.

Both ${}^L H$ and \mathcal{H} are thus extensions of W_F by \widehat{H} , but they need not be isomorphic. If they are, we fix arbitrarily an isomorphism $\eta_1 : \mathcal{H} \rightarrow {}^L H$ of extensions. Then $\phi^s = \eta_1 \circ \phi$ is a supercuspidal parameter for H .

In the general case we need to introduce a z -pair $\mathfrak{z} = (H_1, \eta_1)$ as in [KS99, §2]. It consists of a z -extension $H_1 \rightarrow H$ (recall this means that H_1 has a simply connected derived subgroup and the kernel of $H_1 \rightarrow H$ is an induced torus) and $\eta_1 : \mathcal{H} \rightarrow {}^L H_1$ is an L -embedding that extends the natural embedding $\widehat{H} \rightarrow \widehat{H}_1$. As is shown in [KS99, §2.2], such a z -pair always exists. Again we set $\phi^s = \eta_1 \circ \phi$ and obtain a supercuspidal parameter for H_1 . In the situation where an isomorphism $\eta_1 : \mathcal{H} \rightarrow {}^L H$ does exist, we will allow ourselves to take $H = H_1$ and so regard $\mathfrak{z} = (H, \eta_1)$ as a z -pair, even though in general H will not have a simply connected derived subgroup.

The virtual character on H_1 that the above virtual characters on G and J_b are to be related to is

$$S\Theta_{\phi^s} := \sum_{\pi^s \in \Pi_{\phi^s}(H_1)} \dim(\tau_{\pi^s}) \Theta_{\pi^s}.$$

Here $\pi^s \mapsto \tau_{\pi^s}$ is a bijection $\Pi_{\phi^s}(H_1) \rightarrow \text{Irr}(\pi_0(\text{Cent}(\phi^s, \widehat{H}_1)/Z(\widehat{H}_1)^\Gamma))$ determined by an arbitrary choice of Whittaker datum for H_1 . The argument in the proof of Lemma 2.3.3 shows the independence of $\dim(\tau_{\pi^s})$ of the choice of a Whittaker datum for H_1 .

The relationship between the virtual characters on G , J_b , and H_1 , is expressed in terms of the Langlands-Shelstad transfer factor $\Delta'_{\text{abs}}[\dot{\mathbf{c}}, \mathfrak{z}, \mathfrak{w}, (\psi, z)]$ for the pair of groups (H_1, G) and the corresponding Langlands-Shelstad

transfer factor $\Delta'_{\text{abs}}[\dot{\mathbf{e}}, \mathfrak{z}, \mathfrak{w}, (\xi \circ \psi, \psi^{-1}(z_b) \cdot z)]$ for the pair of groups (H_1, J_b) , both of which are defined by [Kalb, (5.10)]. We will abbreviate both of them to just Δ . It is a simple consequence of the Weyl integration formula that the character relation [Kalb, (5.11)] can be restated in terms of character functions (rather than character distributions) as

$$e(G) \sum_{\pi \in \Pi_\phi(G)} \text{tr } \tau_{z, \mathfrak{w}, \pi}(\dot{s}) \Theta_\pi(g) = \sum_{h_1 \in H_1(F)/\text{st.}} \Delta(h_1, g) S \Theta_{\phi^s}(h_1) \quad (2.4.2)$$

for any strongly regular semi-simple element $g \in G(F)$. The sum on the right runs over stable conjugacy classes of strongly regular semi-simple elements of $H_1(F)$. We also have the analogous identity for J_b :

$$e(J_b) \sum_{\rho \in \Pi_\phi(J_b)} \text{tr } \tau_{z, \mathfrak{w}, \rho}(\dot{s}) \Theta_\rho(j) = \sum_{h_1 \in H_1(F)/\text{st.}} \Delta(h_1, j) S \Theta_{\phi^s}(h_1). \quad (2.4.3)$$

We are only interested in the right hand sides of these two equations as a bridge between their left-hand sides. Essential for this bridge is a certain compatibility between the transfer factors appearing on both right-hand sides.

Definition 2.4.1. Two strongly regular semi-simple elements $g \in G(F)$ and $j \in J_b(F)$ are called *stably conjugate*, or *related*, if there exists $y \in G(G^{\text{nr}})$ such that $j = \xi(ygy^{-1})$. In that case, letting $T = \text{Cent}(g, G)$ the element $y^{-1}by^\sigma$ belongs to $T(F^{\text{nr}})$ and its image in $B(T)$ is called $\text{inv}[b](g, j)$.

Note that, according to Steinberg's theorem, the existence of $y \in G(F^{\text{nr}})$ with $j = \xi(ygy^{-1})$ is equivalent to the existence of $y \in G(\bar{F})$ with the same property. We work here with $G(F^{\text{nr}})$ to facilitate the definition of the invariant. It is straightforward to check that the image in $B(T)$ of $y^{-1}by^\sigma$ is independent of the choice of y .

The compatibility satisfied by the transfer factors is then the following.

Lemma 2.4.2.

$$\Delta(h_1, j) = \Delta(h_1, g) \cdot \langle \text{inv}[b](g, j), s_{h,g} \rangle. \quad (2.4.4)$$

We need to explain the second factor. Given maximal tori $T_H \subset H$ and $T \subset G$, there is a notion of an admissible isomorphism $T_H \rightarrow T$, for which we refer the reader to [Kala, §1.3]. Two strongly regular semi-simple elements $h \in H(\mathbf{Q}_p)$ and $g \in G(\mathbf{Q}_p)$ are called *related* if there exists an admissible isomorphism $T_h \rightarrow T_g$ between their centralizers mapping h to g . If such an

isomorphism exists, it is unique, and in particular defined over F , and shall be called $\varphi_{h,g}$. An element $h_1 \in H_1(F)$ is called related to $g \in G(F)$ if and only if its image $h \in H(F)$ is so. Since g and j are stably conjugate, an element $h_1 \in H_1(F)$ is related to g if and only if it is related to j . If that is not the case, both $\Delta(h_1, j)$ and $\Delta(h_1, g)$ are zero and (2.4.4) is trivially true. Thus assume that h_1 is related to both g and j . Let $s^\natural \in S_\phi$ be the image of \dot{s} under (2.3.2). Note that $s^\natural \in s \cdot Z(\widehat{G})^{\circ, \Gamma}$ and hence the preimage of s^\natural under η belongs to $Z(\widehat{H})^\Gamma$, which in turns embeds naturally into \widehat{T}_h^Γ . Using the admissible isomorphism $\varphi_{h,g}$ we transport s^\natural into \widehat{T}_g^Γ and denote it by $s_{h,g}$. It is then paired with $\text{inv}[b](g, j)$ via the isomorphism $B(T_g) \cong X^*(\widehat{T}_g^\Gamma)$ of [Kot85, §2.4].

Proof. For every finite subgroup $Z \subset Z(G) \subset T_g$ one obtains from $\varphi_{h,g}$ an isomorphism $T_h/\varphi_{h,g}^{-1}(Z) \rightarrow T_g/Z$. Using the subgroups Z_n from the previous subsection we form the quotients $T_{h,n} = T_h/\varphi_{h,g}^{-1}(Z_n)$ and $T_{g,n} = T_g/Z_n$. From $\varphi_{h,g}$ we obtain an isomorphism

$$\widehat{T}_h \rightarrow \widehat{T}_g$$

between the limits over n of the tori dual to $T_{h,n}$ and $T_{g,n}$. Let $\dot{s}_{h,g} \in [\widehat{T}_g]^+$ be the image of \dot{s} under this isomorphism. Let $\text{inv}[z_b](g, j) \in H^1(u \rightarrow W, Z(G) \rightarrow T_g)$ be the invariant defined in [Kalb, §5.1]. If we replace $\langle \text{inv}[b](g, j), s_{h,g} \rangle$ by $\langle \text{inv}[z_b](g, j), \dot{s}_{h,g} \rangle$ then the lemma follows immediately from the defining formula [Kalb, (5.10)] of the transfer factors. The lemma follows from the equality $\langle \text{inv}[b](g, j), s_{h,g} \rangle = \langle \text{inv}[z_b](g, j), \dot{s}_{h,g} \rangle$ proved in [Kalc, §4.2]. \square

3 Geometric Preparations

3.1 Diamonds

We give a brief review of Scholze's theory of diamonds [Sch17]. We will work extensively with perfectoid rings and spaces which have no specified field of scalars, as in [Fon13] or [SW14, Definition 7.1.2].

Definition 3.1.1. A morphism $f: X \rightarrow Y$ of perfectoid spaces is *pro-étale* if it is locally (on the source) of the form $\text{Spa}(A_\infty, A_\infty^+) \rightarrow \text{Spa}(A, A^+)$, where A and A_∞ are perfectoid algebras, and

$$(A_\infty, A_\infty^+) = [\varinjlim (A_i, A_i^+)]^\wedge$$

is a filtered colimit of affinoid perfectoid algebras (A_i, A_i^+) , such that

$$\mathrm{Spa}(A_i, A_i^+) \rightarrow \mathrm{Spa}(A, A^+)$$

is étale for each i .

Let k be a perfect field of characteristic p .

Definition 3.1.2. Let (Perf) be the category of perfectoid spaces. A collection of morphisms $\{f_i: X_i \rightarrow X\}_{i \in I}$ is a *pro-étale covering* if all the f_i are pro-étale, and if for all quasi-compact open $U \subset X$, there exists a finite subset $I_U \subset I$ and a quasi-compact open $U_i \subset X_i$ for each $i \in I_U$, such that $U = \cup_{i \in I_U} f_i(U_i)$. We endow (Perf) with the structure of the site generated by pro-étale covers.

For an object X of (Perf) , define a pre-sheaf h_X on (Perf) by $h_X(Y) = \mathrm{Hom}(Y, X)$. For a perfectoid Huber pair (R, R^+) , we use the abbreviation $\mathrm{Spd}(R, R^+)$ (or just $\mathrm{Spd} R$ if $R^+ = R^\circ$) for $h_{\mathrm{Spa}(R^b, R^{b+})}$.

Proposition 3.1.3 ([SW14, Proposition 8.2.7]). *The pre-sheaf h_X is a sheaf.*

We remark that in the category of sheaves on (Perf) , morphisms $h_X \rightarrow \mathcal{G}$ correspond to elements of $\mathcal{G}(X)$. That is, the functor $X \mapsto h_X$ from (Perf) into the category of sheaves on (Perf) is fully faithful. Therefore we are justified in referring to the representable sheaf h_X simply as X .

- Definition 3.1.4.**
1. If $f: \mathcal{F} \rightarrow \mathcal{G}$ is a morphism of sheaves on (Perf) , say that f is *étale* if for all objects X of (Perf) and all morphisms $X \rightarrow \mathcal{G}$, the pull-back $X \times_{\mathcal{G}} \mathcal{F}$ is representable by an object Y of (Perf) , and $Y \rightarrow X$ is étale.
 2. A *diamond* is a sheaf X on (Perf) of the form X'/R , where X' and R are perfectoid spaces and $R \rightarrow X' \times X'$ is an equivalence relation for which both projections $R \rightarrow X'$ are pro-étale. The *underlying topological space* of X is defined as the quotient of $|X'|$ by the equivalence relation $|R| \rightarrow |X'| \times |X'|$.
 3. A morphism $X' \rightarrow X$ of diamonds is an *open immersion* if for all objects U of (Perf) mapping to X , the pullback $X' \times_X U \rightarrow U$ is representable by an open immersion of perfectoid spaces. In this case we say X' is an *open sub-diamond* of X .

4. A diamond X is *quasi-separated* if for all quasi-compact $U, V \rightarrow X$, the fiber product $U \times_X V$ is quasi-compact. X is *spatial* if it is quasi-compact and quasi-separated, and if $|X|$ admits a basis of open subsets of the form $|U|$, where $U \rightarrow X$ is a quasi-compact open immersion. X is *locally spatial* if it admits an open cover by spatial sub-diamonds.
5. A morphism $X \rightarrow Y$ of diamonds is *proper* if it is quasi-compact, quasi-separated, and universally closed.
6. Let X be a locally spatial diamond. Define $X_{\text{ét}}$ to be the category of étale morphisms $X' \rightarrow X$, endowed with the topology where covers are defined as jointly surjective maps.
7. The *v-topology* on (Perf) is the topology where a cover consists of a collection of maps $X_i \rightarrow X$ such that for any quasi-compact open subset $U \subset X$, there are finitely many i and quasi-compact open subsets $U_i \subset X_i$ such that the U_i jointly cover U .
8. A *small v-sheaf* is a sheaf Y on the v-topology on (Perf) , for which there exists a surjective map of v-sheaves $X \rightarrow Y$, for some perfectoid space² X .
9. A *small v-stack* is a sheaf of groupoids Y on the v-topology on (Perf) , for which there exists a surjective map of stacks $X \rightarrow Y$, for some perfectoid space X , for which $R = X \times_Y X$ is a small v-sheaf.

3.2 The six functor formalism

We will require the “six functor formalism” in the setting of small v-stacks, as described in the introduction to [Sch17].

Let Λ be a ring which is n -torsion for some integer n prime to p .

- Definition 3.2.1.**
1. For a locally spatial diamond Y we let $D(Y_{\text{ét}}, \Lambda)$ be the derived category of complexes of sheaves of Λ -modules on $Y_{\text{ét}}$.
 2. For a small v-stack X , let X_v denote the site of all perfectoid spaces over X , with the v-topology. Let $D_{\text{ét}}(X, \Lambda) \subset D(X_v, \Lambda)$ be the full subcategory whose objects are those $A \in D(X_v, \Lambda)$ such that for all $f: Y \rightarrow X$ from a locally spatial diamond Y , f^*A lies in the left-completion of $D(Y_{\text{ét}}, \Lambda)$

²In [Sch17] there is a “smallness” hypothesis applied to X , which has to do with cutoff cardinals; we will be ignoring such set-theoretic issues.

Then $D_{\acute{e}t}(X, \Lambda)$ admits a derived tensor product, denoted \otimes , and a derived internal Hom, denoted $\underline{\mathbf{RHom}}$. We remark that if when X is a spatial diamond, Scholze defines a site $X_{\acute{e}t}$, and then $D_{\acute{e}t}(X, \Lambda)$ and $D(X_{\acute{e}t}, \Lambda)$ are equivalent.

We need the following constructions:

1. [Sch17, Definition 1.7] For any map $f: X \rightarrow Y$ of small v-stacks that is compactifiable, representable in locally spatial diamonds, and with finite geometric transcendence degree, there is a lower shriek functor³ $f_!: D_{\acute{e}t}(X, \Lambda) \rightarrow D_{\acute{e}t}(Y, \Lambda)$, which agrees with f_* when f is proper, and an upper shriek functor $f^!: D_{\acute{e}t}(Y, \Lambda) \rightarrow D_{\acute{e}t}(X, \Lambda)$ which is right adjoint to $f_!$.
2. [Sch17, Theorem 1.8(iii)] A projection formula $\text{proj}: f_!(\mathcal{F} \otimes f^*\mathcal{G}) \xrightarrow{\cong} f_!\mathcal{F} \otimes \mathcal{G}$. For $\mathcal{F}, \mathcal{G} \in D(Y_{\acute{e}t}, \Lambda)$, the composition

$$f_!(f^!\mathcal{F} \otimes f^*\mathcal{G}) \xrightarrow{\text{proj}} f_!f^!\mathcal{F} \otimes \mathcal{G} \xrightarrow{\text{adj}} \mathcal{F} \otimes \mathcal{G}$$

induces by adjunction a morphism

$$t_{f^!}: f^!\mathcal{F} \otimes f^*\mathcal{G} \rightarrow f^!(\mathcal{F} \otimes \mathcal{G}).$$

3. [Sch17, Theorem 1.8(iv)] The first local Verdier duality isomorphism:

$$\underline{\mathbf{RHom}}(f_!\mathcal{F}, \mathcal{G}) \cong f_*\underline{\mathbf{RHom}}(\mathcal{F}, f^!\mathcal{G}), \quad (3.2.1)$$

4. [Sch17, Theorem 1.8(v)] The second local Verdier duality isomorphism:

$$f^!\underline{\mathbf{RHom}}(\mathcal{F}, \mathcal{G}) \cong \underline{\mathbf{RHom}}(f^*\mathcal{F}, f^!\mathcal{G}). \quad (3.2.2)$$

5. [Sch17, Theorem 1.9] Let

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{g} & Y, \end{array} \quad (3.2.3)$$

be a cartesian diagram of small v-stacks. Assume g is compactifiable, representable in locally spatial diamonds, and has finite geometric transcendence degree. There are base change isomorphisms $g^!f_*\mathcal{F} \cong f'_*(g')^!\mathcal{F}$ and $f^*g_!\mathcal{F} \cong g'_!(f')^*\mathcal{F}$. We will denote both of these by BC.

³We have decided to use the notation $f_*, f_!, f^!, \dots$ for functors between derived categories which are often denoted $Rf_*, Rf_!, Rf^!, \dots$

For $\pi: X \rightarrow \mathrm{Spd} C$ a morphism of small v-stacks, let $K_X = \pi^! \Lambda$ be the dualizing complex. For an object \mathcal{F} of $D(X_{\acute{e}t}, \Lambda)$, let $\mathbf{D}\mathcal{F} = \underline{\mathrm{RHom}}(\mathcal{F}, K_X)$ be the Verdier dual. We write

$$\mathrm{ev}: \mathcal{F} \otimes \mathbf{D}\mathcal{F} \rightarrow K_X$$

for the evaluation morphism. Using (3.2.1) and (3.2.2), a morphism $f: X \rightarrow Y$ interacts with the Verdier dual as follows:

$$f_* \mathbf{D} \cong \mathbf{D} f_! \quad (3.2.4)$$

$$f^! \mathbf{D} \cong \mathbf{D} f^*. \quad (3.2.5)$$

3.3 Exterior tensor products and the Künneth isomorphism

We recall the notion of an exterior tensor product (relative to the base $\mathrm{Spa} C$). For morphisms of small v-stacks $X_1, X_2 \rightarrow S$, consider the cartesian diagram

$$\begin{array}{ccc} & X_1 \times_S X_2 & \\ p_1 \swarrow & & \searrow p_2 \\ X_1 & & X_2 \\ \pi_1 \searrow & & \swarrow \pi_2 \\ & S & \end{array}$$

For objects $\mathcal{F}_i \in D_{\acute{e}t}(X_i, \Lambda)$ ($i = 1, 2$) we define

$$\mathcal{F}_1 \boxtimes_S \mathcal{F}_2 = p_1^* \mathcal{F}_1 \otimes p_2^* \mathcal{F}_2 \in D_{\acute{e}t}(X_1 \times X_2, \Lambda).$$

Lemma 3.3.1. *There is a natural isomorphism*

$$(\pi_1 \times \pi_2)_!(\mathcal{F}_1 \boxtimes_S \mathcal{F}_2) \cong \pi_{1!} \mathcal{F}_1 \otimes \pi_{2!} \mathcal{F}_2.$$

Proof. Combining the projection formula and base change isomorphisms gives:

$$\begin{aligned} p_{2!}(\mathcal{F}_1 \boxtimes_S \mathcal{F}_2) &= p_{2!}(p_1^* \mathcal{F}_1 \otimes p_2^* \mathcal{F}_2) \\ &\xrightarrow{\mathrm{proj}} p_{2!} p_1^* \mathcal{F}_1 \otimes \mathcal{F}_2 \\ &\xrightarrow{\mathrm{BC}} \pi_2^* \pi_{1!} \mathcal{F}_1 \otimes \mathcal{F}_2, \end{aligned}$$

so that we have an isomorphism

$$p_{2!}(\mathcal{F}_1 \boxtimes_S \mathcal{F}_2) \cong \pi_2^* \pi_{1!} \mathcal{F}_1 \otimes \mathcal{F}_2 \quad (3.3.1)$$

Now apply $\pi_{2!}$, and note that $\pi_1 \times \pi_2 = \pi_2 \circ p_2$:

$$\begin{aligned} (\pi_1 \times \pi_2)_!(\mathcal{F}_1 \boxtimes_S \mathcal{F}_2) &= \pi_{2!} p_{2!}(\mathcal{F}_1 \boxtimes_S \mathcal{F}_2) \\ &\cong \pi_{2!}(\pi_2^* \pi_1! \mathcal{F}_1 \otimes \mathcal{F}_2) \\ &\xrightarrow{\text{proj}} \pi_1! \mathcal{F}_1 \otimes \pi_{2!} \mathcal{F}_2, \end{aligned}$$

as claimed. \square

In the special case $S = \text{Spd } C$, we write \boxtimes for \boxtimes_S . In the case $\mathcal{F}_1 = K_{X_1} = \pi_1! \Lambda$, we have a morphism

$$\begin{aligned} p_{2!}(K_{X_1} \boxtimes \mathcal{F}_2) &\xrightarrow{3.3.1} \pi_2^* \pi_1! K_{X_1} \otimes \mathcal{F}_2 \\ &= \pi_2^* \pi_1! \pi_1! \Lambda \otimes \mathcal{F}_2 \\ &\xrightarrow{\text{adj.}} \pi_2^* \Lambda \otimes \mathcal{F}_2 \cong \Lambda \otimes \mathcal{F}_2 \cong \mathcal{F}_2, \end{aligned}$$

which induces by adjunction a morphism

$$K_{X_1} \boxtimes \mathcal{F}_2 \rightarrow p_2! \mathcal{F}_2. \quad (3.3.2)$$

If $\mathcal{F}_2 = K_{X_2}$, (3.3.2) becomes

$$K_{X_1} \boxtimes K_{X_2} \rightarrow K_{X_1 \times X_2}. \quad (3.3.3)$$

3.4 Reflexive sheaves

Let $X \rightarrow \text{Spa } C$ be a small v-stack. For an object \mathcal{F} of $D_{\text{ét}}(X, \Lambda)$, the evaluation morphism induces by adjunction a morphism $\mathcal{F} \rightarrow \mathbf{D}\mathcal{F}$. Also, for two objects $\mathcal{F}_1, \mathcal{F}_2$ of $D_{\text{ét}}(X, \Lambda)$ we have a morphism

$$(\mathbf{D}\mathcal{F}_1 \boxtimes \mathcal{F}_2) \otimes (\mathcal{F}_1 \boxtimes \mathbf{D}\mathcal{F}_2) \xrightarrow{\text{ev} \boxtimes \text{ev}} K_X \boxtimes K_X \xrightarrow{(3.3.3)} K_{X \times X}$$

which induces by adjunction a morphism

$$(\mathbf{D}\mathcal{F}_1) \boxtimes \mathcal{F}_2 \rightarrow \mathbf{D}(\mathcal{F}_1 \boxtimes \mathbf{D}\mathcal{F}_2). \quad (3.4.1)$$

From now on we will write $\mathbf{D}\mathcal{F}_1 \boxtimes \mathcal{F}_2$ instead of $(\mathbf{D}\mathcal{F}_1) \boxtimes \mathcal{F}_2$.

Definition 3.4.1. An object $\mathcal{F} \in D_{\text{ét}}(X, \Lambda)$ is *reflexive* if it is isomorphic to a bounded complex, and if $\mathcal{F} \rightarrow \mathbf{D}\mathcal{F}$ is an isomorphism. \mathcal{F} is *strongly reflexive* if it is reflexive and if the morphism $\mathbf{D}\mathcal{F} \boxtimes \mathcal{F} \rightarrow \mathbf{D}(\mathcal{F} \boxtimes \mathbf{D}\mathcal{F})$ from (3.4.1) is an isomorphism.

Proposition 3.4.2. *Assume that $\Lambda = \mathcal{O}_E/\ell^n$, where $\ell \neq p$ is prime and E/\mathbf{Q}_ℓ is a finitely ramified algebraic extension with ring of integers \mathcal{O}_E . Let $X = \mathrm{Spd} C$, so that $D_{\acute{\mathrm{e}}\mathrm{t}}(X, \Lambda)$ is the derived category of the category of Λ -modules. If $\mathcal{F} \in D_{\acute{\mathrm{e}}\mathrm{t}}(X, \Lambda)$ is reflexive, then each $R^i\Gamma(X, \mathcal{F})$ is finitely generated. If \mathcal{F} is strongly reflexive, then each $R^i\Gamma(X, \mathcal{F})$ is free of finite rank.*

Proof. An object \mathcal{F} of $D(X_{\acute{\mathrm{e}}\mathrm{t}}, \Lambda)$ may be represented by a complex M^\bullet of Λ -modules, with $R^i\Gamma(X, \mathcal{F}) = H^i(M^\bullet)$. Since Λ is self-injective, $\mathbf{D}\mathcal{F} = \mathbf{R}\mathrm{Hom}(\mathcal{F}, \Lambda)$ may be represented by the complex $(M^\bullet)^*$ whose degree- i term is $\mathrm{Hom}(M^{-i}, \Lambda)$, and so $R^i\Gamma(X, \mathbf{D}\mathcal{F}) = H^{-i}(M^\bullet)^*$. Suppose \mathcal{F} is reflexive. Since $\mathcal{F} \rightarrow \mathbf{D}\mathbf{D}\mathcal{F}$ is an isomorphism, the Λ -module $N = R^i\Gamma(X, \mathcal{F})$ has the property that the natural morphism $N \rightarrow N^{**}$ is an isomorphism. If in addition \mathcal{F} is strongly reflexive, then N has the additional property that the natural morphism $(N^* \otimes N) \rightarrow (N \otimes N^*)^*$ is an isomorphism. The claims now follow from Lemma A.4.1. \square

Lemma 3.4.3. *1. For a proper morphism $f: X \rightarrow Y$ of v -stacks over $\mathrm{Spd} C$ and a reflexive (respectively, strongly reflexive) object $\mathcal{F} \in D_{\acute{\mathrm{e}}\mathrm{t}}(X, \Lambda)$, $f_*\mathcal{F}$ is also reflexive (respectively, strongly reflexive). In particular if $X \rightarrow \mathrm{Spa} C$ is proper and $\Lambda = \mathcal{O}_E/\ell^n$ is as in Lemma 3.4.2, then $R^i\Gamma(X, \mathcal{F}) = R^i f_*\mathcal{F}$ is finitely generated (respectively, free of finite rank).*

2. For an étale morphism $f: X \rightarrow Y$ and a reflexive (respectively, strongly reflexive) object $\mathcal{F} \in D_{\acute{\mathrm{e}}\mathrm{t}}(X, \Lambda)$, $f^\mathcal{F}$ is also reflexive (respectively, strongly reflexive).*

Proof. The reflexivity statement (1) follows from (3.2.4) and the relation $f_* = f_!$: we have $\mathbf{D}\mathbf{D}f_*\mathcal{F} \cong \mathbf{D}\mathbf{D}f_!\mathcal{F} \cong \mathbf{D}f_*\mathbf{D}\mathcal{F} \cong \mathbf{D}f_!\mathbf{D}\mathcal{F} \cong f_*\mathbf{D}\mathbf{D}\mathcal{F} \cong f_*\mathcal{F}$, so $f_*\mathcal{F}$ is reflexive. For strong reflexivity, we apply $(f \times f)_*$ to the isomorphism $\mathbf{D}\mathcal{F} \boxtimes \mathcal{F} \cong \mathbf{D}(\mathcal{F} \boxtimes \mathbf{D}\mathcal{F})$, use the Künneth isomorphism (Lemma 3.3.1) and the fact that f_* commutes with \mathbf{D} .

For (2), suppose $\mathcal{F} \in D_{\acute{\mathrm{e}}\mathrm{t}}(Y, \Lambda)$ is reflexive and $f: X \rightarrow Y$ is étale. Using the relation (3.2.5) and $f^* = f^!$ we have $\mathbf{D}\mathbf{D}f^*\mathcal{F} \cong \mathbf{D}f^!\mathbf{D}\mathcal{F} \cong \mathbf{D}f^*\mathbf{D}\mathcal{F} \cong f^!\mathbf{D}\mathbf{D}\mathcal{F} \cong f^*\mathbf{D}\mathbf{D}\mathcal{F} \cong f^*\mathcal{F}$, so \mathcal{F} is reflexive. The argument for strong reflexivity is similar, using the fact that f^* commutes with tensor products. \square

As we establish further properties of reflexive sheaves, the following “two out of six” lemma will be used repeatedly.

Lemma 3.4.4. *Let $A \rightarrow B \rightarrow C \rightarrow D$ be a sequence of morphisms in any category. If the composites $A \rightarrow C$ and $B \rightarrow D$ are isomorphisms, then all morphisms in the sequence are isomorphisms.*

For any $\mathcal{F}_1, \mathcal{F}_2 \in D(X_{\text{ét}}, \Lambda)$ we have a morphism

$$\underline{\text{RHom}}(\mathcal{F}_1, \mathcal{F}_2) \otimes \mathbf{D}\mathcal{F}_2 \rightarrow \mathbf{D}\mathcal{F}_1,$$

which induces by adjunction a morphism

$$\underline{\text{RHom}}(\mathcal{F}_1, \mathcal{F}_2) \rightarrow \underline{\text{RHom}}(\mathbf{D}\mathcal{F}_2, \mathbf{D}\mathcal{F}_1). \quad (3.4.2)$$

Lemma 3.4.5. *If \mathcal{F}_1 and \mathcal{F}_2 are reflexive then (3.4.2) is an isomorphism.*

Proof. Repeatedly applying (3.4.2) gives a sequence of morphisms:

$$\begin{array}{ccc} \underline{\text{RHom}}(\mathcal{F}_1, \mathcal{F}_2) & \longrightarrow & \underline{\text{RHom}}(\mathbf{D}\mathcal{F}_2, \mathbf{D}\mathcal{F}_1) \\ & & \downarrow \\ \underline{\text{RHom}}(\mathbf{DDD}\mathcal{F}_1, \mathbf{DDD}\mathcal{F}_2) & \longleftarrow & \underline{\text{RHom}}(\mathbf{DD}\mathcal{F}_1, \mathbf{DD}\mathcal{F}_2) \end{array}$$

The composite of the first and the second morphism is an isomorphism: If we use $\mathbf{DD}\mathcal{F}_i \cong \mathcal{F}_i$ to identify $\underline{\text{RHom}}(\mathbf{DD}\mathcal{F}_1, \mathbf{DD}\mathcal{F}_2)$ with $\underline{\text{RHom}}(\mathcal{F}_1, \mathcal{F}_2)$ then it becomes the identity. Analogously, the composite of the second and third morphisms is an isomorphism. Therefore by Lemma 3.4.4 all morphisms are isomorphisms, and in particular the first one is. \square

Lemma 3.4.6. *Let X_1 and X_2 be small v -stacks over $\text{Spd } C$, let $p_i: X_1 \times X_2 \rightarrow X_i$ be the projections, and let $\mathcal{F}_i \in D_{\text{ét}}(X_i, \Lambda)$. Assume that \mathcal{F}_1 and \mathcal{F}_2 are reflexive, and that the morphism $\mathbf{D}\mathcal{F}_1 \boxtimes \mathcal{F}_2 \rightarrow \mathbf{D}(\mathcal{F}_1 \boxtimes \mathbf{D}\mathcal{F}_2)$ of (3.4.1) is an isomorphism. Then we have an isomorphism in $D_{\text{ét}}(X_1 \times X_2, \Lambda)$:*

$$\mathbf{D}\mathcal{F}_1 \boxtimes \mathcal{F}_2 \xrightarrow{\sim} \underline{\text{RHom}}(p_1^* \mathcal{F}_1, p_2^! \mathcal{F}_2)$$

Remark 3.4.7. If $X_1 = X_2 = \text{Spa } C$, then Lemma 3.4.6 reduces to the statement that $V^* \otimes W \rightarrow \text{Hom}(V, W)$ is an isomorphism for free modules V and W of finite rank.

Proof. We will describe a sequence of morphisms

$$\mathbf{D}\mathcal{F}_1 \boxtimes \mathcal{F}_2 \rightarrow \underline{\text{RHom}}(p_1^* \mathcal{F}_1, p_2^! \mathcal{F}_2) \rightarrow \mathbf{D}(\mathcal{F}_1 \boxtimes \mathbf{D}\mathcal{F}_2) \rightarrow \underline{\text{RHom}}(\mathbf{D}p_2^! \mathcal{F}_2, \mathbf{D}p_1^* \mathcal{F}_1) \quad (3.4.3)$$

with the intent of applying the two-out-of-six lemma once again.

The first morphism in (3.4.3) is adjoint to the composition

$$(\mathbf{D}\mathcal{F}_1 \boxtimes \mathcal{F}_2) \otimes p_1^*\mathcal{F}_1 \cong (\mathbf{D}\mathcal{F}_1 \otimes \mathcal{F}_1) \boxtimes \mathcal{F}_2 \xrightarrow{\text{ev}^{\otimes 1}} K_{X_1} \boxtimes \mathcal{F}_2 \xrightarrow{3.3.2} p_2^!\mathcal{F}_2 \quad (3.4.4)$$

The second morphism in (3.4.3) is

$$\begin{aligned} \underline{\text{RHom}}(p_1^*\mathcal{F}_1, p_2^!\mathcal{F}_2) &\xrightarrow{-\otimes^1} \underline{\text{RHom}}(p_1^*\mathcal{F}_1 \otimes p_2^*\mathbf{D}\mathcal{F}_2, p_2^!\mathcal{F}_2 \otimes p_2^*\mathbf{D}\mathcal{F}_2) \\ &\xrightarrow{t_{p_2^!}^!} \underline{\text{RHom}}(p_1^*\mathcal{F}_1 \otimes p_2^*\mathbf{D}\mathcal{F}_2, p_2^!(\mathcal{F}_2 \otimes \mathbf{D}\mathcal{F}_2)) \\ &\xrightarrow{\text{ev}} \underline{\text{RHom}}(p_1^*\mathcal{F}_1 \otimes p_2^*\mathbf{D}\mathcal{F}_2, K_{X_1 \times X_2}) \\ &= \mathbf{D}(\mathcal{F}_1 \boxtimes \mathbf{D}\mathcal{F}_2). \end{aligned}$$

For the third morphism in (3.4.3), we start with evaluation morphism

$$\mathbf{D}(\mathcal{F}_1 \boxtimes \mathbf{D}\mathcal{F}_2) \otimes p_1^*\mathcal{F}_1 \otimes p_2^*\mathbf{D}\mathcal{F}_2 = \mathbf{D}(\mathcal{F}_1 \boxtimes \mathbf{D}\mathcal{F}_2) \otimes \mathcal{F}_1 \boxtimes \mathbf{D}\mathcal{F}_2 \rightarrow K_{X_1 \times X_2},$$

which induces by adjunction

$$\mathbf{D}(\mathcal{F}_1 \boxtimes \mathbf{D}\mathcal{F}_2) \otimes p_2^*\mathbf{D}\mathcal{F}_2 \rightarrow \mathbf{D}p_1^*\mathcal{F}_1,$$

which in turn induces by adjunction

$$\mathbf{D}(\mathcal{F}_1 \boxtimes \mathbf{D}\mathcal{F}_2) \rightarrow \underline{\text{RHom}}(p_2^*\mathbf{D}\mathcal{F}_2, \mathbf{D}p_1^*\mathcal{F}_1).$$

In (3.4.3), the composition of the first and second morphisms is an isomorphism by hypothesis, and the composition of the second and third morphisms is an isomorphism by Lemma 3.4.5. Thus we conclude by the two-out-of-six lemma. \square

Consider now a morphism $c: Y \rightarrow X \times X$. We call such a morphism a *correspondence* on X , and its compositions with the two projections $X \times X \rightarrow X$ are denoted by $c_1, c_2: Y \rightarrow X$. Combining (3.2.2) with Lemma 3.4.6 gives the following corollary.

Corollary 3.4.8. *Let $c: Y \rightarrow X \times X$ be a correspondence. Let $\mathcal{F} \in D(X_{\text{ét}}, \Lambda)$ be strongly reflexive. Then we have an isomorphism*

$$\underline{\text{RHom}}(c_1^*\mathcal{F}, c_2^!\mathcal{F}) \xrightarrow{\sim} c^!(\mathbf{D}\mathcal{F} \boxtimes \mathcal{F}).$$

3.5 The Lefschetz fixed-point formula for diamonds

We follow [Var07, §1]. Let X be a proper v-stack over $\mathrm{Spd} C$. Suppose we are given the following data:

1. A correspondence $c: Y \rightarrow X \times X$:

$$\begin{array}{ccc} & Y & \\ c_1 \swarrow & & \searrow c_2 \\ X & & X \end{array},$$

where we assume c_2 is proper. Thus, in particular, Y is proper over $\mathrm{Spd} C$;

2. A strongly reflexive sheaf \mathcal{F} in $D_{\acute{e}t}(X, \Lambda)$.

Let $\Delta: X \rightarrow X \times X$ be the diagonal map, and let $\mathrm{Fix}(c)$ be the fixed point locus of c , defined as the fiber product

$$\begin{array}{ccc} \mathrm{Fix}(c) & \xrightarrow{c'} & X \\ \Delta' \downarrow & & \downarrow \Delta \\ Y & \xrightarrow{c} & X \times X. \end{array} \quad (3.5.1)$$

The composition $\Delta^*(\mathbf{D}\mathcal{F} \boxtimes \mathcal{F}) = \mathbf{D}\mathcal{F} \otimes \mathcal{F} \xrightarrow{\mathrm{ev}} K_X$ induces by adjunction a morphism

$$\mathbf{D}\mathcal{F} \boxtimes \mathcal{F} \rightarrow \Delta_* K_X. \quad (3.5.2)$$

We have a series of morphisms

$$\begin{aligned} \underline{\mathrm{RHom}}(c_1^* \mathcal{F}, c_2^! \mathcal{F}) &\xrightarrow{\mathrm{Cor. 3.4.8}} c^!(\mathbf{D}\mathcal{F} \boxtimes \mathcal{F}) \\ &\xrightarrow{(3.5.2)} c^! \Delta_* K_X \\ &\xrightarrow{\mathrm{BC}} \Delta'_* c'^! K_X \\ &= \Delta'_* K_{\mathrm{Fix}(c)} \end{aligned}$$

whose composition we will call tr_c .

Definition 3.5.1. Let β be a connected component of $\mathrm{Fix}(c)$, and let $p: \beta \rightarrow \mathrm{Spd} C$ be the structure map. (It follows from the properness of c_2 that p is proper.) Let $U \subset Y$ be an open subdiamond containing $\Delta'(\beta)$, and let $u: c_1^* \mathcal{F}|_U \rightarrow c_2^! \mathcal{F}|_U$ be a morphism. We refer to u as a *local cohomological*

correspondence *lifting* c defined in a neighborhood of β . The *local term* $\text{loc}_\beta(u, \mathcal{F})$ is the image of u under

$$\begin{aligned}
\text{Hom}_U(c_1^* \mathcal{F}, c_2^! \mathcal{F}) &= H^0(U, \underline{\text{RHom}}(c_1^* \mathcal{F}, c_2^! \mathcal{F})) \\
&\xrightarrow{\text{tr}_c(U)} H^0(U, \Delta'_* K_{\text{Fix}(c)}) \\
&\cong H^0(\Delta'^{-1}(U), K_{\text{Fix}(c)}) \\
&\rightarrow H^0(\beta, K_\beta) \\
&= p_! p^! \Lambda \\
&\xrightarrow{\text{adj.}} \Lambda
\end{aligned}$$

Observe that $\text{loc}_\beta(u, \mathcal{F})$ does not depend on the choice of the neighborhood U of $\Delta'(\beta)$; that is, the local term really is local.

Example 3.5.2. If $b: \text{Spd } C \times \text{Spd } C \rightarrow \text{Spd } C$ is the trivial correspondence, and \mathcal{F} is a strongly reflexive sheaf on $\text{Spa } C$ corresponding to the free finite-rank Λ -module $V = H^0(\text{Spd } C, \mathcal{F})$, then $\text{Hom}(b_1^* \mathcal{F}, b_2^! \mathcal{F})$ can be identified with $\text{End } V$, and tr_b corresponds to the trace map $\text{End } V \rightarrow \Lambda$.

The structure maps to $\text{Spd } C$ determine a proper morphism of correspondences $\pi: c \rightarrow b$ in the sense of [Var07, Definition 1.1.1(b)], which induces a diagram

$$\begin{array}{ccc}
\text{Hom}(c_1^* \mathcal{F}, c_2^! \mathcal{F}) & \xrightarrow{\text{tr}_c} & H^0(\text{Fix}(c), K_{\text{Fix}(c)}) \\
\pi_! \downarrow & & \downarrow \pi_! \\
\text{End } R\Gamma_c(X, \mathcal{F}) & \xrightarrow{\text{tr}} & \Lambda
\end{array} \tag{3.5.3}$$

where the vertical arrows are the evident morphisms induced by π .

Proposition 3.5.3. *The diagram in (3.5.3) commutes.*

Proof. This is the analogue of [Var07, Proposition 1.2.5]. The proof of the latter ((4.3.4) in [Var07]) relies only upon the tools we've developed so far (especially the base change theorems), which are available in the situation of small v-stacks. \square

Corollary 3.5.4. *Keep the assumption that $\pi: c \rightarrow b$ is proper, where b is the trivial correspondence on $\text{Spa } C$. For $u \in \text{Hom}(c_1^* \mathcal{F}, c_2^! \mathcal{F})$ we have*

$$\text{tr}(\pi_! u | R\Gamma(X, \mathcal{F})) = \sum_{\beta \in \pi_0(\text{Fix}(c))} \text{loc}_\beta(u).$$

3.6 Pro-étale local systems

For a profinite topological space S , let $\underline{S} = \mathrm{Spa}(A, A^\circ)$, where A is the ring of continuous functions $S \rightarrow \mathbb{C}$. Then \underline{S} is a perfectoid space whose underlying topological space is S . By gluing, one can define \underline{S} for any locally profinite topological space. Note that for any small v-stack X over $\mathrm{Spd} \mathbb{C}$, a morphism $X \rightarrow \underline{S}$ is the same thing as a continuous map $|X| \rightarrow S$. In the case that $S = G$ is a locally profinite *group*, it makes sense to talk about a \underline{G} -action on a diamond X : this is a morphism $\underline{G} \times X \rightarrow X$ satisfying the appropriate axioms.

Definition 3.6.1. Let G be a locally profinite group. A *pro-étale \underline{G} -torsor* is G -equivariant morphism $X' \rightarrow X$, where X' and X are small v-stacks and G acts on X' and trivially on X , such that $X = X'/\underline{G}$, and such that for all compact open subgroups $H \subset G$, $X'/\underline{H} \rightarrow X$ is an étale surjection.

Definition 3.6.2. Let X be a small v-stack.

1. For a free Λ -module V , the *constant sheaf* V_X on $X_{\acute{e}t}$ is defined by $U \mapsto \mathrm{Cont}(|U|, V)$ for $U \in X_{\acute{e}t}$. (Here V is given the discrete topology.)
2. A sheaf \mathcal{L} of Λ -modules on $X_{\acute{e}t}$ is a *local system* if there exists an étale cover $\{U_i\}$ of X such that $\mathcal{L}|_{U_i}$ is a constant sheaf for all i .
3. Let H be a profinite group, and let $X' \rightarrow X$ be a pro-étale \underline{H} -torsor. Let $\pi: H \rightarrow \mathrm{GL}(V)$ be a smooth action of H on a free Λ -module V . Let \mathcal{L}_π be the sheaf on $X_{\acute{e}t}$ defined by

$$U \mapsto \mathrm{Cont}_H(|U \times_X X'|, V),$$

where Cont_H means the module of continuous H -equivariant maps. A sheaf \mathcal{L} of Λ -modules on $X_{\acute{e}t}$ is a *pro-étale local system* if there exists an étale cover $\{U_i\}$ of X such that for all i , $\mathcal{L}|_{U_i}$ is isomorphic to a sheaf of the form \mathcal{L}_{π_i} . The sheaf \mathcal{L} is *admissible* if each π_i can be taken to be an admissible representation.

Example 3.6.3. Let G be a locally profinite group, with profinite open subgroup $H \subset G$.

Let $X' \rightarrow X$ be a \underline{G} -torsor of locally spatial diamonds. Let π be a smooth representation of G on a free Λ -module V , and let \mathcal{L}_π be the sheaf on $X_{\acute{e}t}$ defined by $U \mapsto \mathrm{Cont}_G(|U \times_X X'|, V)$. Then \mathcal{L}_π is a pro-étale local system. Indeed, $X'/\underline{H} \rightarrow X$ is an étale cover, and the restriction of \mathcal{L}_π to X'/\underline{H} is $\mathcal{L}_{\pi|_H}$. If π is admissible, then so is \mathcal{L}_π .

Lemma 3.6.4. *Let H be a profinite group, and let $X' \rightarrow X$ be a pro-étale \underline{H} -torsor of locally spatial diamonds. Let (π_i, V_i) be a directed system of smooth representations of H on free Λ -modules V_i . Assume that (π, V) is another such representation, which is the colimit of the (π_i, V_i) . Then $\mathcal{L}_\pi \cong \varinjlim \mathcal{L}_{\pi_i}$ in the category of sheaves on $X_{\text{ét}}$.*

Proof. Since X is locally spatial, $X_{\text{ét}}$ admits a basis consisting of morphisms $U \rightarrow X$ where U is quasi-compact. It suffices to show that $H^0(U, \mathcal{L}_\pi) \cong \varinjlim H^0(U, \mathcal{L}_{\pi_i})$ for such a U . By definition, $H^0(U, \mathcal{L}_\pi) = \text{Cont}_H(|U \times_X X'|, V)$. The morphism $U \times_X X' \rightarrow U$ is an \underline{H} -torsor, and so $U \times_X X'$ is also quasi-compact. Therefore a continuous map $|U \times_X X'| \rightarrow V$ must have finite image, and so it must factor through some V_i . Therefore a section in $H^0(U, \mathcal{L}_\pi)$ must factor through a section in $H^0(U, \mathcal{L}_{\pi_i})$. \square

Lemma 3.6.4 shows that an admissible pro-étale local system is locally isomorphic to a colimit of local systems with finite fibers. If H is a first-countable pro- p group, then the following discussion shows that the colimit can be replaced with a direct sum.

Lemma 3.6.5. *Let π be an admissible representation of a first-countable pro- p group H on a free Λ -module V . Then π is the direct sum of countably many representations on free Λ -modules of finite rank.*

Proof. Since H is pro- p , and p is invertible in Λ , there exists a normalized Haar measure ν on H with coefficients in Λ . For any compact open normal subgroup $K \subset H$, let V^K be the submodule of K -fixed vectors in V ; it is H -stable and finitely generated. We have the Hecke operator $\pi(1_K): V \rightarrow V$, defined by

$$\pi(1_K)(v) = [H : K] \int_{k \in K} \pi(k)v \, dk.$$

Then $\pi(1_K)$ is an H -equivariant section of the inclusion $V^K \rightarrow V$. Therefore V^K is a free Λ -module, since it is a summand of a free module and Λ is local.

Let $K_1 \supset K_2 \supset \dots$ be a basis of compact open normal subgroups. For $i \geq 1$, the operator $\pi(1_{K_i})$ can be used to split $V^{K_i} \rightarrow V^{K_{i+1}}$; let W_i be the complement. Also let $W_0 = V^{K_1}$. Then $V = \bigoplus_{i \geq 0} W_i$. \square

Theorem 3.6.6. *Let X be a small v -stack, and let $\mathcal{F} \in D_{\text{ét}}(X, \Lambda)$. Let $X' \rightarrow X$ be a G -torsor, where G is a first-countable locally pro- p group, let π be an admissible representation of G on a free Λ -module, and let \mathcal{L}_π be the corresponding pro-étale local system on X . Assume that $\mathcal{L}_\pi \otimes \mathcal{F}$ is reflexive, and assume that for all quasi-compact $U \in X_{\text{ét}}$, $R^i\Gamma(U, \mathcal{F})$ is finitely generated.*

1. Suppose that G is pro- p . We have an isomorphism

$$\mathcal{L}_\pi \otimes \mathcal{F} \cong \varprojlim_H \mathcal{L}_{\pi H} \otimes \mathcal{F},$$

where H runs over a basis of compact open normal subgroups of G , and the transition maps are induced by the operators $\pi(1_H)$.

2. $\mathbf{D}(\mathcal{L}_\pi \otimes \mathcal{F}) \cong \mathcal{L}_{\pi^\vee} \otimes \mathbf{D}\mathcal{F}$, where π^\vee is the smooth dual of π .

3. We have an isomorphism

$$\mathrm{RHom}(R\Gamma_c(X, \mathcal{L}_\pi \otimes \mathcal{F}), \Lambda) \cong \mathrm{RHom}_G(R\Gamma_c(X', \mathcal{F}), \pi^\vee).$$

Proof. For (1), it suffices to show that the two objects have the same derived global sections over a quasi-compact $U \in X_{\text{ét}}$. By Lemma 3.6.4 we have $R\Gamma(U, \mathcal{L}_\pi \otimes \mathcal{F}) = \varinjlim_H R\Gamma(U, \mathcal{L}_{\pi H} \otimes \mathcal{F})$. By hypothesis, each $R\Gamma(U, \mathcal{L}_{\pi H} \otimes \mathcal{F})$ is a bounded complex of finitely generated modules, and so $R\Gamma(U, \mathcal{L}_\pi \otimes \mathcal{F})$ is countably generated. On the other hand, since $\mathcal{L}_\pi \otimes \mathcal{F}$ is reflexive, we have

$$R\Gamma(U, \mathcal{L}_\pi \otimes \mathcal{F}) \cong R\Gamma(U, \mathbf{D}\mathbf{D}(\mathcal{L}_\pi \otimes \mathcal{F}))$$

is dual to $R\Gamma_c(U, \mathbf{D}(\mathcal{L}_\pi \otimes \mathcal{F}))$, and therefore it is finitely generated by Lemma A.4.2. Thus there exists H for which $R\Gamma(U, \mathcal{L}_{\pi H} \otimes \mathcal{F}) \rightarrow R\Gamma(U, \mathcal{L}_\pi \otimes \mathcal{F})$ is an isomorphism, whose inverse is $\pi(1_H)$.

For (2), it is a statement which is local on $X_{\text{ét}}$, and so we may assume that G is pro- p . By (1) we have

$$\begin{aligned} \mathbf{D}(\mathcal{L}_\pi \otimes \mathcal{F}) &\cong \mathbf{D}(\varprojlim_H \mathcal{L}_{\pi H} \otimes \mathcal{F}) \\ &\cong \varinjlim_H \mathbf{D}(\mathcal{L}_{\pi H} \otimes \mathcal{F}) \\ &\cong \varinjlim_H \mathcal{L}_{(\pi H)^\vee} \otimes \mathbf{D}\mathcal{F} \\ &\cong \mathcal{L}_{\pi^\vee} \otimes \mathbf{D}\mathcal{F}, \end{aligned}$$

where the last step uses Lemma 3.6.4.

For (3): We evaluate $R\Gamma(X, \mathcal{L}_{\pi^\vee} \otimes \mathbf{D}\mathcal{F})$ in two ways. On the one hand, by (1) we have $\mathcal{L}_{\pi^\vee} \otimes \mathbf{D}\mathcal{F} \cong \mathbf{D}(\mathcal{L}_\pi \otimes \mathcal{F})$, and so by (3.2.4) $R\Gamma(X, \mathcal{L}_{\pi^\vee} \otimes \mathbf{D}\mathcal{F}) \cong \underline{\mathrm{RHom}}(R\Gamma_c(X, \mathcal{L}_\pi \otimes \mathcal{F}), \Lambda)$.

On the other hand we claim $R\Gamma(X, \mathcal{L}_{\pi^\vee} \otimes \mathbf{D}\mathcal{F}) \cong \underline{\mathrm{RHom}}_G(R\Gamma_c(X', \mathcal{F}), \pi^\vee)$. Again, the statement is local on X , so it suffices to prove this in the case

that G is pro- p . Applying (1) we have

$$\begin{aligned} R\Gamma(X, \mathcal{L}_{\pi^\vee} \otimes \mathbf{D}\mathcal{F}) &\cong \varprojlim_H R\Gamma(X, \mathcal{L}_{(\pi^\vee)^H} \otimes \mathbf{D}\mathcal{F}) \\ &\cong \varprojlim_H R\Gamma(X'/\underline{H}, \mathcal{L}_{(\pi^\vee)^H} \otimes \mathbf{D}\mathcal{F})^{G/H}, \end{aligned}$$

where we are taking derived G/H invariants. Since $\mathcal{L}_{(\pi^\vee)^H}$ is constant on X'/\underline{H} with finite fibers:

$$\begin{aligned} R\Gamma(X, \mathcal{L}_{\pi^\vee} \otimes \mathbf{D}\mathcal{F}) &\cong \varprojlim_H (R\Gamma(X'/\underline{H}, \mathbf{D}\mathcal{F}) \otimes (\pi^\vee)^H)^{G/H} \\ &\cong \varprojlim_H \mathrm{RHom}_{G/H}(R\Gamma_c(X'/\underline{H}, \mathcal{F}), (\pi^\vee)^H) \\ &\cong \mathrm{RHom}_G(\varinjlim_H R\Gamma_c(X'/\underline{H}, \mathcal{F}), \pi^\vee) \\ &\cong \mathrm{RHom}_G(R\Gamma_c(X', \mathcal{F}), \pi^\vee) \end{aligned}$$

□

Example 3.6.7. We can now give an example of a reflexive local system with infinite fibers. Let D/C be the rigid unit disk around 1, considered as a group under multiplication, and let $\tilde{D} = \varprojlim_p D$. Then $\tilde{D}^* = \tilde{D} \setminus 0$ is a perfectoid space admitting a continuous action of \mathbf{Q}_p^\times without geometric fixed points, so that $\tilde{D}^*/\underline{\mathbf{Q}}_p^\times$ is a diamond. (This diamond appears in [Wei16]).

We have the pro-étale \mathbf{Z}_p -torsor $\tilde{D}^*/p^{\mathbf{Z}} \rightarrow \tilde{D}^*/\mathbf{Q}_p^\times$. Let $\Lambda = \overline{\mathbf{F}}_\ell$, and let π be an infinite-dimensional admissible representation of \mathbf{Z}_p^\times on a Λ -vector space (namely, the direct sum of a sequence of characters of increasing conductor).

For $U \in (\tilde{D}^*/\mathbf{Q}_p^\times)_{\text{ét}}$ qcqs, let $U' \in (\tilde{D}^*/p^{\mathbf{Z}})_{\text{ét}}$ be its preimage, a qcqs perfectoid space with a continuous \mathbf{Z}_p^\times -action. Then U' is the completed perfection of a quasi-compact rigid space. By [Hub96, Proposition 6.1.1], each $H^i(U'_{\text{ét}}, \Lambda)$ is finite, and so there exists an open subgroup $H \subset \mathbf{Z}_p^\times$ which acts trivially on each one. From this one can deduce as in Theorem 3.6.6 that $\mathcal{L}_\pi \cong \varprojlim_H \mathcal{L}_{\pi^H}$, and so $\mathbf{D}\mathcal{L}_\pi \cong \varinjlim_H \mathbf{D}\mathcal{L}_{\pi^H} \cong \varinjlim_H \mathcal{L}_{(\pi^\vee)^H}[2](1) \cong \mathcal{L}_{\pi^\vee}[2](1)$.

3.7 Interaction between local terms and pro-étale local systems

Let X be a small v-stack, let $c = (c_1, c_2): Y \rightarrow X \times X$ be a proper correspondence, and let $\mathcal{F} \in D_{\text{ét}}(X, \Lambda)$ be strongly reflexive. We assume that c_2

is étale. Then we have $c_2^! = c_2^*$. Let $y \in \text{Fix}(c)$ be an isolated fixed point with image $x = c_1(y) = c_2(y)$, and let $u_{\mathcal{F}}$ be a cohomological correspondence lifting c , i.e. $u_{\mathcal{F}} \in \text{Hom}(c_1^* \mathcal{F}, c_2^! \mathcal{F})$. We have defined the local term $\text{loc}_y(u_{\mathcal{F}})$ in Definition 3.5.1. Now assume given an admissible pro-étale local system \mathcal{L} on X and a cohomological correspondence $u_{\mathcal{L}}: c_1^* \mathcal{L} \rightarrow c_2^* \mathcal{L}$. Assume further that $\mathcal{L} \otimes \mathcal{F}$ is again strongly reflexive. We shall need an expression of $\text{loc}_y(u_{\mathcal{L}} \otimes u_{\mathcal{F}})$ as a product of $\text{loc}_y(u_{\mathcal{F}})$ by a suitably defined trace of $u_{\mathcal{L}}$ on the fiber \mathcal{L}_x . Note that the fiber of $u_{\mathcal{L}}$ at y is an endomorphism $u_{\mathcal{L},y}: \mathcal{L}_x \rightarrow \mathcal{L}_x$.

We build up this result in stages, starting with the case that \mathcal{L} has finite fibers.

Lemma 3.7.1. *Assume that \mathcal{L} has finite fibers. Then $\mathcal{L} \otimes \mathcal{F} \in D(X_{\text{ét}}, \Lambda)$ is strongly reflexive and*

$$\text{loc}_y(u_{\mathcal{L}} \otimes u_{\mathcal{F}}) = \text{tr}(u_{\mathcal{L},y} | \mathcal{L}_x) \text{loc}_y(u_{\mathcal{F}}).$$

Proof. The property strongly reflexive is local on X . If $U \rightarrow X$ is an étale morphism for which $\mathcal{L}|_U \cong V_U$ for a finite free Λ -module V , then $(\mathcal{L} \otimes \mathcal{F})|_U \cong V_U \otimes \mathcal{F}|_U$ is a direct sum of finitely many copies of $\mathcal{F}|_U$. But a direct sum of strongly reflexive sheaves is again strongly reflexive.

The relation involving local terms is also local on X . Choose an étale neighborhood U of x which trivializes \mathcal{L} , so that we have an isomorphism $\mathcal{L}_U \cong \mathcal{L}_x \otimes \Lambda_U$. Since \mathcal{L}_x is a free Λ -module of finite rank, and all of our six functors preserve finite direct sums, we have an isomorphism

$$\underline{\text{RHom}}(c_1^*(\mathcal{L} \otimes \mathcal{F}), c_2^!(\mathcal{L} \otimes \mathcal{F})) \cong \text{End } \mathcal{L}_x \otimes \underline{\text{RHom}}(c_1^* \mathcal{F}, c_2^! \mathcal{F})$$

which carries $u_{\mathcal{L}} \otimes u_{\mathcal{F}}$ onto $u_{\mathcal{L},x} \otimes u_{\mathcal{F}}$. Tracing through the construction of tr_c , one sees that the diagram

$$\begin{array}{ccc} \text{Hom}_U(c_1^*(\mathcal{L} \otimes \mathcal{F}), c_2^!(\mathcal{L} \otimes \mathcal{F})) & \longrightarrow & \text{End } \mathcal{L}_x \otimes \text{Hom}_U(c_1^* \mathcal{F}, c_2^! \mathcal{F}) \\ \text{tr}_c \downarrow & & \downarrow \text{tr} \otimes \text{tr}_c \\ \Lambda & \xrightarrow{=} & \Lambda \end{array}$$

commutes. □

We now want to drop the assumption that \mathcal{L} has finite fibers. Let G be a locally pro- p group which admits a countable basis of open normal subgroups, let $X' \rightarrow X$ be a G -torsor, and let π be an admissible representation of G on a free (possibly infinite-rank) Λ -module V . Let $\mathcal{L} = \mathcal{L}_{\pi}$ be the corresponding pro-étale local system on X .

We assume that $\mathcal{L} \otimes \mathcal{F}$ is strongly reflexive. Under this assumption we'd like to give a generalization of the formula for the local terms in Lemma 3.7.1. The difficulty is that $u_{\mathcal{L},y}: \mathcal{L}_x \rightarrow \mathcal{L}_x$ has no *a priori* well-defined trace. The idea is to show that a sort of Harish-Chandra trace exists. After passing to an étale neighborhood of y , we may assume that G is pro- p . For each pro- p compact open normal subgroup $H \subset G$, we have the operator $\pi(1_H): V \rightarrow V$, which commutes with the action of G . Therefore it induces a morphism $\pi(1_H): \mathcal{L} \rightarrow \mathcal{L}$ with image lying in $\mathcal{L}_{\pi H}$. Since V^H is a free Λ -module of finite rank, we may define the trace of $\pi(1_H)_y \circ u_{\mathcal{L},y}$ as the trace of the restriction of this operator to $\mathcal{L}_{\pi H,x}$.

Proposition 3.7.2. *Assume that $\mathcal{L} \otimes \mathcal{F}$ is strongly reflexive. Then for all sufficiently small open subgroups $H \subset G$ we have*

$$\mathrm{loc}_y(u_{\mathcal{L}} \otimes u_{\mathcal{F}}) = \mathrm{tr}(\pi(1_H) \circ u_{\mathcal{L},y}|_{\mathcal{L}_x}) \mathrm{loc}_y(u_{\mathcal{F}}).$$

Proof. Let $\mathcal{F}_\pi = \mathcal{L}_\pi \otimes \mathcal{F}|_U$. By Corollary 3.4.8, $u_{\mathcal{L}} \otimes u_{\mathcal{F}}$ determines an element of $H^0(Y, c^!(\mathbf{D}\mathcal{F}_\pi \boxtimes \mathcal{F}_\pi)) = \mathrm{Hom}_Y(\Lambda, c^!(\mathbf{D}\mathcal{F}_\pi \boxtimes \mathcal{F}_\pi))$. Tracing through the definition of tr_c , the local term morphism $\mathrm{loc}_y: \mathrm{Hom}(c_1^* \mathcal{F}_\pi, c_2^! \mathcal{F}_\pi) \rightarrow \Lambda$ factors through the following series of morphisms:

$$\begin{aligned} \mathrm{Hom}(c_1^* \mathcal{F}_\pi, c_2^! \mathcal{F}_\pi) &\rightarrow \mathrm{Hom}_Y(\Lambda, c_U^!(\mathbf{D}\mathcal{F}_\pi \boxtimes \mathcal{F}_\pi)) \\ &\rightarrow \mathrm{Hom}_Y(\Lambda, c_U^! \Delta_*(\mathbf{D}\mathcal{F}_\pi \otimes \mathcal{F}_\pi)) \\ &\xrightarrow{\sim} \mathrm{Hom}_Y(\Lambda, i_{y*} i_x^!(\mathbf{D}\mathcal{F}_\pi \otimes \mathcal{F}_\pi)) \\ &\xrightarrow{\sim} \mathrm{Hom}_{\{y\}}(\Lambda, i_x^!(\mathbf{D}\mathcal{F}_\pi \otimes \mathcal{F}_\pi)) \\ &\xrightarrow{\sim} \mathrm{Hom}_X(i_x! \Lambda, \mathbf{D}\mathcal{F}_\pi \otimes \mathcal{F}_\pi) \\ &\rightarrow \mathrm{Hom}_X(j_{V!} \Lambda, \mathbf{D}\mathcal{F}_\pi \otimes \mathcal{F}_\pi), \end{aligned}$$

where in the last step, $j_V: V \rightarrow X$ is a quasi-compact étale neighborhood of x , so that there is a morphism $j_{V!} \Lambda \rightarrow i_x! \Lambda$ adjoint to $\Lambda \cong i_x^* j_{V!} \Lambda \rightarrow \Lambda$. By Lemma 3.6.4 \mathcal{L}_π is a colimit of the local systems $\mathcal{L}_{\pi H}$, where H runs over compact subgroups of G . The functor $j_{V!}$ has a right adjoint $j_V^! = j_V^*$ which is also a left adjoint, so that $j_V^!$ preserves colimits. Therefore $j_{V!} \Lambda$ is a compact object of $D^b(X, \Lambda)$, and hence

$$\mathrm{Hom}_X(j_{V!} \Lambda, \mathbf{D}\mathcal{F}_\pi \otimes \mathcal{F}_\pi) \cong \varinjlim_H \mathrm{Hom}_X(j_{V!} \Lambda, \mathbf{D}\mathcal{F}_\pi \otimes \mathcal{F}_{\pi H}).$$

Given an element $u \in \mathrm{Hom}(c_1^* \mathcal{F}_\pi, c_2^! \mathcal{F}_\pi)$, the local term $\mathrm{loc}_y(u, \mathcal{F}_\pi)$ is the image of u under

$$\mathrm{Hom}_X(j_{V!} \Lambda, \mathbf{D}\mathcal{F}_\pi \otimes \mathcal{F}_{\pi H}) \rightarrow \mathrm{Hom}_X(j_{V!} \Lambda, K_X) \rightarrow \Lambda$$

for all sufficiently small open subgroups $H \subset G$. But u and $\pi(1_H) \circ u$ have the same image in $\mathrm{Hom}_X(j_{V!}\Lambda, \mathbf{DF}_\pi \otimes \mathcal{F}_{\pi H})$, and therefore they have the same local term. The local term of $\pi(1_H) \circ u$ is the same as the local term of the restriction of $\pi(1_H) \circ u$ to $\mathcal{L}_{\pi H} \otimes \mathcal{F}$.

Now suppose $u = u_{\mathcal{L}} \otimes u_{\mathcal{F}}$ as in the lemma. Since $\mathcal{L}_{\pi H}$ has finite fibers, Lemma 3.7.1 applies to give $\mathrm{loc}_y(\pi(1_H) \circ u) = \mathrm{tr}(\pi(1_H) \circ u_{\mathcal{L},y}) \mathrm{loc}_y(u_{\mathcal{F}})$. \square

3.8 Quotients by pro- p -groups

For our applications we will need to gather some results for small v-stacks of the form $[X/J]$, where X is a locally spatial diamond and J is a locally pro- p group which acts continuously on X . (The continuity of the action means that the action map comes from a morphism of diamonds $\alpha: J \times X \rightarrow X$.)

Some of the following material is adapted from [Sch15, Section 2].

Definition 3.8.1. Let X be a spatial diamond admitting a continuous action by a profinite group H . Let $[X/H]_{\acute{e}t}$ be the site whose objects are diamonds Y equipped with a continuous action of H and an H -equivariant étale morphism $Y \rightarrow X$. Morphisms are H -equivariant maps over H , and covers are topological covers.

Proposition 3.8.2. *Let X be a qcqs spatial diamond admitting a continuous action by a profinite group H . Let Y be a qc separated diamond, and let $Y \rightarrow X$ be an étale morphism. Then there exists an open subgroup $K \subset H$ and a continuous action of K on Y such that $Y \rightarrow X$ is K -equivariant.*

Proof. For an open subgroup $K \subset H$, let α_K and pr_K denote the action and projection morphisms $\underline{K} \times X \rightarrow X$, respectively. We are seeking an open subgroup $K \subset H$ together with an isomorphism $\alpha_K^* Y \cong \mathrm{pr}_K^* Y$ in $(\underline{K} \times X)_{\acute{e}t}$ satisfying the appropriate cocycle condition over $\underline{K} \times \underline{K} \times X$.

As $K \subset H$ runs through open subgroups, $\underline{K} \times X$ constitutes a cofiltered inverse system of qcqs diamonds, whose inverse limit is X . That is, the inclusions $i_K: X \rightarrow \underline{K} \times X$ along the origin induce an isomorphism $i: X \xrightarrow{\sim} \varprojlim_K \underline{K} \times X$. By [Sch17, Proposition 9.23], the functor

$$i^*: 2\text{-}\varprojlim_K (\underline{K} \times X)_{\acute{e}t, \mathrm{qc}, \mathrm{sep}} \rightarrow X_{\acute{e}t, \mathrm{qc}, \mathrm{sep}} \quad (3.8.1)$$

is an equivalence, where the subscripts indicate the category of étale morphisms from diamonds which are quasicompact and separated. The morphism $Y \rightarrow X$ is an object of $X_{\acute{e}t, \mathrm{qc}, \mathrm{sep}}$. Since $\mathrm{pr}_H \circ i = \alpha_H \circ i$, the objects $i^* \mathrm{pr}_H^* Y$ and $i^* \alpha_H^* Y$ are isomorphic, and so (since i^* is an equivalence) there

exists an open subgroup $K \subset H$ such that $\mathrm{pr}_K^* Y$ and $\mathrm{pr}_K^* Y$ have isomorphic images in $(\underline{K} \times X)_{\acute{\mathrm{e}}\mathrm{t}}$, which is to say that there is an isomorphism $\alpha_K^* Y \cong \mathrm{pr}_K^* Y$. A similar argument shows that (possibly after shrinking K) this isomorphism satisfies the cocycle condition. \square

Let \mathcal{F}_H be an object of $D([X/H]_{\acute{\mathrm{e}}\mathrm{t}}, \Lambda)$. It follows formally from Proposition 3.8.2 that

$$H^i(X_{\acute{\mathrm{e}}\mathrm{t}}, \mathcal{F}) \cong \varinjlim_K H^i([X/K]_{\acute{\mathrm{e}}\mathrm{t}}, \mathcal{F}_K), \quad (3.8.2)$$

where $K \subset H$ runs through open subgroups, and \mathcal{F}_K and \mathcal{F} denote the obvious pullbacks, cf. [Sch15, Proposition 2.8].

Lemma 3.8.3. *Suppose H is a pro- p group acting on a qcqs spatial diamond X , and let \mathcal{F}_H be an object of $D([X/H]_{\acute{\mathrm{e}}\mathrm{t}}, \Lambda)$. Then the map*

$$H^i([X/H]_{\acute{\mathrm{e}}\mathrm{t}}, \mathcal{F}_H) \cong H^i(X, \mathcal{F})^H$$

is an isomorphism. In particular the action of H on $H^i(X_{\acute{\mathrm{e}}\mathrm{t}}, \mathcal{F})$ is smooth.

Proof. Let $K \subset H$ be an open normal subgroup. Consider the Čech spectral sequence corresponding to the cover $[X/K] \rightarrow [X/H]$:

$$H^i(H/K, H^j([X/K]_{\acute{\mathrm{e}}\mathrm{t}}, \mathcal{F}_K)) \implies H^{i+j}([X/H]_{\acute{\mathrm{e}}\mathrm{t}}, \mathcal{F}_H).$$

Since H/K is a p -group and Λ is prime-to- p torsion, the left-hand side vanishes for $i > 0$, and so

$$H^i([X/K]_{\acute{\mathrm{e}}\mathrm{t}}, \mathcal{F}_K)^H \cong H^i([X/H]_{\acute{\mathrm{e}}\mathrm{t}}, \mathcal{F}_H).$$

Taking the colimit over $K \subset H$ and applying (3.8.2) gives the result. \square

Lemma 3.8.4. *Suppose J is a locally pro- p group acting on a qcqs diamond X , and let \mathcal{F}_J be an object of $D([X/J]_{\acute{\mathrm{e}}\mathrm{t}}, \Lambda)$. If \mathcal{F}_J is reflexive (respectively, strongly reflexive), then so is \mathcal{F}_K for each open subgroup $K \subset J$, and then $H^i(X, \mathcal{F})$ is an admissible representation of J (respectively, an admissible representation of J on a free Λ -module).*

Proof. Since \mathcal{F}_J is reflexive (respectively, strongly reflexive), then by Lemma 3.4.3 so is its pullback through the étale map $[X/K] \rightarrow [X/J]$, which is \mathcal{F}_K . Then $H^i([X/K], \mathcal{F}_K) \cong H^i(X, \mathcal{F})^K$ is a finitely generated Λ -module (respectively, finite rank free); since this is true for a basis of open subgroups $K \subset J$, and since $H^i(X, \mathcal{F})^K$ is a summand of $H^i(X, \mathcal{F})^{K'}$ for an open subgroup $K' \subset K$, we see that $H^i(X, \mathcal{F})$ is admissible (respectively, admissible and free). \square

If X is proper, and if \mathcal{F}_J is strongly reflexive, we may apply the Lefschetz-Verdier fixed-point formula to compute the Euler characteristic of $R\Gamma(X, \mathcal{F})$ as a virtual J -module, by computing the traces of a Hecke operators. To wit, let $g \in J$, and let $K \subset J$ be an open pro- p subgroup. Let $K' \subset J$ be an open subgroup contained in $K \cap g^{-1}Kg$, and consider the correspondence c_g defined by

$$\begin{array}{ccc} & [X/K'] & \\ \alpha_g \swarrow & & \searrow \alpha_1 \\ [X/K] & & [X/K], \end{array} \quad (3.8.3)$$

where $\alpha_g: [X/K'] \rightarrow [X/K]$ is the descent of the multiplication-by- g map on X , and α_1 is the projection. The J -equivariance of \mathcal{F} induces a morphism $u_g: \alpha_g^* \mathcal{F}_K \rightarrow \alpha_1^* \mathcal{F}_K$. Since α_1 is étale, $\alpha_1^* = \alpha_1^!$. Thus u_g is a cohomological correspondence lying over c_g , and therefore it induces an endomorphism of each $H^i([X/K]_{\text{ét}}, \mathcal{F}_K)$.

Let μ be a Λ -valued Haar measure on J : this exists because J is locally pro- p and p is invertible in Λ .

Lemma 3.8.5. *Under the isomorphism $H^i([X/K]_{\text{ét}}, \mathcal{F}_K) \cong H^i(X_{\text{ét}}, \mathcal{F})^K$, the endomorphism of $H^i([X/K]_{\text{ét}}, \mathcal{F}_K)$ induced by u_g corresponds to the restriction of the operator on $H^i(X_{\text{ét}}, \mathcal{F})$ defined by*

$$v \mapsto \frac{1}{\mu(K')} \int_{k \in K_g} k(v) d\mu.$$

Proof. Let $T \in \text{End } H^i(X_{\text{ét}}, \mathcal{F})$ be the displayed operator. For $v \in H^i(X_{\text{ét}}, \mathcal{F})^K$ we have

$$T(v) = \sum_{[k] \in K/K'} kg(v),$$

which corresponds to the operator on $H^i([X/K]_{\text{ét}}, \mathcal{F}_K)$ induced by u_g . \square

Finally, we need an auxiliary result concerning pro-étale local systems on $[X/J]$. Let G be a first-countable locally pro- p group. Let $X' \rightarrow X$ be a G -torsor which is J -equivariant, let ρ be an admissible representation of G on a free Λ -module, let $\mathcal{L} = \mathcal{L}_\rho$ be the corresponding pro-étale local system on X . Then \mathcal{L} is J -equivariant; let \mathcal{L}_K be its descent to $[X/K]$. There is a cohomological correspondence $u_{g, \mathcal{L}}$ on \mathcal{L}_K lying over the correspondence c_g of (3.8.3).

For $x \in X$, let us write $[x]_{K'} \in [X/K']$ for its image. For an isolated (stacky) point $[x]_{K'} \in \text{Fix}(c_g)$, there is a corresponding operator $u_{g, \mathcal{L}, [x]_{K'}}$

on $\mathcal{L}_{[x]_K}$. With an eye towards applying Proposition 3.7.2, we spell out what this operator does.

Choose a representative $x \in X$ which is fixed by some element in Kg . Let $J_x \subset J$ be the stabilizer of x . Then $[x]_{K'} \cong [\mathrm{Spa} C'/\overline{J_x \cap K'}]$. The fiber $\mathcal{L}_{[x]_K}$ is \mathcal{L}_x with its action of $J_x \cap K'$ remembered, and $H^0([x]_K, \mathcal{L}_{[x]_K}) = \mathcal{L}_x^{J_x \cap K}$. Then for $v \in \mathcal{L}_x^{J_x \cap K}$ we have

$$u_{g, \mathcal{L}, [x]_{K'}}(v) = \sum_{k \in (J_x \cap K'g)/(J_x \cap K'g)} kv. \quad (3.8.4)$$

4 Geometric Langlands for a p -adic field

4.1 The B_{dR}^+ -affine Grassmannian

In this section we recall the B_{dR}^+ -affine Grassmannian from [SW14]. Recall that the usual affine Grassmannian for a reductive group G over a field k is the ind-scheme representing the functor $R \mapsto G(R((t)))/G(R[[t]])$ for a k -algebra R . In the B_{dR}^+ -version, the power series ring $R((t))$ is replaced with the de Rham period ring $B_{\mathrm{dR}}(R)$, which we review below.

Recall that F is our fixed p -adic field. Write \mathcal{O}_F for its valuation ring and \mathbf{F}_q for its residue field. For a perfect \mathbf{F}_q -algebra R , we have the \mathcal{O}_F -Witt vectors $W_{\mathcal{O}_F}(R) = W(R) \otimes_{W(\mathbf{F}_q)} \mathcal{O}_F$.

Definition 4.1.1. Let R be a perfectoid F -algebra, and let $\theta: W_{\mathcal{O}_F}(R^\flat)[1/p] \rightarrow R$ be the usual map; let $\xi \in W_{\mathcal{O}_F}(R^\flat)$ generate the kernel. Let $B_{\mathrm{dR}}^+(R)$ be the ξ -adic completion of $W_{\mathcal{O}_F}(R^\flat)[1/p]$, and let $B_{\mathrm{dR}}(R) = B_{\mathrm{dR}}^+(R)[1/\xi]$.

In the case that $R = C$ is a perfectoid field, $B_{\mathrm{dR}}^+(C)$ is a discrete valuation ring with uniformizer ξ , which contains an algebraic closure of F . Therefore we have the Cartan decomposition

$$G(B_{\mathrm{dR}}^+(C)) \backslash G(B_{\mathrm{dR}}(C)) / G(B_{\mathrm{dR}}^+(C)) \xrightarrow{\sim} X_*(T)/W,$$

where T is any maximal torus of G defined over \bar{F} and W is its Weyl group. Since any two maximal tori of G defined over \bar{F} are conjugate in $G(\bar{F}) \subset G(B_{\mathrm{dR}}^+(C))$, this decomposition is independent of the choice of maximal torus T . We can thus take T to be the universal maximal torus, which is part of the universal Borel pair (T, B) (i.e. the limit over all Borel pairs of G).

Definition 4.1.2. Let C be an algebraically closed perfectoid field containing F and let $\mu \in X_*(T)/W$. Define sheaves Gr_G and $\mathrm{Gr}_{G, \mu}$ on Perf_C as follows.

1. Let Gr_G be the sheaf on Perf_C associated to the presheaf

$$\mathrm{Spa}(R, R^+) \mapsto G(B_{\mathrm{dR}}(R))/G(B_{\mathrm{dR}}^+(R)).$$

2. Let $\mathrm{Gr}_{G, \leq \mu}$ be the subsheaf of Gr_G whose points over (R, R^+) consist of those $L \in \mathrm{Gr}_G(R, R^+)$ such that, for every geometric point $x: \mathrm{Spa}(C', (C')^\circ) \rightarrow \mathrm{Spa}(R, R^+)$, the pullback $x^*L \in \mathrm{Gr}_G(C', (C')^\circ)$ corresponds under the Cartain decomposition to $\lambda \in X_*(T)/W$ with $\lambda \leq \mu$.

It is possible to define Gr_G over $\mathrm{Spd} F$ and $\mathrm{Gr}_{G, \leq \mu}$ over $\mathrm{Spd} E$ (where E is a reflex field for μ), but since we are not considering the action of the Weil group of F , we are content with working over $\mathrm{Spd} C$.

Theorem 4.1.3 ([SW14, Theorem 21.3.6]). *$\mathrm{Gr}_{G, \leq \mu}$ is a proper spatial diamond.*

Example 4.1.4. If $G = T$ is a torus, then there is an isomorphism of diamonds over $\mathrm{Spd} C$:

$$\begin{aligned} X_*(T) &\rightarrow \mathrm{Gr}_T \\ \nu &\mapsto L_\nu := \nu(\xi)T(B_{\mathrm{dR}}^+). \end{aligned}$$

4.2 Geometric Satake equivalence

In this section we explain Assumption 2 of Theorem 1.0.4. Essentially, it is that the geometric Satake equivalence of [MV07] holds for the B_{dR}^+ -Grassmannian Gr . Note that [Zhu17] introduces a different sort of mixed-characteristic affine Grassmannian and proves a geometric Satake equivalence for it, but unfortunately it does not seem obvious to us how to derive what we need directly from this.

Let $\ell \neq p$, and let $\overline{\mathbf{Q}}_\ell\text{-Vect}$ be the category of vector spaces over $\overline{\mathbf{Q}}_\ell$. Let $\mathcal{P}_{G(B_{\mathrm{dR}}^+)}(\mathrm{Gr}_G)$ denote the category of perverse $\overline{\mathbf{Q}}_\ell$ -sheaves on $\mathrm{Gr}_{G, \acute{\mathrm{e}}\mathrm{t}}$ which are $G(B_{\mathrm{dR}}^+)$ -equivariant, and let

$$\begin{aligned} H: \mathcal{P}_{G(B_{\mathrm{dR}}^+)}(\mathrm{Gr}_G) &\rightarrow \overline{\mathbf{Q}}_\ell\text{-Vect} \\ \mathcal{F} &\rightarrow \bigoplus_{i \in \mathbf{Z}} H^i(\mathrm{Gr}_{G, \acute{\mathrm{e}}\mathrm{t}}, \mathcal{F}) \end{aligned}$$

be the global cohomology functor.

Assumption 2 is the following:

1. $\mathcal{P}_{G(B_{\text{dR}}^+)}(\text{Gr}_G)$ is preserved under the convolution product. The convolution and fusion products are equivalent, so that $\mathcal{P}_{G(B_{\text{dR}}^+)}(\text{Gr}_G)$ is even a commutative tensor category.
2. H is an exact faithful tensor functor. By Tannakian duality, $\widehat{G} = \text{Aut}^{\otimes} H$ is a pro-algebraic group over $\overline{\mathbf{Q}}_{\ell}$, for which there is an exact tensor equivalence $\text{Rep}_{\widehat{G}} \cong \mathcal{P}_{G(B_{\text{dR}}^+)}(\text{Gr}_G)$ which pulls back H to the canonical fiber functor on $\text{Rep}_{\widehat{G}}$.
3. Given $\mu \in X_*(T)$, let r_{μ} be the algebraic representation of $\text{Rep}_{\widehat{G}}$ with highest weight μ , let L_{μ} be the image of μ under $X_*(T) \rightarrow \text{Gr}_T \rightarrow \text{Gr}_G$, and let IC_{μ} be the intersection complex associated to the $G(B_{\text{dR}}^+)$ -orbit of L_{μ} . The above tensor equivalence carries IC_{μ} onto r_{μ} .
4. (The weight decomposition, [MV07, Theorems 3.5 and 3.6].) Fix a Borel pair (T, B) of G_C and write $B = TN$. For $\nu \in X_*(T)$, let $S_{\nu} \subset \text{Gr}_G$ denote the $N(B_{\text{dR}})$ -orbit of L_{ν} . This is an ind-diamond; its intersection with any $\text{Gr}_{\leq \mu}$ is a diamond. We have

$$\overline{S}_{\nu} = \bigcup_{\nu' \leq \nu} S_{\nu'}$$

[MV07, Proposition 3.1(a)].

For $\mathcal{F} \in \mathcal{P}_{G(B_{\text{dR}}^+)}$, the cohomology $H_c^i(S_{\nu}, \mathcal{F})$ is zero unless $i = 2\rho(\nu)$.

We have a natural equivalence of functors $\mathcal{P}_{G(B_{\text{dR}}^+)} \rightarrow \overline{\mathbf{Q}}_{\ell}\text{-Vect}$:

$$H \cong \bigoplus_{\nu \in X_*(T)} H_c^{2\rho(\nu)}(S_{\nu}, -): \mathcal{P}_{G(B_{\text{dR}}^+)}(\text{Gr}_G) \rightarrow \overline{\mathbf{Q}}_{\ell}\text{-Vect}$$

The ν -weight space of H is $H_c^{2\rho(\nu)}(S_{\nu}, -)$.

5. (The Demazure resolution, [NP01, Proposition 9.4].) Let Λ be a ring which is ℓ^n -torsion for some n , and let $\text{IC}_{\mu, \Lambda}$ be the intersection complex with coefficients in Λ . Then $\text{IC}_{\mu, \Lambda}$ is a direct summand of $r_*\Lambda$, where $r: \text{Gr}'_{G, \leq \mu} \rightarrow \text{Gr}_G$ is a *Demazure resolution*; $\text{Gr}'_{G, \leq \mu} \rightarrow \text{Spa } C$ is smooth and proper.

We need the last item for the following lemma.

Lemma 4.2.1. *Let $U \rightarrow \text{Gr}_{G, \leq \mu}$ be an étale morphism from a quasi-compact diamond. Then each $R^i\Gamma(U_{\text{ét}}, \text{IC}_{\mu, \Lambda})$ is finitely generated.*

Proof. Considering item (5) above, this follows from the fact that if $f: U \rightarrow \mathrm{Spa} C$ is a smooth and quasicompact spatial diamond, then $R\Gamma(U, \Lambda) = Rf_*\Lambda$ is finitely generated in each degree. By Poincaré duality, $R\Gamma(U, \Lambda)$ is dual to $R\Gamma_c(U, K_U)$. By smoothness, K_U is v -locally a shift and twist of the constant sheaf Λ . Now the result follows from the more general fact that if f is any smooth morphism of spatial diamonds, then $f_!$ preserves constructible sheaves; this follows from [Sch17, Proposition 18.9.ii]. \square

4.3 The Fargues-Fontaine curve

The following is a review of [Far, §1]. All of the following constructions are relative to our local field F (which is E in [Far]); often we will suppress the F from the notation. We remark that all of the constructions have analogues in the case that F has equal characteristic.

Let $F_0 \subset F$ denote the maximal subfield which is unramified over \mathbf{Q}_p . Choose a uniformizer $\pi_F \in \mathcal{O}_F$.

For a perfectoid \mathbf{F}_q -algebra R with pseudo-uniformizer ϖ , we define

$$\begin{aligned} \mathcal{Y}_R &= \mathrm{Spa} W_{\mathcal{O}_F}(R^\circ) \setminus \{\pi_F[\varpi] = 0\} \\ \mathcal{X}_R &= Y_R / \phi_R^{\mathbf{Z}}, \end{aligned}$$

where $\phi_R: R \rightarrow R$ is the q th power Frobenius automorphism. \mathcal{X}_R is the *adic Fargues-Fontaine curve*. We also need the schematic Fargues-Fontaine curve X_R , defined as

$$X_R = \mathrm{Proj} \bigoplus_{d=0}^{\infty} H^0(\mathcal{Y}_R, \mathcal{O}_{\mathcal{Y}_R})^{\phi_R = \pi_F^d}.$$

When R is a perfectoid field, X_R is a noetherian scheme of degree 1.

Now suppose R^\sharp is a perfectoid F -algebra equipped with an isomorphism ι of \mathbf{F}_q -algebras $R^\sharp \cong R$. The pair (R^\sharp, ι) is called an *untilt* of R . The kernel of the homomorphism $\theta: W(R^\circ)[1/p] \rightarrow R^\sharp$ is a primitive element ξ of degree 1, which determines a Cartier divisor $D_{R^\sharp} \hookrightarrow X_R$. The completion of X_R along D_{R^\sharp} is $\mathrm{Spf} B_{dR}^+(R^\sharp)$ [Far, Proposition 1.33].

4.4 Kottwitz's theory of σ -conjugacy classes

We recall here some facts about Kottwitz's set $B(G)$ of σ -conjugacy classes in $G(\check{F})$ [Kot85]. Associated to such a class $[b] \in B(G)$ there are two invariants: $\kappa([b]) \in \pi_1(G)_\Gamma$ and $\nu_{[b]} \in \check{C}_\mathbf{Q}$. Let us explain the notation.

First, $\pi_1(G)$ is the algebraic fundamental group introduced by Borovoi [Bor98]. It can be described as follows. Given a maximal torus $S \subset G$ defined over F let S_{sc} be the preimage of S in the simply connected cover of the derived subgroup of G . The natural map $S_{\text{sc}} \rightarrow S$ induces an injective group homomorphism $X_*(S_{\text{sc}}) \rightarrow X_*(S)$. If S' is another maximal torus of G defined over F , then the isomorphism $X_*(S)/X_*(S_{\text{sc}}) \rightarrow X_*(S')/X_*(S'_{\text{sc}})$ induced by conjugation by any $g \in G(\bar{F})$ with $gSg^{-1} = S'$ is independent of the choice of g , and in particular Γ -equivariant. Taking the limit over all possible maximal tori S of the quotient $X_*(S)/X_*(S_{\text{sc}})$ thus gives a finitely generated abelian group with Γ -action, and this is the definition of $\pi_1(G)$. The assignment of $\kappa([b]) \in \pi_1(G)_\Gamma$ to $[b]$ is explained in [Kot97, §4.9, §7.5]. It produces a map

$$\kappa : B(G) \rightarrow \pi_1(G)_\Gamma$$

functorial with respect to all homomorphisms of reductive groups. Note that if \widehat{S} is the torus dual to S we have $X_*(S)/X_*(S_{\text{sc}}) = X^*(\widehat{S})/X^*([\widehat{S}]_{\text{ad}}) = X^*(Z(\widehat{G}))$ and in this way one obtains $\pi_1(G)_\Gamma = X^*(Z(\widehat{G}))^\Gamma$. The invariant $\kappa([b])$ was initially taken to lie in $X^*(Z(\widehat{G}))^\Gamma$, but following [RR96] we use $\pi_1(G)_\Gamma$ instead because it is obviously functorial in G , while the functoriality of $X^*(Z(\widehat{G}))$ is less obvious, due to the fact that $G \mapsto \widehat{G}$ is functorial only with respect to homomorphisms with normal image.

Next we turn to the *Newton point* $\nu_{[b]} \in \bar{C}_{\mathbf{Q}}$. Let \mathbb{D} be the pro-torus determined by $X^*(\mathbb{D}) = \mathbf{Q}$. Then associated to $[b]$ is a σ -stable $G(\check{F})$ -conjugacy class of homomorphisms $\mathbb{D} \rightarrow G$ defined over \check{F} , called *slope morphisms*. There are different ways to think about them. One way, following [Kot85, §4.2] is to obtain from a representative b of $[b]$ and a finite-dimensional rational representation $\rho : G \rightarrow \text{GL}(V)$ the structure of a σ - \check{F} -space on V (an isocrystal in the classical sense when $F = \mathbf{Q}_p$) and use its slope decomposition to obtain a homomorphism from \mathbb{D} into $\text{GL}(V)$ defined over \check{F} . For varying ρ these homomorphisms splice together to a homomorphism $\mathbb{D} \rightarrow G$ defined over \check{F} . Varying the representative b replaces this homomorphism by a $G(\check{F})$ -conjugate.

A second way to think about the slope morphism is via the reinterpretation of $B(G)$ as the set of cohomology classes of algebraic 1-cocycles of the Galois gerbe $1 \rightarrow \mathbb{D} \rightarrow \mathcal{E} \rightarrow \Gamma$ with values in $G(\bar{F})$, alluded to in §2.3. The restriction of such a 1-cocycle to \mathbb{D} is by definition an algebraic homomorphism $\mathbb{D} \rightarrow G$ defined over \bar{F} .

In either interpretation, we may compose the slope morphism with $\psi^{-1} : G \rightarrow G^*$, for some inner twist $\psi \in \Psi$ (which can be chosen to be defined over F^{ur} by Steinberg's theorem on the vanishing of $H^1(F^{\text{ur}}, G^*(\bar{F}))$), and

thereby obtain a σ -stable $G^*(\check{F})$ -conjugacy class of morphisms $\mathbb{D} \rightarrow G^*$ defined over \check{F} , respectively a Γ -stable $G^*(\bar{F})$ -conjugacy class of morphisms $\mathbb{D} \rightarrow G^*$ defined over \bar{F} . Fix a Borel pair (T, B) of G^* defined over F . Up to conjugation such a morphism can be arranged to take values in T . Then it corresponds to a map $X^*(T) \rightarrow X^*(\mathbb{D}) = \mathbf{Q}$, hence to an element of $X_*(T) \otimes \mathbf{Q}$. Up to further conjugation by the Weyl group of T this element can be made B -dominant. This B -dominant element is then unique. It is therefore Γ -stable. Let $\bar{C}_{T, \mathbf{Q}}$ be the subset of B -dominant elements in $[X_*(T) \otimes \mathbf{Q}]^\Gamma = X_*(T)^\Gamma \otimes \mathbf{Q} = X_*(A_T) \otimes \mathbf{Q}$, where $A_T \subset T$ is the maximal split torus in T . If another Borel pair (T', B') is chosen, there exists $g \in G^*(F)$ such that $g(T, B)g^{-1} = (T', B')$ and such a g is well defined up to right multiplication by $T(F)$, so the induced Γ -equivariant isomorphism $X_*(T) \rightarrow X_*(T')$ is independent of g . It further induces an isomorphism $\bar{C}_{T, \mathbf{Q}} \rightarrow \bar{C}_{T', \mathbf{Q}}$ and we take $\bar{C}_{\mathbf{Q}}$ to be limit over all possible (T, B) . Note that $\bar{C}_{\mathbf{Q}}$ can alternatively be described as the limit over all possible T of the quotients $X_*(A_T) \otimes \mathbf{Q}/W(T)(F)$, where $W(T)$ is the Weyl group of T .

We have thus described both invariants $\kappa([b]) \in \pi_1(G)_\Gamma$ and $\nu_{[b]} \in \bar{C}_{\mathbf{Q}}$ of an element $[b] \in B(G)$. The inclusion $Z(G^*) \rightarrow G^*$ induces a map $X_*(Z(G^*)) \otimes \mathbf{Q} \rightarrow \bar{C}_{\mathbf{Q}}$. The elements $[b] \in B(G)$ for which $\nu_{[b]}$ lies in the image of this map are called *basic* and the subset of $B(G)$ consisting of basic elements is denoted by $B(G)_{\text{bas}}$. Equivalently, an element $[b]$ is basic if its slope morphism $\mathbb{D} \rightarrow G$ takes values in $Z(G)$. Kottwitz shows [Kot85, Proposition 5.6] that $B(G)_{\text{bas}}$ is a section of κ , that is, $\kappa : B(G)_{\text{bas}} \rightarrow \pi_1(G)_\Gamma$ is a bijection. Furthermore, the product map $\kappa \times \nu : B(G) \rightarrow \pi_1(G)_\Gamma \times \bar{C}_{\mathbf{Q}}$ is injective [Kot97, §4.13].

Let now $\mu \in X_*(T)_\Gamma$. Kottwitz defines in [Kot97, §6] a subset $B(G, \mu) \subset B(G)$ as follows. Let $\mu_1 \in \pi_1(G)_\Gamma$ be the image of μ under $X_*(T)_\Gamma = \pi_1(T)_\Gamma \rightarrow \pi_1(G)_\Gamma$. Let $\mu_2 \in X_*(A_T) \otimes \mathbf{Q}$ be the image of μ under the normalized norm map $X_*(T)_\Gamma \rightarrow X_*(T)^\Gamma \otimes \mathbf{Q}$. The $W(T)(F)$ -orbit of μ_2 contains a unique member of $\bar{C}_{\mathbf{Q}}$, which we take in place of μ_2 . Then $B(G, \mu)$ is the subset of $B(G)$ consisting of those $[b]$ for which $\kappa([b]) = \mu_1$ and $\nu_{[b]} \leq \mu_2$. The latter condition can be reformulated by saying that $\nu_{[b]}$ lies in the convex hull of the Weyl orbit of μ_2 . The subset $B(G, \mu)$ contains a unique basic element.

4.5 Vector bundles and G -bundles

Here we review the theory of vector bundles and G -bundles on the Fargues-Fontaine curve, developed in the absolute setting in [FF] and [Far15]. As

usual F/\mathbf{Q}_p is a finite extension with residue field \mathbf{F}_q and uniformizer π , and \check{F} is the completion of a maximal unramified extension of F .

Let $\sigma: \check{F} \rightarrow \check{F}$ be the Frobenius automorphism induced by the q th power Frobenius map on $\overline{\mathbf{F}}_q$. Recall that an *isocrystal* is a pair (N, ϕ_N) , where N is a finite-dimensional \check{F} -vector space and $\phi_N: N \rightarrow N$ is a σ -linear automorphism. By the Dieudonné-Manin classification, every isocrystal (N, ϕ_N) admits a canonical \mathbf{Q} -grading: $(N, \phi_N) \cong \bigoplus_{\lambda \in \mathbf{Q}} (N_\lambda, \phi_{N_\lambda})$, where each $(N_\lambda, \phi_{N_\lambda})$ is isoclinic of slope λ . Morphisms between isocrystals preserve this grading.

Let (N, ϕ_N) be an isocrystal. For every perfectoid ring $R/\overline{\mathbf{F}}_q$, we have the vector bundle $\mathcal{O}_{\mathcal{Y}_R} \otimes_{\check{F}} N$ on \mathcal{Y}_R , which comes equipped with a ϕ_R -equivariant automorphism, namely $\phi = \phi_R \otimes \phi_N$. Therefore $\mathcal{O}_{\mathcal{Y}_R} \otimes_{\check{F}} N$ can be descended to $\mathcal{X}_R = \mathcal{Y}_R/\phi_R$. We will need a schematic version of this descent to the schematic curve X_R , namely

$$\mathcal{E}_{N, \phi_N} = \text{Proj} \bigoplus_{d \geq 0} (H^0(\mathcal{Y}_R, \mathcal{O}_{\mathcal{Y}_R}) \otimes_{\check{F}} N)^{\phi_R \otimes \phi_N = \pi^d}.$$

It is clear that $(N, \phi_N) \mapsto \mathcal{E}_{N, \phi_N}$ is a functor.

Let us discuss the absolute case, when $R = C$ is an algebraically closed perfectoid field. We have a Harder-Narasimhan theory for vector bundles on X_C : for a vector bundle \mathcal{E} on X_C , the slope of \mathcal{E} is $\deg \mathcal{E} / \text{rk } \mathcal{E}$. Then every vector bundle admits a canonical \mathbf{Q} -filtration by slopes. In fact this \mathbf{Q} -filtration is split, so each vector bundle has a \mathbf{Q} -grading as well. This applies in particular to the vector bundle \mathcal{E}_{N, ϕ_N} , in which case the slopes of \mathcal{E}_{N, ϕ_N} are -1 times the slopes of (N, ϕ_N) .

Theorem 4.5.1 ([FF]). *Let C/\mathbf{F}_q be an algebraically closed perfectoid field.*

1. *Every vector bundle on X_C is isomorphic to \mathcal{E}_{N, ϕ_N} for an isocrystal (N, ϕ_N) .*
2. *Morphisms between vector bundles preserve the \mathbf{Q} -filtration (but not necessarily the \mathbf{Q} -grading).*
3. *The morphisms between \mathcal{E}_{M, ϕ_M} and \mathcal{E}_{N, ϕ_N} that preserve the \mathbf{Q} -grading are precisely the functorial images of the morphisms between (M, ϕ_M) and (N, ϕ_N) .*

We now generalize to G -bundles. Let G be a connected reductive group over F and let Rep_G be the F -linear category of algebraic representations of G . A G -isocrystal is a \otimes -functor from Rep_G to the category of isocrystals.

An element $b \in G(\check{F})$ determines a G -isocrystal $N_b: V \mapsto (V \otimes_F \check{F}, b(1 \otimes \sigma))$ whose isomorphism class only depends on the class of b in $B(G)$. Recall that $J_b(F)$ is the automorphism group of N_b .

Since isocrystals admit a canonical \mathbf{Q} -grading, the element $b \in G(\check{F})$ induces a \mathbf{Q} -grading on $\text{Rep}_{\check{F}}$, which in turn induces a filtration. Let M (respectively, P) be the stabilizer in $G_{\check{F}}$ of this grading (respectively, this filtration). Then $P \subset G_{\check{F}}$ is a parabolic subgroup with Levi factor M . Since morphisms between isocrystals preserve the \mathbf{Q} -grading, $J_b(F)$ is a subgroup of $M(\check{F})$.

A G -bundle on a scheme X/F is a \otimes -functor from Rep_G to the category of vector bundles on X . We will write \mathcal{E}_1 for the trivial G -bundle, which is $V \mapsto V \otimes_F \mathcal{O}_X$. Given an element $b \in G(\check{F})$, we get a G -bundle \mathcal{E}_b on X_R , via $V \mapsto \mathcal{E}_{N_b(V)}$. (Note that there is no conflict of notation with \mathcal{E}_1 .) Then the isomorphism class of \mathcal{E}_b only depends on the class of b in $B(G)$.

Theorem 4.5.2 ([Far15]). *Let C/\mathbf{F}_q be an algebraically closed perfectoid field. Every G -bundle \mathcal{E} on X_C is isomorphic to \mathcal{E}_b for some $b \in G(\check{F})$. Thus the set of isomorphism classes of G -bundles on X_C is in bijection with $B(G)$.*

This theorem allows us to define invariants $\kappa(\mathcal{E}) \in \pi_1(G)_\Gamma$ and $\nu_{\mathcal{E}} \in \bar{C}_{\mathbf{Q}}$ for an arbitrary G -bundle \mathcal{E} on X_C .

If (R^\sharp, ι) is an untilt of R , then the G -bundle \mathcal{E}_b on X_R comes equipped with a trivialization along the completion of the divisor $D_{R^\sharp} \hookrightarrow X_R$. This is because the adic version of \mathcal{E}_b is descended from the trivial G -bundle on \mathcal{Y}_R , and because \mathcal{Y}_R has a distinguished R^\sharp -point lying above (the adic version of) D_{R^\sharp} .

We use this to derive a result about automorphisms of a G -bundle on the absolute Fargues-Fontaine curve. Let C/F be an algebraically closed perfectoid field, and let $\infty \in X_{C^b}$ be the closed point corresponding to the untilt C of C^b . Let $b \in G(\check{F})$. We write $\text{Aut } \mathcal{E}_b$ for the group of all automorphisms of \mathcal{E}_b and $\text{Aut}^{\text{gr}} \mathcal{E}_b$ for the subgroup of those automorphisms that preserve the \mathbf{Q} -grading. According to Theorem 4.5.1 the homomorphism $J_b(F) \rightarrow \text{Aut } \mathcal{E}_b$ induced by functoriality of $(N, \phi_N) \rightarrow \mathcal{E}_{N, \phi_N}$ is an isomorphism of $J_b(F)$ onto $\text{Aut}^{\text{gr}} \mathcal{E}_b$. Note that $\text{Aut } \mathcal{E}_b$ may be much larger than $\text{Aut}^{\text{gr}} \mathcal{E}_b$. For example, if $G = \text{GL}_2$ and $b = \text{diag}(1, p^{-1})$, then $J_b(F) \cong F^\times \times F^\times$, whereas $\text{Aut } \mathcal{E}_b$ is a semidirect product of this group by $H^0(X_{C^b}, \mathcal{O}(1))$.

By the remark of the previous paragraph, we have a distinguished trivialization of the stalk $\mathcal{E}_{b, \infty}$ as a G -bundle over $\text{Spec } B_{\text{dR}}^+(C)$. The \mathbf{Q} -filtration of \mathcal{E}_b induces a filtration of $\mathcal{E}_{b, \infty}$, whose stabilizer in $G(B_{\text{dR}}^+(C))$ is $P(B_{\text{dR}}^+(C))$.

Since automorphisms of \mathcal{E}_b must preserve this \mathbf{Q} -filtration, we have a homomorphism $\text{Aut } \mathcal{E}_b \rightarrow P(B_{\text{dR}}^+(C))$.

Lemma 4.5.3. $J_b(F) \rightarrow \text{Aut } \mathcal{E}_b$ admits a section, which makes the following diagram commute:

$$\begin{array}{ccc} \text{Aut } \mathcal{E}_b & \longrightarrow & P(B_{\text{dR}}^+(C)) \\ \downarrow & & \downarrow \\ J_b(F) & \longrightarrow & M(\check{F}) \longrightarrow M(B_{\text{dR}}^+(C)) \end{array}$$

Proof. For every object $V \in \text{Rep}_F$, there is the evident section $\text{Aut } \mathcal{E}_{N_b(V)} \rightarrow \text{Aut}^{\text{gr}} \mathcal{E}_{N_b(V)}$. Since these constructions are functorial in V , we have a section $\text{Aut } \mathcal{E}_b \rightarrow \text{Aut}^{\text{gr}} \mathcal{E}_b \cong \text{Aut } N_b = J_b(F)$ as required. The commutativity of the diagram comes from tracing through the definitions of P and M . \square

4.6 G -torsors

Here we remind the reader of some standard material concerning G -torsors and their relation to G -bundles. Let $G \rightarrow X$ be a group scheme. For our purposes, a G -torsor on X is a faithfully flat F -morphism $\mathcal{T} \rightarrow X$ together with an action $G \times \mathcal{T} \rightarrow \mathcal{T}$ lying over the trivial action on X , such that fppf-locally on X we have a G -equivariant isomorphism $\mathcal{T} \cong G \times X$.

If $G' \rightarrow X$ is another group scheme there is the evident notion of a (G, G') -bitorsor on X , which receives commuting actions of G and G' and which is a torsor for each. The categories of torsors and bitorsors are groupoids. If \mathcal{T} is a G -torsor and \mathcal{T}' is a (G, G') -bitorsor, we have the contracted product $\mathcal{T} \times_G \mathcal{T}' = (\mathcal{T} \times \mathcal{T}')/G$, a G' -torsor.

Lemma 4.6.1. *Let $G \rightarrow X$ be a reductive group scheme. The category of G -bundles on X is equivalent to the category of G -torsors over X .*

Proof. We sketch the equivalence. If \mathcal{E} is a G -bundle on X , let \mathcal{T} be the functor which assigns to an X -scheme Y the set of trivializations of the base change of \mathcal{E} to Y ; then \mathcal{T} is representable by a G -torsor over X . Conversely, if $\mathcal{T} \rightarrow X$ is a G -torsor, the corresponding G -bundle is $V \mapsto \mathcal{T} \times_G V$. \square

We now return to the situation of the Fargues-Fontaine curve X_R . It may be helpful to spell out the equivalence in Lemma 4.6.1 for the G -bundle \mathcal{E}_b on X_R , where $b \in G(\check{F})$. Before doing so we introduce some notation: let λ_g (respectively, ρ_g) denote left multiplication by g on G (respectively, right multiplication by g^{-1}).

The G -torsor on X_R corresponding to \mathcal{E}_b is

$$\mathcal{T}_b = \text{Proj} \bigoplus_{d \geq 0} (H^0(\mathcal{Y}_R, \mathcal{O}_{\mathcal{Y}_R}) \otimes_{\check{F}} \check{F}[G])^{\phi_R \otimes \lambda_b \sigma = \pi^d}, \quad (4.6.1)$$

where $\check{F}[G]$ is the coordinate ring of $G_{\check{F}}$. The G -action on \mathcal{T}_b is induced from the action of G on itself by right translation. (Indeed, one checks that $\mathcal{T}_b \times_G V \cong \mathcal{E}_b(V)$ as tensor functors on $V \in \text{Rep}_G$.) It may be useful (if only for psychological reasons) to record the adic version of \mathcal{T}_b :

$$\mathcal{T}_b^{\text{ad}} = (\mathcal{Y}_R \times_{\check{F}} G_{\check{F}}^{\text{ad}}) / (\phi_R \times \lambda_b \sigma)$$

(here G^{ad} means the adic space over F attached to the scheme G).

Now suppose that b is basic. Recall that J_b is the inner form of G corresponding to $\text{ad } b$. For $h \in J_b(\check{F}) = G(\check{F})$ we write $\check{\mathcal{T}}_h$ for the associated J_b -torsor. It is easily verified that $h \mapsto hb$ induces a bijection $B(J_b) \xrightarrow{\sim} B(G)$. The following lemma shows that this bijection can be upgraded to an equivalence of groupoids.

Lemma 4.6.2. *Let $b \in G(\check{F})$ be basic.*

1. *There is a natural (G, J_b) -bitorsor structure extending the G -torsor structure on \mathcal{T}_b .*
2. *For $h \in G(\check{F})$ we have an isomorphism of G -torsors $\check{\mathcal{T}}_h \times_{J_b} \mathcal{T}_b \cong \mathcal{T}_{hb}$.*
3. *The groupoids of J_b -torsors and G -torsors on X_R are equivalent, via $\mathcal{T} \mapsto \mathcal{T} \times_{J_b} \mathcal{T}_b$.*

Proof. The isomorphism $\check{F}[G] \cong \check{F}[J_b]$ carries $(\text{ad } b)\sigma$ onto σ . Applying (4.6.1) to the element $b^{-1} \in J(\check{F})$ shows that

$$\check{\mathcal{T}}_{b^{-1}} = \text{Proj} \bigoplus_{d \geq 0} (H^0(\mathcal{Y}_R, \mathcal{O}_{\mathcal{Y}_R}) \otimes_{\check{F}} \check{F}[G])^{\phi_R \otimes (\lambda_{b^{-1}} \circ \text{ad } b)\sigma = \pi^d}.$$

Using the identity $\lambda_{b^{-1}} \circ \text{ad } b = \rho_b$, we see that inversion on G gives an isomorphism of X_R -schemes $\check{\mathcal{T}}_{b^{-1}} \cong \mathcal{T}_b$. This isomorphism endows \mathcal{T}_b with the structure of a J_b -torsor. The actions of J_b and G on \mathcal{T}_b commute, because they are given by the action of G on $\check{F}[G]$ by left and right translations, respectively. This completes (1).

For (2), observe that the multiplication morphism μ on G fits into a commutative diagram

$$\begin{array}{ccc} G_{\check{F}} \times G_{\check{F}} & \xrightarrow{\mu} & G_{\check{F}} \\ \lambda_h \circ \text{ad } b \times \lambda_b \downarrow & & \downarrow \lambda_{hb} \\ G_{\check{F}} \times G_{\check{F}} & \xrightarrow{\mu} & G_{\check{F}}. \end{array}$$

In light of (4.6.1), the co-multiplication $\mu : \check{F}[G] \rightarrow \check{F}[G] \otimes_{\check{F}} \check{F}[G]$ induces the required isomorphism $\check{\mathcal{T}}_h \times_{J_b} \mathcal{T}_b \cong \mathcal{T}_{hb}$.

For (3), we need to show that $\mathcal{T} \mapsto \mathcal{T} \times_{J_b} \mathcal{T}_b$ is an equivalence of groupoids. By symmetry we have a functor $\mathcal{T} \mapsto \mathcal{T} \times_G \check{\mathcal{T}}_{b^{-1}} \cong \mathcal{T} \times_G \mathcal{T}_b$ going in the other direction. The composition of these functors applied to a G -torsor \mathcal{T} gives

$$(\mathcal{T} \times_G \check{\mathcal{T}}_{b^{-1}}) \times_{J_b} \mathcal{T}_b \cong \mathcal{T} \times_G (\check{\mathcal{T}}_{b^{-1}} \times_{J_b} \mathcal{T}_b) \cong \mathcal{T} \times_G \mathcal{T}_1 \cong \mathcal{T}.$$

The other composition is similar. □

4.7 Beauville-Laszlo gluing and modifications

Let R/\mathbf{F}_q be a perfectoid ring, and let (R^\sharp, ι) be an untilt of R to F . Recall that it determines a Cartier divisor $D_{R^\sharp} \hookrightarrow X_R$. We have noted that the completion of X_R along D_{R^\sharp} is $\text{Spf } B_{\text{dR}}^+(R^\sharp)$. Let $B_e(R) = H^0(X_R \setminus D_{R^\sharp}, \mathcal{O}_{X_R})$.

Given a G -bundle \mathcal{E} , let \mathcal{E}_e be its restriction to $X_R \setminus D_{R^\sharp} = \text{Spec } B_e(R)$, and let $\mathcal{E}_{\text{dR}}^+$ be its completion along D_{R^\sharp} , so that $\mathcal{E}_{\text{dR}}^+$ is a G -bundle over $\text{Spf } B_{\text{dR}}^+(R^\sharp)$. By the following gluing lemma, \mathcal{E} is determined by \mathcal{E}_e and $\mathcal{E}_{\text{dR}}^+$.

Lemma 4.7.1. *[BL95] The category of G -bundles on X_R is equivalent to the category of triples $(\mathcal{E}_e, \mathcal{E}_{\text{dR}}^+, \iota)$, where \mathcal{E}_e is a G -bundle over $B_e(R)$, $\mathcal{E}_{\text{dR}}^+$ is a G -bundle over $B_{\text{dR}}^+(R^\sharp)$, and $\iota : \mathcal{E}_e \otimes_{B_e(R)} B_{\text{dR}}(R^\sharp) \xrightarrow{\sim} \mathcal{E}_{\text{dR}}^+ \otimes_{B_{\text{dR}}^+(R^\sharp)} B_{\text{dR}}(R^\sharp)$ is an isomorphism.*

We now consider a pair (\mathcal{E}, β) consisting of a G -bundle \mathcal{E} on X_R and a trivialization of \mathcal{E} over $B_{\text{dR}}^+(R^\sharp)$. In terms of the gluing lemma this corresponds to a pair (\mathcal{E}_e, ι) , where \mathcal{E}_e is a G -bundle over $B_e(R)$ and $\iota : \mathcal{E}_e \otimes_{B_e(R)} B_{\text{dR}}(R^\sharp) \xrightarrow{\sim} \mathcal{E}_{1, B_{\text{dR}}(R^\sharp)}$ – to obtain a triple as in the gluing lemma we take for $\mathcal{E}_{\text{dR}}^+$ the trivial $B_{\text{dR}}^+(R^\sharp)$ -lattice $\mathcal{E}_{1, B_{\text{dR}}^+(R^\sharp)}$ inside of $\mathcal{E}_{1, B_{\text{dR}}(R^\sharp)}$. An isomorphism $(\mathcal{E}_e, \iota) \rightarrow (\mathcal{E}'_e, \iota')$ is given by a pair (α_e, h) with $\alpha_e : \mathcal{E}_e \xrightarrow{\sim} \mathcal{E}'_e$ and $h \in G(B_{\text{dR}}^+(R^\sharp))$ satisfying $\iota' \circ (\alpha_e \otimes \text{id}_{B_{\text{dR}}(R^\sharp)}) = h \circ \iota$.

Definition 4.7.2. Let (\mathcal{E}, β) be a G -bundle on X_R trivialized over $B_{\text{dR}}(R^\sharp)$, corresponding to the pair (\mathcal{E}_e, ι) . Let $g \in G(B_{\text{dR}}(R^\sharp))$. The *modification of (\mathcal{E}, β) at R^\sharp via g* is the pair $(\mathcal{E}[g], \beta)$ corresponding to the pair $(\mathcal{E}_e, g^{-1}\iota)$.

Note that the isomorphism class of $\mathcal{E}[g]$ only depends on the class of g in $\text{Gr}_G(R)$, and this is the motivation for the use of g^{-1} instead of g .

In this paper we are particularly interested in the G -bundles \mathcal{E}_b for $b \in G(\bar{F})$. The untilt (R^\sharp, ι) provides a point of \mathcal{Y}_R and thus, as explained earlier, a canonical trivialization of \mathcal{E}_b over $B_{\text{dR}}^+(R^\sharp)$. For every $g \in G(B_{\text{dR}}(R^\sharp))$ we thus obtain the modified bundle $\mathcal{E}_b[g]$, whose isomorphism class depends only on the image of g in $\text{Gr}_G(R)$. In the special when $g = L_\nu = \nu(\xi)$ for a cocharacter $\nu \in X_*(T)$ of an F -rational maximal torus $T \subset G$ we will write $(\mathcal{E}[\nu], \beta)$ for the modification of a G -bundle (\mathcal{E}, β) by g , and in particular $\mathcal{E}_b[\nu]$ for the modification $\mathcal{E}_b[g]$.

Lemma 4.7.3. *Let \mathcal{E} be a G -bundle on X_G , let $T \subset G$ be a maximal torus, let $\nu \in X_*(T)$ be a cocharacter, and let $\hat{\nu} \in X^*(\hat{T})$ be the corresponding character. In the group $X^*(Z(\hat{G})^\Gamma)$ we have*

$$\kappa(\mathcal{E}[\nu]) = \kappa(\mathcal{E}) - \hat{\nu}|_{Z(\hat{G})^\Gamma}.$$

Proof. The proof is a devissage argument based on the functoriality of the Kottwitz map κ and the following easily established fact: If $f: G \rightarrow H$ is a homomorphism of reductive groups, write f_* for the functor carrying G -bundles to H -bundles; then for a G -bundle E we have

$$f_*(\mathcal{E}[\nu]) \cong (f_*\mathcal{E})[f \circ \nu]. \quad (4.7.1)$$

Step 0: $G = \mathbf{G}_m$. In this case $\kappa: B(\mathbf{G}_m) \rightarrow \mathbf{Z}$ is an isomorphism, which agrees with the degree map on vector bundles; the claim reduces to the fact that modifying a line bundle by $t \mapsto t^n$ reduces its degree by n .

Step 1: $G = T = \text{Res}_{E/F} \mathbf{G}_m$ for a finite extension E/F . In this case $X^*(\hat{T}) = X_*(T)$ is the group ring $\mathbf{Z}[\Gamma_{E/F}]$. The norm maps $N: T \rightarrow \mathbf{G}_m$ and $N: \mathbf{Z}[\Gamma_{E/F}] \rightarrow \mathbf{Z}$ fit into the commutative diagram

$$\begin{array}{ccc} B(T) & \xrightarrow{N} & B(\mathbf{G}_m) \\ \kappa \downarrow & & \downarrow \kappa \\ X^*(\hat{T}^\Gamma) & \xrightarrow{N} & \mathbf{Z} \end{array}$$

and all four maps are isomorphisms. The claim follows from (4.7.1) and Step 0.

Step 2: $G = T$ is a torus. Let E/F be the splitting field of T and M a free $\mathbf{Z}[\Gamma_{E/F}]$ -module together with a Γ -equivariant surjection $M \rightarrow X_*(T)$. If S is the torus with $X_*(S) = M$ then S is a product of tori of the form $\text{Res}_{E/F} \mathbf{G}_m$ and we have a surjection $S \rightarrow T$ with connected kernel, and hence [Kot85, §1.9] a surjection $B(S) \rightarrow B(T)$, as well as a surjection $X_*(S)_\Gamma \rightarrow X_*(T)_\Gamma$. We have $\mathcal{E} \cong \mathcal{E}_b$ for some $b \in B(T)$; let $b_S \in B(S)$ be a lift of b and let $\nu_S \in X_*(S)$ be a lift of ν . The claim follows from (4.7.1) and Step 1 applied to S .

Step 3: G_{der} is simply connected. According to [Kot97, §7.5] the map κ is given by $B(G) \rightarrow B(D) \rightarrow X^*(\widehat{D}^\Gamma) = X^*(Z(\widehat{G})^\Gamma)$, where $D = G/G_{\text{der}}$, and the claim follows from (4.7.1) and Step 2 applied to D .

Step 4: General G . Let $1 \rightarrow K \rightarrow \widetilde{G} \rightarrow G \rightarrow 1$ be a z -extension. Again we have a surjection $B(\widetilde{G}) \rightarrow B(G)$ as well as surjections $X_*(\widetilde{T}) \rightarrow X_*(T)$ for any maximal torus $\widetilde{T} \subset \widetilde{G}$ with image $T \subset G$. This allows us to lift both b and ν to elements $\widetilde{b} \in \widetilde{G}(\check{F})$ and $\widetilde{\nu} : \mathbf{G}_m \rightarrow \widetilde{G}$. The claim follows from (4.7.1) and Step 3 applied to \widetilde{G} . \square

4.8 The admissible locus, and spaces of shtukas

Definition 4.8.1. Let $b \in G(\check{F})$ and let $\mu \in X_*(T)$. The b -admissible locus in $\text{Gr}_{G, \leq \mu}$ is the subfunctor $\text{Gr}_{G, \leq \mu}^{b\text{-adm}} \subset \text{Gr}_{G, \leq \mu}$ assigning to a perfectoid C -algebra R the set of $g \in \text{Gr}_{G, \leq \mu}(R)$ such that for every geometric point $x: \text{Spa } C' \rightarrow \text{Spa } R$, $x^* \mathcal{E}_b[g]$ is isomorphic to the trivial G -bundle \mathcal{E}_1 on $X_{(C')^\flat}$.

Proposition 4.8.2. $\text{Gr}_{G, \leq \mu}^{b\text{-adm}} \subset \text{Gr}_{G, \leq \mu}$ is an open subfunctor, and thus is a diamond. It is empty if $b \notin B(G, \mu)$.

Proof. The locus of $g \in \text{Gr}_{G, \leq \mu}$ where $\mathcal{E}_b[g]$ is semistable (that is, where $\mathcal{E}_b[g]$ corresponds to a class in $B(G)_{\text{bas}}$) is open, by the ‘‘semicontinuity of the slope polygon’’ [KL15, Theorem 7.4.5]. Furthermore, the locus where $\kappa(\mathcal{E}_b[\nu]) = 0$ is open (and closed) by [Far, Theorem 2.15]. Since $\kappa: B(G)_{\text{bas}} \rightarrow \pi_1(G)_\Gamma$ is a bijection, we have (over a geometric point) $\mathcal{E}_b[g] \cong \mathcal{E}_1$ if and only if $\mathcal{E}_b[g]$ is basic and $\kappa(\mathcal{E}_b[g]) = 0$. Therefore $\text{Gr}_{G, \leq \mu}^{b\text{-adm}}$ is open.

The second assertion follows from Lemma 4.7.3. \square

If \mathcal{E} is a G -bundle on X_R whose pullback to every geometric point is trivial, then the space of trivializations of \mathcal{E} is a pro-étale $G(F)$ -torsor over R [KL15, Theorem 9.3.13]. Trivializing $\mathcal{E}_b[g]$ for the family of b -admissible g gives the space of (infinite-level) local G -shtukas.

Definition 4.8.3. Let $\mathcal{M}_{G,b} \rightarrow \mathrm{Spd}C$ denote the sheafification of the presheaf which assigns to a perfectoid C -algebra R the set of isomorphisms

$$\alpha: \mathcal{E}_b|_{X_{R^b} \setminus D_R} \rightarrow \mathcal{E}_1|_{X_{R^b} \setminus D_R},$$

Recall from §4.5 that \mathcal{E}_b comes equipped with a trivialization over the formal neighborhood of D_R , i.e. an isomorphism $\beta: \mathcal{E}_b \otimes B_{\mathrm{dR}}^+(R) \rightarrow \mathcal{E}_1 \otimes B_{\mathrm{dR}}^+(R)$. Then $(\alpha \otimes B_{\mathrm{dR}}(R))^{-1} \circ (\beta \otimes B_{\mathrm{dR}}(R))$ is an automorphism of $\mathcal{E}_b \otimes B_{\mathrm{dR}}(R)$, hence an element of $J_b(B_{\mathrm{dR}}(R)) = G(B_{\mathrm{dR}}(R))$. The assignment sending α to the $G(B_{\mathrm{dR}}^+(R))$ -coset of this element gives a morphism $\mathcal{M}_{G,b} \rightarrow \mathrm{Gr}_G$ (the Grothendieck-Messing period map). Note that $\mathcal{E}_b[g] \cong \mathcal{E}_1$ via α , so that this morphism factors through the admissible locus. For a cocharacter μ , let $\mathcal{M}_{G,b,\mu} \subset \mathcal{M}_{G,b}$ be the pullback of $\mathrm{Gr}_{G,b,\leq\mu} \subset \mathrm{Gr}_{G,b}$.

The sheaf $\mathcal{M}_{G,b,\mu}$ admits an action of $J_b(F)$ lying over the action on $\mathrm{Gr}_{G,\leq\mu}^{b\text{-adm}}$, via $\alpha \mapsto \alpha \circ g^{-1}$ for $g \in J_b(F)$. It also admits an action of $G(F)$, via $\alpha \mapsto g \circ \alpha$ for $g \in G(F)$. This action is clearly simple and preserves each fiber of the Grothendieck-Messing period map. The Beauville-Laszlo gluing lemma 4.7.1 implies that this action is transitive on each fiber, thus the Grothendieck-Messing period map is a $G(F)$ -torsor. We record this observation as follows.

Proposition 4.8.4. $\mathcal{M}_{G,b,\mu} \rightarrow \mathrm{Gr}_{G,\leq\mu}^{b\text{-adm}}$ is a $G(F)$ -torsor. In particular $\mathcal{M}_{G,b,\mu}$ is a locally spatial diamond.

In situations where there exists a tower of Rapoport-Zink spaces attached to (G, b, μ) , the inverse limit along the tower will be $\mathcal{M}_{G,b,\mu}$; see [SW13] for the case of $G = \mathrm{GL}_n$ and μ minuscule. In general, the $\mathcal{M}_{G,b,\mu}/K$ (for $K \subset G(F)$ compact open) will play the role of the tower of “local Shimura varieties” expected by [RV14].

Example 4.8.5. When $G = T$ is a torus, $\mathrm{Gr}_T \cong X_*(T)$. If $\mu \in X_*(T)$, then $\mathrm{Gr}_{T,\mu} = \{\mu\}$. We have an isomorphism $B(T) \cong X_*(T)_\Gamma$ (where $\Gamma = \mathrm{Gal}(\overline{F}/F)$), and $B(T, \mu) = \{b\}$ for the class $b \in B(T)$ identified with the image of μ in $X_*(T)_\Gamma$. Then $\mathcal{M}_{T,b,\mu}$ is a principal homogenous space for $T(F)$. Note that $J_b(F) = T(F)$, and the actions of $J_b(F)$ and $T(F)$ on $\mathcal{M}_{T,b,\mu}$ agree.

4.9 Duality for spaces of shtukas in the basic case

For $b \in G(\check{F})$ basic, [RV14, Conjecture 5.8] predicts an isomorphism between local Shimura varieties attached to the groups G and J_b . Indeed, there is an isomorphism $\mathcal{M}_{G,b,\mu} \cong \mathcal{M}_{J_b,b^*,\mu^*}$, as we explain below. This is a generalization of the duality theorem in [SW13, Theorem 7.2.3], which treats the case $G = \mathrm{GL}_n$ and μ minuscule.

Recall the map $G(\check{F}) \rightarrow G(\check{F}) = J_b(\check{F})$ given by $h \mapsto h^* = hb^{-1}$. Let $\check{b} = b^{-1} = 1^*$, considered as an element of $J_b(\check{F})$. For a cocharacter μ of G , define $\check{\mu}$ as the composite of μ^{-1} with $G_{\check{F}} \cong J_{b,\check{F}}$.

Proposition 4.9.1. *Let $b \in G(\check{F})$ be basic. There is an isomorphism $\mathcal{M}_{G,b,\mu} \rightarrow \mathcal{M}_{J_b,\check{b},\check{\mu}}$ which respects the actions of $G(F) \times J_b(F)$.*

Proof. By Lemma 4.6.1, $\mathcal{M}_{G,b}$ is isomorphic to the sheafification of the presheaf which assigns to a perfectoid C -algebra R the set of isomorphisms of G -torsors $\alpha: \mathcal{T}_b \rightarrow \mathcal{T}_1$ over $X_{R^{\flat}} \setminus D_R$. Given such an α , we obtain an isomorphism of J_b -torsors $\mathcal{T}_b \times_G \mathcal{T}_b \rightarrow \mathcal{T}_1 \times_G \mathcal{T}_b$. Applying Lemma 4.6.2, this amounts to an isomorphism $\alpha^*: \check{\mathcal{T}}_1 \rightarrow \check{\mathcal{T}}_{\check{b}}$. Then $\alpha \mapsto \check{\alpha} = (\alpha^*)^{-1}$ induces an isomorphism of sheaves $\mathcal{M}_{G,b} \rightarrow \mathcal{M}_{J_b,\check{b}}$. To finish the proof, one has to observe that α is a modification of type $\leq \mu$ if and only if $\check{\alpha}$ is a modification of type $\leq \check{\mu}$; this is a matter of unraveling the definitions. \square

4.10 Definition of $\mathcal{H}(G, b, \mu)[\pi]$, and Assumption 3

For a finite local ring Λ which is ℓ^n -torsion for some $n \geq 1$, we have the intersection complex $\mathrm{IC}_{\mu,\Lambda} \in D(\mathrm{Gr}_{G,b,\leq\mu,\acute{e}t}, \Lambda)$. Let us write $\mathrm{IC}_{\mu,\Lambda}$ again for the pullback of this to $\mathcal{M}_{G,b,\mu}$. Then we have the compactly supported cohomology $H_c^i(\mathcal{M}_{G,b,\mu}, \mathrm{IC}_{\mu,\Lambda})$, a Λ -module with an action of $J_b(F) \times G(F)$.

Definition 4.10.1. Fix a finite extension E/\mathbf{Q}_ℓ . For each open compact subgroup $K \subset G(F)$ we define

$$R\Gamma_c(G, b, \mu, K) = \varprojlim_n R\Gamma_c(\mathcal{M}_{G,b,\mu}/K, \mathrm{IC}_{\mu,\mathcal{O}_E/\ell^n}) \otimes_{\mathcal{O}_E} E.$$

For a smooth irreducible admissible representation ρ of $J_b(F)$ with coefficients in E we define

$$H^i(G, b, \mu)[\rho] = \varinjlim_K \mathrm{Ext}_{J_b(F)}^i(R\Gamma_c(G, b, \mu, K), \rho),$$

a smooth representation of $G(F)$.

It will be helpful to reinterpret these representations in terms of objects living on the proper diamond $\mathrm{Gr}_{G, \leq \mu}$. Let $\Lambda = \mathcal{O}_E/\ell^n$, and let $\pi: G(F) \rightarrow \mathrm{GL}(V)$ be a smooth admissible representation of $G(F)$ on a free Λ -module V . Let \mathcal{L}_π be the sheaf on $\mathrm{Gr}_{G, \mu}^{b\text{-adm}}$ formed by descent of the constant sheaf V through $\mathcal{M}_{G, b, \mu} \rightarrow \mathrm{Gr}_{G, \mu}^{b\text{-adm}}$ via π , as in Example 3.6.3. Let $j: \mathrm{Gr}_{G, \leq \mu}^{b\text{-adm}} \rightarrow \mathrm{Gr}_{G, \leq \mu}$ denote the inclusion.

Definition 4.10.2. We define an object $\mathcal{F}_\pi \in D(\mathrm{Gr}_{G, \leq \mu, \text{ét}}, \Lambda)$ by

$$\mathcal{F}_\pi := j_! \mathcal{L}_\pi \otimes \mathrm{IC}_{\mu, \Lambda}.$$

Since $\mathcal{M}_{G, b, \mu} \rightarrow \mathrm{Gr}_{G, \leq \mu}$ and $\mathrm{IC}_{\mu, \Lambda}$ are $J_b(F)$ -equivariant, \mathcal{F}_π descends to a sheaf on $[\mathrm{Gr}_{G, \leq \mu} / J_b(F)]_{\text{ét}}$, which we still call \mathcal{F}_π .

Assumption 3. \mathcal{F}_π is strongly reflexive on $[\mathrm{Gr}_{G, \leq \mu} / J_b(F)]_{\text{ét}}$.

The following lemma reduces the main theorem to proving a certain trace identity.

Lemma 4.10.3. *Work under Assumptions 1-3. In order to prove Theorem 1.0.4, it suffices to show the following. Let $\phi: W_F \rightarrow {}^L G$ be a discrete Langlands parameter, and let $\pi \in \Pi_\phi(G)$. Assume that V contains a $G(F)$ -invariant lattice V_0 ; let π_n be the representation of $G(F)$ on V_0/ℓ^n . Let $f \in C_c^\infty(J_b(F), \Lambda)$ be supported on the regular elliptic locus. Then*

$$\mathrm{tr}(f | H^*(\mathrm{Gr}_{G, \leq \mu}, \mathcal{F}_{\pi_n})) = (-1)^d \sum_{\rho \in \Pi_\phi(J_b)} \dim \mathrm{Hom}_{S_\phi}(\delta_{\pi, \rho}, r_\mu) \mathrm{tr} \rho(f). \quad (4.10.1)$$

Proof. First we claim that Theorem 1.0.4 is invariant under replacing a representation π of $G(F)$ with $\pi \otimes \chi$, where χ is a character of $G^{\mathrm{ab}}(F)$ and G^{ab} is the maximal abelian quotient of G . We have a $G(F) \times J_b(F)$ -equivariant morphism $\mathcal{M}_{G, b, \mu} \rightarrow \mathcal{M}_{G^{\mathrm{ab}}, b^{\mathrm{ab}}, \mu^{\mathrm{ab}}}$, where b^{ab} and μ^{ab} are the images of b and μ respectively. From Example 4.8.5, $\mathcal{M}_{G^{\mathrm{ab}}, b^{\mathrm{ab}}, \mu^{\mathrm{ab}}}$ is a principal homogenous space for $G^{\mathrm{ab}}(F)$, on which $G(F)$ and $J_b(F)$ both act through their common map to $G^{\mathrm{ab}}(F)$. This implies that each $H^i(\mathcal{M}_{G, b, \mu}, \mathrm{IC}_{\mu, n})$ is induced from a representation of the group $(G(F) \times J_b(F))^1$ consisting of pairs with the same image in $G^{\mathrm{ab}}(F)$. This in turn implies that whenever a representation ρ of $J_b(F)$ is contained in $H^*(G, b, \mu)[\pi]$, then $\rho \otimes \chi$ is contained in $H_c^*(G, b, \mu)[\pi \otimes \chi]$ for each character of $G^{\mathrm{ab}}(F)$, which is the claim.

We fix an isomorphism $\mathbf{C} \rightarrow \bar{\mathbf{Q}}_l$ and use it to interpret π as having $\bar{\mathbf{Q}}_l$ -coefficients. By Corollary A.2.2 and the above argument we may assume that π has a $G(F)$ -invariant lattice. Let π_n be the reduction of a $G(F)$ -invariant

lattice in π modulo ℓ^n . Let $K \subset J_b(F)$ be a compact open subgroup. Under Assumption (3), \mathcal{F}_{π_n} is reflexive on $[\mathrm{Gr}_{G,b,\mu}/\underline{K}]_{\acute{e}t}$. By Theorem 3.6.6(3) we have an isomorphism

$$\begin{aligned} & \mathrm{RHom}(R\Gamma_c([\mathrm{Gr}_{G,b,\mu}^{b\text{-adm}}/\underline{K}]_{\acute{e}t}, \mathcal{L}_{\pi_n^\vee} \otimes \mathrm{IC}_{\mu,n}) \\ & \cong \mathrm{RHom}_{G(F)}(R\Gamma_c((\mathcal{M}_{G,b,\mu}/\underline{K})_{\acute{e}t}, \mathrm{IC}_{\mu,n}), \pi_n). \end{aligned}$$

Each cohomology group of this object is free of finite rank over $\Lambda = \mathcal{O}_E/\ell^n$. By hypothesis we have an expression for the trace of a Hecke operator $f \in C_c^\infty(J_b(F), \Lambda)$ on the right hand side whenever f is K -bi-invariant and supported on the regular elliptic locus. That expression persists when we apply \varprojlim_n and invert ℓ . We can then apply \varinjlim_K , so that we have an expression for the trace of any $f \in C_c^\infty(J_b(F), E)$ on

$$\varinjlim_K \mathrm{RHom}_{G(F)}(R\Gamma_c((\mathcal{M}_{G,b,\mu}/\underline{K})_{\acute{e}t}, \mathrm{IC}_{\mu,E}), \pi),$$

at least when f has support on the regular elliptic locus. By Proposition 4.9.1 the above is $R\Gamma(J_b, \check{b}, \check{\mu})[\pi]$. We have shown that

$$\mathrm{tr}(f|H^*(J_b, \check{b}, \check{\mu})[\pi]) = (-1)^d \sum_{\rho \in \Pi_{\phi^\vee}(J_b)} \dim \mathrm{Hom}_{S_{\phi^\vee}}(\delta_{\pi^\vee, \rho}, r_\mu) \mathrm{tr} \rho(f).$$

Since $H^*(J_b, \check{b}, \check{\mu})[\pi]$ is admissible, this shows that

$$H^*(J_b, \check{b}, \check{\mu})[\pi] = (-1)^d \sum_{\rho \in \Pi_{\phi^\vee}(J_b)} [\dim \mathrm{Hom}_{S_\phi}(\delta_{\pi^\vee, \rho}, r_\mu)] \rho$$

as elements of $\mathrm{Groth}(J_b(F))^{\mathrm{ell}}$. This is Theorem 1.0.4 for the triple $(J_b, \check{b}, \check{\mu})$ and the representation π playing the role of (G, b, μ) and ρ . \square

4.11 Remarks on Assumption 3

Assumption 3 is related to Scholze's program to interpret Langlands parameters in terms of reflexive sheaves on Bun_G , the moduli stack of G -bundles on the Fargues-Fontaine curve. Let us briefly explain the connection.

Scholze shows that an object \mathcal{F} of $D(\mathrm{Bun}_{G,\acute{e}t}, \Lambda)$ is reflexive if and only if for all points $x \in \mathrm{Bun}_G$, (all cohomology groups of) the stalk \mathcal{F}_x is an admissible representation of $\mathrm{Aut} x$. In particular there are the points coming from basic isocrystals: for basic $b \in B(G)$, we have the substack $j_b: \mathrm{Bun}_G^b \subset \mathrm{Bun}_G$ which classifies G -bundles that are isomorphic to \mathcal{E}_b on every geometric

point; then $\text{Bun}_B^b = BJ_b(F)$. For an admissible representation ρ of $J_b(F)$, let L_ρ be the corresponding sheaf on $\text{Bun}_{G,\text{ét}}^b$; then $j_b^!L_\rho$ is reflexive.

The stack Bun_G comes equipped with a family of Hecke correspondences for each cocharacter μ [Far]:

$$\begin{array}{ccc} & \text{Hecke}_{\leq \mu} & \\ h_1 \swarrow & & \searrow h_2 \\ \text{Bun}_G & & \text{Bun}_G \times \text{Spd } C. \end{array}$$

The Hecke operator $\mathcal{H}_\mu: D(\text{Bun}_G, \Lambda) \rightarrow D(\text{Bun}_G \times \text{Spd } C, \Lambda)$ is defined by $\mathcal{F} \mapsto h_{2!}(h_1^*\mathcal{F} \otimes \text{IC}_{\mu,\Lambda})$. Scholze shows that if \mathcal{F} is reflexive, then so is $h_2^*\mathcal{F} \otimes \text{IC}_{\mu,\Lambda}$. (A special case occurs when μ is minuscule, so that $\text{IC}_{\mu,\Lambda}$ is a shift and twist of the constant sheaf; then h_1 is a “smooth” morphism of diamond stacks, and reflexive sheaves are preserved under smooth pullbacks.) Since h_2 is proper, \mathcal{H}_μ preserves reflexive sheaves.

There is a cartesian diagram

$$\begin{array}{ccc} [\text{Gr}_{G,\leq \mu} / J_b(F)] & \xrightarrow{\alpha} & \text{Hecke}_{\leq \mu} \\ \downarrow & & \downarrow h_2 \\ \text{Bun}_G^b \times \text{Spa } C & \xrightarrow{j_b \times \text{id}} & \text{Bun}_G \times \text{Spa } C, \end{array}$$

in which the horizontal maps are open immersions. Since $h_1^*j_{1!}L_\pi \otimes \text{IC}_{\mu,\Lambda}$ is reflexive, so is $\alpha^*h_1^*j_{1!}L_\pi \otimes \text{IC}_{\mu,\Lambda}$. We also have a cartesian diagram

$$\begin{array}{ccc} [\text{Gr}_{G,\leq \mu}^{b\text{-adm}} / J_b(F)] & \xrightarrow{\beta} & \text{Bun}_G^1 \\ \downarrow & & \downarrow j_{1!} \\ [\text{Gr}_{G,\leq \mu} / J_b(F)] & \xrightarrow{h_1 \circ \alpha} & \text{Bun}_G, \end{array}$$

which gives us a base change isomorphism $\alpha^*h_1^*j_{1!}L_\pi \otimes \text{IC}_{\mu,\Lambda} \cong j_1^*\alpha^*L_\pi \otimes \text{IC}_\mu$. The morphism α onto $\text{Bun}_G^1 \cong BG(F)$ corresponds to the $G(F)$ -torsor $\mathcal{M}_{G,b,\mu}$, so that $\alpha^*L_\pi \cong \mathcal{L}_\pi$. Therefore \mathcal{F}_π is reflexive. (Strong reflexivity requires a further argument.)

5 Proof of the main theorem

5.1 Elliptic fixed points on Gr_G

In this section, C is a complete algebraically closed field containing F . Write B_{dR}^+ and B_{dR} for the corresponding rings of Fontaine. Then B_{dR}^+ is a com-

plete discrete valuation ring containing \check{F} with residue field C . It also contains a unique copy of \bar{F} . We have $\mathrm{Gr}_G(C) = G(B_{\mathrm{dR}})/G(B_{\mathrm{dR}}^+)$.

Remark 5.1.1. We can consider the extended Bruhat-Tits building of G over the discretely valued field B_{dR} and identify the $G(B_{\mathrm{dR}})$ -set $\mathrm{Gr}_G(C)$ with a piece of this building as follows. The inclusions $F \rightarrow \bar{F} \rightarrow B_{\mathrm{dR}}^+ \rightarrow B_{\mathrm{dR}}$ show that the base change $G \times B_{\mathrm{dR}}^+$ is a split reductive group scheme with generic fiber $G \times B_{\mathrm{dR}}$. Let \mathcal{B} be the (reduced) Bruhat-Tits building of the split reductive group $G \times B_{\mathrm{dR}}$ and let $K = G(B_{\mathrm{dR}}^+)$. Then by [BT84, 5.1.40] there exists a hyperspecial point $\bar{o} \in \mathcal{B}$ such that $K = \mathfrak{G}_{\bar{o}}^0(B_{\mathrm{dR}}^+)$, where $\mathfrak{G}_{\bar{o}}^0$ is the connected parahoric B_{dR}^+ -group scheme with generic fiber $G \times B_{\mathrm{dR}}$ associated to the point \bar{o} . The point \bar{o} can be characterized by [BT84, 4.6.29] as the unique fixed point of K . Let $\mathcal{B}^{\mathrm{ext}}$ be the extended Bruhat-Tits building of $G \times B_{\mathrm{dR}}$. Recall that $\mathcal{B}^{\mathrm{ext}} = \mathcal{B} \times X_*(A_G)_{\mathbf{R}}$, where A_G is the connected center of G (automatically split over B_{dR}). The group $G(B_{\mathrm{dR}})$ acts on $X_*(A_G)_{\mathbf{R}}$ via the isomorphism $X_*(A_G)_{\mathbf{R}} \rightarrow X_*(A'_G)_{\mathbf{R}}$, where A'_G is the maximal abelian quotient of G . Let $o = (\bar{o}, z)$ be any point in $\mathcal{B}^{\mathrm{ext}}$ lying over \bar{o} . Then K can be characterized as the full stabilizer of o in $G(B_{\mathrm{dR}})$ – it is clear that K stabilizes o and the converse inclusion follows from the Cartan decomposition $G(B_{\mathrm{dR}}) = KX_*(T)K$ (which relies on \bar{o} being hyperspecial) and the fact that $X_*(T)$ acts on the apartment of T in $\mathcal{B}^{\mathrm{ext}}$ by translations. It follows that the action of $G(B_{\mathrm{dR}})$ on $\mathcal{B}^{\mathrm{ext}}$ provides a $G(B_{\mathrm{dR}})$ -equivariant bijection from the coset space $G(B_{\mathrm{dR}})/G(B_{\mathrm{dR}}^+)$ to the orbit of $G(B_{\mathrm{dR}})$ through o .

Proposition 5.1.2. *Let $g \in G(\check{F})$ be a regular semisimple element, and let $T \subset G_{\check{F}}$ be its connected centralizer. Then an element of $\mathrm{Gr}_G(C)$ is fixed by g if and only if it is fixed by all of $T(\check{F})$.*

Proof. If $x \in G(B_{\mathrm{dR}})/G(B_{\mathrm{dR}}^+)$ is a g -fixed point, then its image in $\mathcal{B}^{\mathrm{ext}}$ is a g -fixed point belonging to the orbit of o and we can write $x = ho$ for some $h \in G(B_{\mathrm{dR}})$. For every root $\alpha : T \rightarrow \mathbf{G}_m$ we have $\alpha(g) \in \bar{F}^\times$ and hence $\alpha(g) \notin 1 + \ker(\theta)$, where $\ker(\theta)$ is the maximal ideal in B_{dR}^+ . According to [Tit79, 3.6.1] the image of x in \mathcal{B} belongs to the apartment \mathcal{A} of T . At the same time, $g \in G(B_{\mathrm{dR}}^+) = K$ also fixed \bar{o} , so for the same reason $\bar{o} \in \mathcal{A}$. Thus \bar{o} belongs to both apartments \mathcal{A} and $h^{-1}\mathcal{A}$. Since K acts transitively on the apartments containing \bar{o} [BT84, 4.6.28], we can multiply h on the right by an element of K to ensure that $h^{-1}\mathcal{A} = \mathcal{A}$. By [BT72, 7.4.10] we then have $h \in N(T, G)(B_{\mathrm{dR}})$. Since \bar{o} is hyperspecial, every Weyl reflection is realized in K and hence we may again modify h on the right to achieve $h \in T(B_{\mathrm{dR}})$. We see now that $x = ho$ is fixed by all of $T(\check{F}) \subset T(B_{\mathrm{dR}}^+)$.

and that furthermore the coset $x = hG(B_{\mathrm{dR}}^+)$ is the image of the coset $hT(B_{\mathrm{dR}}^+)$. \square

Let Gr_G^g be the fixed point locus of g , in the sense of (3.5.1). Recall from Example 4.1.4 the isomorphism $X_*(T) \rightarrow \mathrm{Gr}_T$ sending μ to $L_\mu = \mu(\xi)$.

Corollary 5.1.3. *The morphism $\mathrm{Gr}_G^g \rightarrow \mathrm{Gr}_G$ factors through an isomorphism $\mathrm{Gr}_G^g \xrightarrow{\sim} \mathrm{Gr}_T$.*

Corollary 5.1.4. *Let $B = TU$ be a Borel subgroup of $G_{\bar{F}}$ containing T , and let $S_\nu = U(B_{\mathrm{dR}}) \cdot L_\nu \subset \mathrm{Gr}_G$. Then $S_\nu(C)$ has a unique g -fixed element, namely L_ν .*

Proof. Let $n \in U(B_{\mathrm{dR}})$ be such that nL_ν is a g -fixed point of $S_\nu(C)$. Then $nL_\nu = L_\mu$ for some $\mu \in X_*(T)$ by Corollary 5.1.3. Using Remark 5.1.1 we obtain

$$K \ni \mu(\xi)^{-1}n\nu(\xi) = (\mu(\xi)^{-1}n\mu(\xi))(\nu - \mu)(\xi).$$

Now $K \cap B(B_{\mathrm{dR}}) = U(B_{\mathrm{dR}}^+) \rtimes T(B_{\mathrm{dR}}^+)$ and $(\nu - \mu)(\xi) \in T(B_{\mathrm{dR}}^+)$ implies $\nu = \mu$, since $\xi \in B_{\mathrm{dR}}^+$ is a uniformizer. \square

5.2 Calculation of local invariants

Let $T \subset G_{\bar{F}}$ be a maximal torus. Let $g \in T(\check{F})$ be a regular semisimple element and, let $\mu, \nu \in X_*(T)$.

Let E/\mathbf{Q}_ℓ be a finite extension, and let $\Lambda = \mathcal{O}_E/\ell^n$ for some n . Let $\mathrm{IC}_{\mu, \Lambda}$ be the intersection complex on $\mathrm{Gr}_{G, \leq \mu}$ with coefficients in Λ . It is $G(B_{\mathrm{dR}}^+)$ -equivariant, so in particular it is g -equivariant. Furthermore $\mathrm{IC}_{\mu, \Lambda}$ is strongly reflexive by Assumption 3 applied to the trivial representation π .

Proposition 5.2.1. *The local term of g at L_ν is given by*

$$\mathrm{loc}_{L_\nu}(g, \mathrm{IC}_\mu) = (-1)^{2\rho(\nu)} \dim r_\mu[\nu].$$

Proof. Recall from Subsection 4.2 the facts about \bar{S}_ν and its cohomology that we are assuming. Let $j: S_\nu \hookrightarrow \bar{S}_\nu$ denote the inclusion, and let $i: \partial S_\nu \rightarrow \bar{S}_\nu$ denote the inclusion of the complement. Consider the exact sequence of $G(B_{\mathrm{dR}}^+)$ -equivariant sheaves on the proper diamond $\bar{S}_\nu \cap \mathrm{Gr}_{\leq \mu}$:

$$0 \rightarrow j_!j^* \mathrm{IC}_\mu \rightarrow \mathrm{IC}_\mu \rightarrow i_*i^* \mathrm{IC}_\mu \rightarrow 0. \quad (5.2.1)$$

We will consider the trace of $g \in G(\check{F})$ acting on the cohomology of each term. The cohomology $H^q(\bar{S}_\nu, j_!j^* \mathrm{IC}_\mu) = H_c^q(S_\nu, \mathrm{IC}_\mu)$ is zero unless $q =$

$2\rho(\nu)$. Since S_ν admits an action of the algebraic group T , and IC_μ is equivariant for this action, we have a morphism from T onto the constant scheme $\mathrm{End} H^{2\rho(\nu)}(S_\nu, \mathrm{IC}_\mu)$. Since T is connected, this morphism must be constant and the action of T is trivial. Therefore $\mathrm{tr}(g|H^*(S_\nu, j_! j^* \mathrm{IC}_\mu)) = (-1)^{2\rho(\nu)} \dim H_c^{2\rho(\nu)}(S_\nu, \mathrm{IC}_\mu) = (-1)^{2\rho(\nu)} \dim r_\mu[\nu]$.

Since \overline{S}_ν and ∂S_ν are proper diamonds, Corollary 3.5.4 can be used to give an expression for the trace of g on the cohomology of sheaves on either space. By Corollary 5.1.4, the fixed points of g on \overline{S}_ν (resp., ∂S_ν) are the points $L_{\nu'}$ for $\nu' \leq \nu$ (resp., $\nu' < \nu$). We get

$$\mathrm{tr}(g|R\Gamma(\overline{S}_\nu, \mathrm{IC}_\mu)) = \sum_{\nu' \leq \nu} \mathrm{loc}_{L_{\nu'}}(g, \mathrm{IC}_\mu)$$

and

$$\mathrm{tr}(g|R\Gamma(\overline{S}_\nu, i_* i^* \mathrm{IC}_\mu)) = \mathrm{tr}(g|R\Gamma(\partial S_\nu, \mathrm{IC}_\mu)) = \sum_{\nu' < \nu} \mathrm{loc}_{L_{\nu'}}(g, \mathrm{IC}_\mu).$$

Combining these observations with (5.2.1) gives the result. \square

5.3 The Beauville-Laszlo morphism on elliptic fixed points

Let $T \subset J_b$ be a F -rational elliptic maximal torus. We fix a complete algebraically closed field C/F ; this determines an (absolute) Fargues-Fontaine curve $X = X(C^b)$ and a closed point $\infty \in X$. The completion $\widehat{\mathcal{O}}_{X, \infty}$ is the Fontaine period ring $B_{\mathrm{dR}}^+ = B_{\mathrm{dR}}^+(C)$; this is a discrete valuation ring with uniformizer ξ , residue field C and fraction field B_{dR} .

Let $\nu: \mathbf{G}_m \rightarrow T$ be a cocharacter defined over C . Then ν determines a geometric point $L_\nu \in \mathrm{Gr}_G(C)$. Let $\mathcal{E}_b[\nu]$ be the modification of \mathcal{E}_b at ∞ via ν as in Definition 4.7.2. Thus we have isomorphisms

$$\mathcal{E}_b[\nu]|_{X \setminus \{\infty\}} \xrightarrow{\sim} \mathcal{E}_b|_{X \setminus \{\infty\}} \quad (5.3.1)$$

$$\mathcal{E}_b[\nu]_\infty \otimes_{B_{\mathrm{dR}}^+} B_{\mathrm{dR}} \xrightarrow{\sim} \mathcal{E}_{b, \infty} \otimes_{B_{\mathrm{dR}}^+} B_{\mathrm{dR}} \quad (5.3.2)$$

of G -bundles over $X \setminus \{\infty\}$ and B_{dR} , respectively; the second identification carries $\mathcal{E}_b[\nu]_\infty$ onto the image of $\mathcal{E}_{b, \infty}$ under $\nu(\xi) \in T(B_{\mathrm{dR}})$.

The group $T(F)$ acts on \mathcal{E}_b and commutes with $\nu(\xi)$, and so by Lemma 4.7.1 it acts on $\mathcal{E}_b[\nu]$.

Proposition 5.3.1. *Let $b \in G(\check{F})$ be an element in the unique basic class $[b] \in B(G, \{\nu\})$.*

1. The modified vector bundle $\mathcal{E}_b[\nu]$ is trivial: there exists an isomorphism $\mathcal{E}_b[\nu] \cong \mathcal{E}_1$. In other words L_ν lies in the b -admissible locus $\mathrm{Gr}_G^{b\text{-adm}}$.
2. There exists a maximal F -rational torus $S \subset G$ and an isomorphism $\iota_{b,\nu}: T \rightarrow S$, such that for $g \in T(F)$, the isomorphism in (1) carries g onto $\iota_{b,\nu}(g)$. The pair $(S, \iota_{b,\nu})$ is well-defined up to conjugacy in $G(F)$.
3. The elements $g \in T(F)$ and $\iota_{b,\nu}(g)$ are related whenever g is strongly regular. Thus we have an element $\mathrm{inv}[b](\iota_{b,\nu}(g), g) \in B(S)$. The image of $\mathrm{inv}[b](\iota_{b,\nu}(g), g)$ under the isomorphism $B(S) \cong X^*(\widehat{S}^\Gamma)$ equals the restriction of $\iota_{b,\nu} \circ \nu \in X^*(\widehat{S})$ to \widehat{S}^Γ .

Proof. Let $b' \in G(\check{F})$ be an element whose class in $B(G)$ corresponds to the isomorphism class of $\mathcal{E}_b[\nu]$, and choose an isomorphism

$$\gamma: \mathcal{E}_b[\nu] \rightarrow \mathcal{E}_{b'}.$$

We first claim that $[b']$ is basic. We have the algebraic group $J_{b'}$, a priori an inner form of a Levi subgroup M^* of G^* , where G^* is as before the quasi-split inner form of G . Showing that $[b']$ is basic is equivalent to showing $M^* = G^*$. Let $g \in T(F)$ be a strongly regular semisimple element, so that $T = \mathrm{Cent}(g, J_b)$, and let $g' \in \mathrm{Aut} \mathcal{E}_{b'}$ be the automorphism induced by g via $\gamma: g' = \gamma g \gamma^{-1}$.

Recall from §4.5 that both $\mathcal{E}_b[\nu]$ and $\mathcal{E}_{b'}$ come equipped with a trivialization over B_{dR}^+ . The stalks of g' , g , and γ at the point $\infty: \mathrm{Spec} C \rightarrow X$ are thus naturally identified with elements of $G(B_{\mathrm{dR}}^+)$. Among them, $g'_\infty \in P(B_{\mathrm{dR}}^+)$, where $P \subset G$ is a parabolic subgroup with Levi factor $J_{b'}$. Let \bar{g}'_∞ be the image of g'_∞ under $P(B_{\mathrm{dR}}^+) \rightarrow J_{b'}(B_{\mathrm{dR}}^+)$. By Lemma 4.5.3, $\bar{g}'_\infty \in J_{b'}(F)$. By Lemma A.3.1 \bar{g}'_∞ is conjugate to g'_∞ , so \bar{g}'_∞ is conjugate to g_∞ in $G(B_{\mathrm{dR}})$. Since g, \bar{g}' are both regular semi-simple \bar{F} -points of G , being conjugate in $G(B_{\mathrm{dR}})$ is the same as being conjugate in $G(\bar{F})$. Their centralizers, being F -rational tori, are thus isomorphic over F . Thus $J_{b'}$ contains a maximal torus that is elliptic for G . Elliptic maximal tori transfer across inner forms [Kot86, §10], which means that the Levi subgroup $M^* \subset G^*$ of which $J_{b'}$ is an inner form contains a maximal torus that is elliptic for G^* . Therefore $M^* = G^*$.

We have shown that $\mathcal{E}_b[\nu]$ is semi-stable, implying that $\mathrm{Aut} \mathcal{E}_{b'} = J_{b'}(F)$ and that $g' \in J_{b'}(F)$. Lemma 4.7.3 shows that $\kappa([b']) = \kappa([b]) - \nu = 0$. Therefore $[b'] = [1]$ by [Kot85, Proposition 5.6], which is (1).

Let us assume that $b' = 1$, so that γ is an isomorphism $\mathcal{E}_b[\nu] \cong \mathcal{E}_1$ and $g' \in G(F) = \mathrm{Aut} \mathcal{E}_1$. Let $S = \mathrm{Cent}(g', G)$. Conjugation by γ induces an

isomorphism of B_{dR} -rational tori $\iota_{b,\nu}: T \rightarrow S$ carrying g onto g' . We claim that this isomorphism respects the F -structures. As argued above there exists $y \in G(\bar{F})$ that conjugates g to g' . By Steinberg's theorem we can even take $y \in G(\check{F})$. The isomorphism $\text{Ad}(y): T \rightarrow S$ maps the F -point g to the F -point g' and is the only such isomorphism. But $\sigma_S \circ \text{Ad}(y) \circ \sigma_T$ also maps g to g' and thus must equal $\text{Ad}(y)$. In other words, $\text{Ad}(y)$ respects the F -structures of T and S . Finally, $\gamma^{-1}y \in G(B_{\text{dR}})$ centralizes g , so lies in $T(B_{\text{dR}})$, so $\text{Ad}(\gamma)$ and $\text{Ad}(y)$ induce the same isomorphism $T \rightarrow S$. This proves (2).

Since $g \in J_b(F)$, we have $g^\sigma = b^{-1}gb$. On the other hand, since $g' \in G(F)$ we have $(g')^\sigma = g'$. Therefore the element $b_S := yby^{-\sigma}$ commutes with g' and so lies in $S(\check{F})$. Recall that the class of b_S in $B(S)$ is the invariant $\text{inv}[b](g', g)$.

The element y^{-1} induces an isomorphism $\mathcal{E}_{b_S} \rightarrow \mathcal{E}_b$, and also an isomorphism between modifications $\mathcal{E}_{b_S}[\iota_{b,\nu} \circ \nu] \rightarrow \mathcal{E}_b[\nu]$. Composing this with γ gives an isomorphism $\gamma y^{-1}: \mathcal{E}_{b_S}[\iota_{b,\nu} \circ \nu] \rightarrow \mathcal{E}_1$. Since $\gamma y^{-1} \in S(B_{\text{dR}})$, this isomorphism descends to an isomorphism of S -bundles. By Lemma 4.7.3, the image of b_S in $B(S) \cong X^*(\widehat{S}^\Gamma)$ is $\iota_{b,\nu} \circ \nu|_{\widehat{S}^\Gamma}$, which establishes (3). \square

5.4 A character identity

Fix an element $b \in G(F^{\text{nr}})$ for which $[b] \in B(G)$ is basic, an elliptic F -rational maximal torus $T \subset J_b$, and a cocharacter $\mu \in X_*(T) = X^*(\widehat{T})$. Assume that $[b]$ is the unique basic class in $B(G, \{\mu\})$.

There is a canonical \widehat{G} -conjugacy class of embeddings $\widehat{T} \rightarrow \widehat{G}$, of which we fix an arbitrary representative and identify \widehat{T} with its image in \widehat{G} . Consider the irreducible representation r_μ of \widehat{G} with highest weight μ .

If $\nu \in X^*(\widehat{T})$ is any weight of r_μ then $\nu|_{Z(\widehat{G})} = \mu|_{Z(\widehat{G})}$ and consequently $[b] \in B(G, \{\nu\})$. Proposition 5.3.1 shows that there exists an isomorphism $\mathcal{E}_b[\nu] \cong \mathcal{E}_1$ and an isomorphism $\iota_{b,\nu}: T \rightarrow S$ onto a F -rational maximal torus $S \subset G$, which translates the action of $T(F)$ on $\mathcal{E}_b[\nu]$ into the action of $S(F)$ on \mathcal{E}_1 .

Proposition 5.4.1. *Let ϕ be a discrete L -parameter for G . Let $g \in T(F)$ be a regular element. For any $\pi \in \Pi_\phi(G)$ we have*

$$e(G) \sum_{\nu \in X^*(\widehat{T})} \Theta_\pi(\iota_{b,\nu}(g)) \dim r_\mu[\nu] = e(J_b) \sum_{\rho \in \Pi_\phi(J_b)} \dim \text{Hom}_{S_\phi}(\delta_{\pi,\rho}, r_\mu) \Theta_\rho(g). \quad (5.4.1)$$

Proof. We will combine the results of Proposition 5.3.1 with the character relations reviewed in Subsection 2.4. Let $s \in S_\phi$ be a semi-simple element, and let $\dot{s} \in S_\phi^+$ be a lift of it. Then we have the refined endoscopic datum $\dot{\mathfrak{e}} = (H, \mathcal{H}, \dot{s}, \eta)$ defined in (2.4.1); we choose as in that section a z -pair $\mathfrak{z} = (H_1, \eta_1)$. Then

$$\begin{aligned}
& e(G) \sum_{\pi' \in \Pi_\phi(G)} \operatorname{tr} \tau_{z, \mathfrak{w}, \pi'}(\dot{s}) \Theta_{\pi'}(\iota_{b, \nu}(g)) \\
\stackrel{(2.4.3)}{=} & \sum_{h_1 \in H_1(F)/\operatorname{st}} \Delta(h_1, \iota_{b, \nu}(g)) S_{\Theta_{\phi^s}}(h_1) \\
\stackrel{(2.4.4)}{=} & \sum_{h_1 \in H_1(F)/\operatorname{st}} \Delta(h_1, g) \langle \operatorname{inv}[b](g, \iota_{b, \nu}(g)), s_{h, g} \rangle S_{\Theta_{\phi^s}}(h_1) \\
\stackrel{(5.3.1)(3)}{=} & \sum_{h_1 \in H_1(F)/\operatorname{st}} \Delta(h_1, g) \nu(s_{h, g})^{-1} S_{\Theta_{\phi^s}}(h_1).
\end{aligned}$$

We multiply on either side by $\dim r_\mu[\nu]$ and sum over $\nu \in X_*(T)$ to obtain

$$\begin{aligned}
& e(G) \sum_{\nu \in X^*(\widehat{T})} \sum_{\pi' \in \Pi_\phi(G)} \operatorname{tr} \tau_{z, \mathfrak{w}, \pi'}(\dot{s}) \Theta_{\pi'}(\iota_{b, \nu}(g)) \dim r_\mu[\nu] \\
= & \sum_{h_1 \in H_1(F)/\operatorname{st}} \Delta(h_1, g) \sum_{\nu \in X^*(\widehat{T})} \nu(s_{h, g}^{-1}) \dim r_\mu[\nu] S_{\Theta_{\phi^s}}(h_1) \\
= & \sum_{h_1 \in H_1(F)/\operatorname{st}} \Delta(h_1, g) \operatorname{tr} r_\mu(s_{h, g}^{-1}) S_{\Theta_{\phi^s}}(h_1) \\
\stackrel{(*)}{=} & \operatorname{tr} \check{r}_\mu(s^\natural) \sum_{h_1 \in H_1(F)/\operatorname{st}} \Delta(h_1, g) S_{\Theta_{\phi^s}}(h_1) \\
\stackrel{(2.4.2)}{=} & \operatorname{tr} \check{r}_\mu(s^\natural) e(J_b) \sum_{\rho \in \Pi_\phi(J_b)} \operatorname{tr} \tau_{z, \mathfrak{w}, \rho}(\dot{s}) \Theta_\rho(g),
\end{aligned}$$

where $(*)$ holds since the image of $s_{h, g}$ under any admissible embedding $\widehat{T} \rightarrow \widehat{G}$ is conjugate to s^\natural in \widehat{G} and r_μ is a representation of \widehat{G} . Recall here that $s^\natural \in S_\phi$ is the image of \dot{s} under (2.3.2).

Multiply both sides of above equation by $\operatorname{tr} \check{r}_{z, \mathfrak{w}, \pi}(\dot{s})$. As functions of $\dot{s} \in S_\phi^+$ both sides then become invariant under $Z(\widehat{G})^+$ and thus become functions of the finite quotient $\bar{S}_\phi = S_\phi^+ / Z(\widehat{G})^+ = S_\phi / Z(\widehat{G})^\Gamma$. Now apply

$|\bar{S}_\phi|^{-1} \sum_{\bar{s} \in \bar{S}_\phi}$ to both sides to obtain an equality between

$$|\bar{S}_\phi|^{-1} e(G) \sum_{\bar{s} \in \bar{S}_\phi} \sum_{\nu \in X^*(\hat{T})} \sum_{\pi' \in \Pi_\phi(G)} \text{tr } \check{\tau}_{z, \mathfrak{w}, \pi}(\dot{s}) \text{tr } \tau_{z, \mathfrak{w}, \pi'}(\dot{s}) \Theta_{\pi'}(\iota_{b, \nu}(g)) \dim r_\mu[\nu]$$

and

$$|\bar{S}_\phi|^{-1} \sum_{\bar{s} \in \bar{S}_\phi} \text{tr } \check{r}_\mu(s^\natural) e(J_b) \sum_{\rho \in \Pi_\phi(J_b)} \text{tr } \check{\tau}_{z, \mathfrak{w}, \pi}(\dot{s}) \text{tr } \tau_{z, \mathfrak{w}, \rho}(\dot{s}) \Theta_\rho(g),$$

where in both formulas \dot{s} is an arbitrary lift of \bar{s} . Executing the sum over \bar{s} in the first of the two expressions gives

$$e(G) \sum_{\nu \in X^*(\hat{T})} \Theta_\pi(\iota_{b, \nu}(g)) \dim r_\mu[\nu],$$

which is the left side of Eq. (5.4.1). To treat the second expression we note that by definition $\check{\tau}_{z, \mathfrak{w}, \pi} \otimes \tau_{z, \mathfrak{w}, \rho}(\dot{s}) = \delta_{\pi, \rho}(s^\natural)$. Furthermore, the composition of the map (2.3.2) with the natural projection $S_\phi \rightarrow S_\phi/Z(\hat{G})^\Gamma$ is equal to the natural projection $S_\phi^+ \rightarrow S_\phi^+/Z(\hat{G})^+ = S_\phi/Z(\hat{G})^\Gamma = \bar{S}_\phi$. Thus s^\natural is simply a lift of \bar{s} to S_ϕ . This implies that

$$|\bar{S}_\phi|^{-1} \sum_{\bar{s} \in \bar{S}_\phi} \text{tr } \check{r}_\mu(s^\natural) \delta_{\pi, \rho}(s^\natural) = \dim \text{Hom}_{\mathbf{C}}(\delta_{\pi, \rho}, r_\mu)^{\bar{S}_\phi} = \dim \text{Hom}_{S_\phi}(\delta_{\pi, \rho}, r_\mu).$$

□

5.5 Application of the Lefschetz-Verdier fixed-point formula

Recall the object $\mathcal{F}_\pi \in D_{\text{ét}}(\text{Gr}_{G, \leq \mu}, \Lambda)$ from Definition 4.10.2. The cohomology $H^i(\text{Gr}_{G, \leq \mu}, \mathcal{F}_\pi)$ is an admissible representation of $J_b(F)$, which is nonzero for finitely many i . Thus if f is a Λ -valued compactly supported smooth function on $J_b(F)$, it makes sense to talk about the trace of f as a Hecke operator on the Euler characteristic $H^*(\text{Gr}_{G, \leq \mu}, \mathcal{F}_\pi)$. We will use the Lefschetz-Verdier fixed-point formula to compute this trace when f is supported on the locus of strongly regular elliptic elements in $J_b(G)$, with the intention of applying Lemma 4.10.3.

Let $g \in J_b(F)$ be a strongly regular elliptic element and let T be its centralizer. The map

$$T_{\text{reg}}(F) \times J_b(F)/T(F) \rightarrow J_b(F), \quad (t, jT) \mapsto jtj^{-1} \quad (5.5.1)$$

is a local homeomorphism whose fibers are torsors for the Weyl group $N(T, J_b)(F)/T(F)$. Thus there exists an open pro- p subgroup $K \subset J_b(F)$ so that every element of Kg is conjugate to an element of $T_{\text{reg}}(F)$, and is in particular strongly regular elliptic. Let $K' \subset K$ be an open subgroup contained in $K \cap g^{-1}Kg$, and consider the correspondence c_g on $[\text{Gr}_{G, \leq \mu} / K]$ defined by

$$\begin{array}{ccc} & [\text{Gr}_{G, \leq \mu} / K'] & \\ \alpha_g \swarrow & & \searrow \alpha_1 \\ [\text{Gr}_{G, \leq \mu} / K] & & [\text{Gr}_{G, \leq \mu} / K], \end{array}$$

where α_g is the descent of the multiplication-by- g map on $\text{Gr}_{G, \leq \mu}$, and α_1 is the projection. The fixed-point locus of this correspondence is $\text{Fix}(c_g) = [\text{Gr}_{G, \leq \mu}^{Kg} / K']$, where $\text{Gr}_{G, \leq \mu}^{Kg}$ is the locus in $\text{Gr}_{G, \leq \mu}$ consisting of points fixed by some element of Kg .

The data G and μ being fixed in this section, we write $\text{Gr} = \text{Gr}_{G, \leq \mu}$.

Lemma 5.5.1. *There are finitely many K' -orbits in Gr^{Kg} .*

Proof. Since the map in (5.5.1) is a local homeomorphism with finite fibers, the preimage of Kg under it is compact open, and projecting to both factors we obtain an open compact neighborhood $L \subset J_b(F)$ of 1 that is invariant under left translation by K' , and an open compact neighborhood $V \subset T_{\text{reg}}(F)$ of g . Then every element of Kg is of the form $l tl^{-1}$ for $l \in L$ and $t \in V$. By Proposition 5.1.2 we have $\text{Gr}^t = \text{Gr}^{T(F)}$. Moreover $\text{Gr}^{lgl^{-1}} = l \text{Gr}^g$. Thus

$$\text{Gr}^{Kg} \subset \bigcup_{l \in L, t \in V} l \text{Gr}^t = \bigcup_{l \in L} l \text{Gr}^{T(F)}.$$

Since the action of K' on L by left translation has finitely many orbits, and $\text{Gr}^{T(F)}$ is the finite set consisting of those elements of $X_*(T)$ lying in the convex hull of the Weyl-orbit of μ , the proof is complete. \square

For an element $x \in \text{Gr}^{Kg}$, we let $J_b(F)_x \subset J_b(F)$ be the stabilizer of x . An element $t \in J_b(F)_x \cap Kg$ is a strongly regular elliptic element stabilizing x . By Corollary 5.1.3 x is fixed by the maximal torus $T_t \subset J_b$ containing t , and there exists a cocharacter $\nu_x \in X_*(T_t)$ for which $L_{\nu_x} = x$. Proposition 5.3.1 gives an element $\iota_{b, \nu_x}(t) \in G(F)$ related to t ; we write $\iota_x(t) = \iota_{b, \nu_x}(t)$.

By Proposition 5.3.1, $\text{Gr}^{Kg} \subset \text{Gr}^{b\text{-adm}}$, so that $x \in \text{Gr}^{Kg}$ lies in the image of the shtuka space $\mathcal{M}_{G, b, \mu} \rightarrow \text{Gr}$. On the fiber over x , t acts as $\iota_x(t)$.

Recall that \mathcal{L}_π is the descent of \underline{V}_π through the $G(F)$ -torsor $\mathcal{M}_{G,b,\mu} \rightarrow \mathrm{Gr}^{b\text{-adm}}$. Thus there is an isomorphism $\mathcal{L}_{\pi,x} \cong V_\pi$ which identifies the action of t with that of $\iota_x(t)$.

We write $[x]_{K'}$ for the image of x in $[\mathrm{Gr}^{Kg}/K']$. Then we have $[x]_{K'} \cong [\mathrm{Spa} C / (J_b(F)_x \cap K')]$.

The sheaf \mathcal{F}_π on Gr is $J_b(F)$ -equivariant. Write u_g for the cohomological correspondence lying over c_g coming from this equivariance. For a fixed point $x \in \mathrm{Gr}^{Kg}$ we would like to compute the local term $\mathrm{loc}_{[x]_{K'}}(u_g, \mathcal{F}_\pi)$. The restriction of \mathcal{F}_π to $\mathrm{Gr}^{b\text{-adm}}$ is $\mathcal{L}_\pi \otimes \mathrm{IC}_\mu$. Applying Propositions 3.7.2 and 5.2.1 gives

$$\mathrm{loc}_{[x]_{K'}}(u_g, \mathcal{F}_\pi) = \mathrm{tr} \left(\pi(1_H) \circ u_{g,\mathcal{L},[x]_{K'}} \Big| \mathcal{L}_{\pi,x} \right) (-1)^{2\rho(\nu_x)} \dim r_\mu[\nu_x] \quad (5.5.2)$$

for all sufficiently small open subgroups $H \subset G(F)$. First we deal with the sign. Since $\rho \in X_*(\widehat{T}_{\mathrm{ad}})$ the value of the pairing $\langle 2\rho, \nu_x \rangle$ modulo 2 depends only on the image of ν_x in $X^*(\widehat{T})/X^*(\widehat{T}_{\mathrm{ad}}) = X^*(Z(\widehat{G}))$. We may assume that ν_x is a weight of r_μ , otherwise $\dim r_\mu[\nu_x] = 0$. But then $\nu_x = \mu$ in $X^*(Z(\widehat{G}))$.

Recall that G^* is a quasi-split inner form of G . Let $\mu_1, \mu_2 \in X^*(Z(\widehat{G}_{\mathrm{sc}})^\Gamma)$ be the elements corresponding to the inner twists $G^* \rightarrow G$ and $G^* \rightarrow J_b$ by Kottwitz's homomorphism [Kot86, Theorem 1.2]. By Lemma A.1.1 we have $e(J_b)e(G) = (-1)^{\langle 2\rho, \mu_2 - \mu_1 \rangle}$. But since J_b is obtained from G by twisting by b , the difference $\mu_2 - \mu_1$ is equal to the image of $\kappa(b) \in X^*(Z(\widehat{G})^\Gamma)$ under the map $X^*(Z(\widehat{G})^\Gamma) \rightarrow X^*(Z(\widehat{G}_{\mathrm{sc}})^\Gamma)$ dual to the natural map $Z(\widehat{G}_{\mathrm{sc}}) \rightarrow Z(\widehat{G})$. Since $b \in B(G, \mu)$ we see that $\mu_2 - \mu_1 = \mu$ and conclude that

$$e(J_b)e(G) = (-1)^{\langle 2\rho, \mu \rangle}.$$

As for the trace factor appearing in (5.5.2):

Lemma 5.5.2. *For sufficiently small $H \subset G(F)$ we have*

$$\mathrm{tr} \left(\pi(1_H) \circ u_{g,\mathcal{L},[x]_{K'}} \Big| \mathcal{L}_{\pi,[x]_K} \right) = \sum_{t \in W_x} \Theta_\pi(\iota_x(t)),$$

where W_x runs through a set of representatives for $(J_b(F)_x \cap Kg) / (J_b(F)_x \cap K'g)$.

Proof. By (3.8.4) the action of $u_{g,\mathcal{L},[x]_{K'}}$ on $\mathcal{L}_{[x]_K} \cong V_\pi^{\iota_x(J_b(F)_x) \cap K}$ is given by the operator $v \mapsto \sum_{t \in W_x} \pi(\iota_x(t))v$. For sufficiently small H we have $\mathrm{tr} \pi(1_H) \circ \pi(\iota_x(t)) = \Theta_\pi(\iota_x(t))$ for all $t \in W_x$, which gives the lemma. \square

Applying the global Lefschetz-Verdier formula to $u_{\mathcal{F}_\pi}$ and using Lemma 3.8.5 gives

$$\frac{1}{\mu(K')} \operatorname{tr}(1_{Kg} | H^*(\operatorname{Gr}, \mathcal{F}_\pi)) = e(G)e(J_b) \sum_{[x]_{K'} \in \operatorname{Gr}^{Kg}/K'} \sum_{t \in W_x} \Theta_\pi(\iota_x(t)) \dim r_\mu[\nu_x] \quad (5.5.3)$$

Define

$$S_K = \left\{ (x, t) \in \operatorname{Gr} \times Kg \mid tx = x \right\},$$

considered as a subdiamond of $\operatorname{Gr} \times Kg$. Then $(x, t) \mapsto \iota_x(t)$ is a continuous map from S_K onto the set of conjugacy classes in $G(F)$.

Lemma 5.5.3. *The projection map $S_K \rightarrow Kg$ is a local homeomorphism with finite fibers.*

Proof. By Proposition 5.1.2 and Corollary 5.1.3, for $t \in Kg$ we have $\operatorname{Gr}^t = \operatorname{Gr}^{T_t(F)}$ and this can be identified with the set of cocharacters of T_t bounded by μ , which is a finite set. Thus $S_K \rightarrow Kg$ has finite fibers.

Let $(x, t) \in S_K$. We can write down a local inverse to $S_K \rightarrow Kg$ in a neighborhood of t : choose compact open neighborhoods $V \subset T_t(F)$ of t and $L \subset J_b(F)/T(F)$ of 1 respectively so that the restriction of (5.5.1) to $V \times L$ is an isomorphism onto its image $U \subset Kg$. Thus each $u \in Kg$ is equal to $lt'l^{-1}$ for unique $t' \in V$ and $l \in J_b(F)/T(F)$ and the desired local inverse sends u to (lx, u) . \square

Recall that Gr^g is finite. Lemma 5.5.3 allows us to choose for each $x \in \operatorname{Gr}^g$ an open subset $U_x \subset S_K$ containing (x, g) such that

- the U_x are disjoint,
- U_x is isomorphic to its image V_x under $S_K \rightarrow Kg$,
- the map $(y, t) \mapsto \Theta_\pi(\iota_y(t))$ is constant on U_x , taking the value $\Theta_\pi(\iota_x(g))$.

Let $L \subset K$ be an open subgroup which is small enough so that $Lg \subset \cap_x V_x$. Then $S_L \subset \coprod_x U_x$, and for each x the projection map $S_L \cap U_x \rightarrow Lg$ is an isomorphism. We have a partition $\operatorname{Gr}^{Lg} = \coprod_{x \in \operatorname{Gr}^g} \operatorname{Gr}^{Lg}[x]$, where $\operatorname{Gr}^{Lg}[x]$ is the image of $S_L[x] := S_L \cap U_x$ under the projection $S_L \rightarrow \operatorname{Gr}^{Lg}$. Applying (5.5.3) with L in place of K gives

$$\begin{aligned} & \frac{1}{\mu(L')} \operatorname{tr}(1_{Lg} | H^*(\operatorname{Gr}, \mathcal{F}_\pi)) \\ = & e(G)e(J_b) \sum_{x \in \operatorname{Gr}^g} \dim r_\mu[\nu_x] \Theta_\pi(\iota_x(g)) \sum_{[y]_{L'} \in \operatorname{Gr}^{Lg}[x]/L'} [J_b(F)_y \cap L : J_b(F)_y \cap L'] \end{aligned}$$

The projection $S_L[x] \rightarrow \mathrm{Gr}^{Lg}[x]$ is surjective, and the fiber over $y \in \mathrm{Gr}^{Lg}[x]$ is $J_b(F)_y \cap Lg$. Similarly, the quotient map $S_L[x]/L' \rightarrow \mathrm{Gr}^{Lg}[x]/L'$ is surjective, and the fiber over $[y]_{L'}$ is $(J_b(F)_y \cap L)/(J_b(F)_y \cap L')$. Therefore the inner sum in the above expression is $\#S_L[x]/L' = [L : L']$. We have arrived at

$$\frac{1}{\mu(L)} \mathrm{tr}(1_{Lg} | H^*(\mathrm{Gr}, \mathcal{F}_\pi)) = e(G)e(J_b) \sum_{x \in \mathrm{Gr}^g} \dim r_\mu[\nu_x] \Theta_\pi(\iota_x(g)).$$

Note that the right-hand side does not depend on L , which means that the Harish-Chandra trace of g on $H^*(\mathrm{Gr}, \mathcal{F}_\pi)$ is well-defined, and equals

$$e(G)e(J_b) \sum_{\nu \in X^*(\widehat{T})} \Theta_\pi(\iota_{b,\nu}(g)) \dim r_\mu[\nu].$$

By Proposition 5.4.1, this equals

$$\sum_{\rho \in \Pi_\phi(J_b)} \dim \mathrm{Hom}_{S_\phi}(\delta_{\pi,\rho}, r_\mu) \Theta_\rho(g).$$

This concludes the proof of Theorem 1.0.4 (via its reduction in Lemma 4.10.3), because the Harish-Chandra traces of regular elliptic elements on both sides of that theorem agree.

A Elementary lemmas

A.1 A calculation of the Kottwitz sign

In this appendix, F is any local field of characteristic zero and G is a connected reductive group defined over F . We will give a formula for the Kottwitz sign $e(G)$ in terms of the dual group \widehat{G} . Fix a quasi-split inner form G^* and an inner twisting $\psi : G^* \rightarrow G$. Let $h \in H^1(\Gamma, G_{\mathrm{ad}}^*)$ be the class of $\sigma \mapsto \psi^{-1}\sigma(\psi)$. Via the Kottwitz homomorphism [Kot86, Theorem 1.2] the class h corresponds to a character $\nu \in X^*(Z(\widehat{G}_{\mathrm{sc}})^\Gamma)$.

Choose an arbitrary Borel pair $(\widehat{T}_{\mathrm{sc}}, \widehat{B}_{\mathrm{sc}})$ of $\widehat{G}_{\mathrm{sc}}$ and let $2\rho \in X_*(\widehat{T}_{\mathrm{sc}})$ be the sum of the $\widehat{B}_{\mathrm{sc}}$ -positive coroots. The restriction map $X^*(\widehat{T}_{\mathrm{sc}}) \rightarrow X^*(Z(\widehat{G}_{\mathrm{sc}}))$ is surjective and we can lift ν to $\dot{\nu} \in X^*(\widehat{T}_{\mathrm{sc}})$ and form $\langle 2\rho, \dot{\nu} \rangle \in \mathbf{Z}$. A different lift $\dot{\nu}$ would differ by an element of $X^*(\widehat{T}_{\mathrm{ad}})$, and since $\rho \in X_*(\widehat{T}_{\mathrm{ad}})$ we see that the image of $\langle 2\rho, \dot{\nu} \rangle$ in $\mathbf{Z}/2\mathbf{Z}$ is independent of the choice of lift $\dot{\nu}$. We thus write $\langle 2\rho, \nu \rangle \in \mathbf{Z}/2\mathbf{Z}$. Since any two Borel pairs in $\widehat{G}_{\mathrm{sc}}$ are conjugate $\langle 2\rho, \nu \rangle$ does not depend on the choice of $(\widehat{T}_{\mathrm{sc}}, \widehat{B}_{\mathrm{sc}})$.

Lemma A.1.1.

$$e(G) = (-1)^{\langle 2\rho, \nu \rangle}.$$

Proof. We fix Γ -invariant Borel pairs $(T_{\text{ad}}, B_{\text{ad}})$ in G_{ad}^* and $(\widehat{T}_{\text{sc}}, \widehat{B}_{\text{sc}})$ in \widehat{G}_{sc} . Then we have the identification $X^*(T_{\text{ad}}) = X_*(\widehat{T}_{\text{sc}})$. Let $(T_{\text{sc}}, B_{\text{sc}})$ be the preimage in G_{sc}^* of $(T_{\text{ad}}, B_{\text{ad}})$.

By definition the Kottwitz sign is the image of h under

$$H^1(\Gamma, G_{\text{ad}}^*) \xrightarrow{\delta} H^2(\Gamma, Z(G_{\text{sc}}^*)) \xrightarrow{\rho} H^2(\Gamma, \{\pm 1\}) \longrightarrow \{\pm 1\},$$

where $\rho \in X^*(T_{\text{sc}})$ is half the sum of the B_{sc} -positive roots and its restriction to $Z(G_{\text{sc}}^*)$ is independent of the choice of $(T_{\text{ad}}, B_{\text{ad}})$. By functoriality of the Tate-Nakayama pairing this is the same as pairing $\delta h \in H^2(\Gamma, Z(G_{\text{sc}}^*))$ with $\rho \in H^0(\Gamma, X^*(Z(G_{\text{sc}}^*)))$. The canonical pairing $X^*(T_{\text{ad}}) \otimes X^*(\widehat{T}_{\text{sc}}) \rightarrow \mathbf{Z}$ induces the perfect pairing $X^*(T_{\text{sc}})/X^*(T_{\text{ad}}) \otimes X^*(\widehat{T}_{\text{sc}})/X^*(\widehat{T}_{\text{ad}}) \rightarrow \mathbf{Q}/\mathbf{Z}$ and hence the isomorphism $X^*(Z(G_{\text{sc}}^*)) \rightarrow \text{Hom}_{\mathbf{Z}}(X^*(Z(\widehat{G}_{\text{sc}})), \mathbf{Q}/\mathbf{Z}) = Z(\widehat{G}_{\text{sc}})$, where the last equality uses the exponential map. Under this isomorphism $\rho \in X^*(Z(G_{\text{sc}}^*))^\Gamma$ maps to the element $(-1)^{2\rho} \in Z(\widehat{G}_{\text{sc}})^\Gamma$ obtained by mapping $(-1) \in \mathbf{C}^\times$ under $2\rho \in X^*(T_{\text{ad}}) = X_*(\widehat{T}_{\text{sc}})$. The lemma now follows from [Kot86, Lemma 1.8]. \square

A.2 Integral supercuspidal representations

In this appendix, F is a non-archimedean local field of residual characteristic p and G is a connected reductive group defined over F . We denote by G^{ab} the maximal abelian quotient of G , a torus. Let $l \neq p$ be a prime. We shall record an immediate consequence of work of Vigneras.

Lemma A.2.1. *Let π be an irreducible representation of $G(F)$ on a $\bar{\mathbf{Q}}_l$ -vector space. There exists a character $\chi : G^{\text{ab}}(F) \rightarrow \bar{\mathbf{Q}}_l^\times$ such that the central character of $\pi \otimes \chi$ takes values in $\bar{\mathbf{Z}}_l$.*

Proof. Let A_G be the maximal split central torus in G . Consider the map $\text{val}_{A_G} : A_G(F) \rightarrow X_*(A_G)$ given by $\langle \text{val}_{A_G}(x), \alpha \rangle = \text{val}_F(\alpha(x))$ for all $\alpha \in X^*(A_G)$, where $\text{val}_F : F^\times \rightarrow \mathbf{Z}$ is the normalized valuation. This map fits into the exact sequence

$$1 \rightarrow A_G(F)_0 \rightarrow A_G(F) \rightarrow X_*(A_G) \rightarrow 0$$

where $A_G(F)_0$ is the maximal bounded subgroup of $A_G(F)$. Let ω_π be the central character of π and let $\text{val}_l : \bar{\mathbf{Q}}_l^\times \rightarrow \mathbf{Q}$ be the normalized valuation of $\bar{\mathbf{Q}}_l$. The composition $\text{val}_l \circ \omega_\pi$ is trivial on $A_G(F)_0$. Thus the restriction of

this composition to $A_G(F)$ becomes a character $X_*(A_G) \rightarrow \mathbf{Q}$. Its image is a sublattice of \mathbf{Q} and we can choose $N \in \mathbf{N}$ so that this image is contained in $\frac{1}{N}\mathbf{Z}$.

Let $X^*(G)_F = X^*(G)^\Gamma = X^*(G^{\text{ab}})^\Gamma$ be the group of F -rational characters of G , $X_*(G)_F = \text{Hom}_{\mathbf{Z}}(X^*(G)_F, \mathbf{Z})$ and let $\text{val}_G : G^{\text{ab}}(F) \rightarrow X_*(G)_F$ be defined just as in the case of A_G . It need not be surjective and we let $\Lambda(G)$ be its image. The restriction to $A_G(F)$ of val_G is not necessarily equal to val_{A_G} . The composition $\text{val}_G \circ \text{val}_{A_G}^{-1}$ gives an inclusion of lattices $X_*(A_G) \rightarrow \Lambda(G)$ with finite cokernel. We choose an extension of $\text{val}_l \circ \omega_\pi : X_*(A_G) \rightarrow \frac{1}{N}\mathbf{Z}$ to a homomorphism $\Lambda(G) \rightarrow \frac{1}{M}\mathbf{Z}$ for a suitable multiple M of N and let χ' be the composition of this extension with val_G . Then $\chi' : G^{\text{ab}}(F) \rightarrow \frac{1}{M}\mathbf{Z}$ is a group homomorphism whose restriction to $A_G(F)$ coincides with $\text{val}_l \circ \omega_\pi$.

Choose an element $y \in \bar{\mathbf{Q}}_l^\times$ with $\text{val}_l(y) = -\frac{1}{M}$ and let $\chi(g) = y^{M\chi'(g)}$. The central character of $\pi \otimes \chi$ is $\omega_\pi \cdot \chi$ and by construction its restriction to $A_G(F)$ takes values in $\bar{\mathbf{Z}}_l^\times$. Since the quotient $Z_G(F)/A_G(F)$ is compact, $\omega_\pi \cdot \chi$ must take values in $\bar{\mathbf{Z}}_l^\times$. \square

Corollary A.2.2. *Let π be an irreducible supercuspidal representation of $G(F)$ on a $\bar{\mathbf{Q}}_l$ vector space. There exists a character $\chi : G^{\text{ab}}(F) \rightarrow \bar{\mathbf{Q}}_l^\times$ such that $\pi \otimes \chi$ has a $\bar{\mathbf{Z}}_l$ -lattice invariant under $G(F)$.*

Proof. This follows immediately from the above Lemma and [Vig96, II.4.12]. \square

A.3 Some group theory

Let G be a connected reductive group over an algebraically closed field, $P \subset G$ a parabolic subgroup with Levi decomposition $P = MN$.

Lemma A.3.1. *If $m \in M$ and $n \in N$ are such that $mn \in P$ is regular semi-simple, then it is G -conjugate to m .*

Proof. Let $t = mn$. Being a semi-simple element of the algebraic group P , it is contained in a maximal torus $T \subset P$. Since P is a parabolic subgroup of G , T is also a maximal torus of G . Since T is contained in P , it normalizes N , and $T \rtimes N$ is solvable, hence contained in a Borel subgroup B of G . If U is the unipotent radical of B , then $N \subset U$. Then $m = tn^{-1}$ is conjugate to t by an element of U , according to [Hum95, 2.4]. \square

A.4 Some basic algebra

Let R be a discrete valuation ring with maximal ideal \mathfrak{m} . Let $\kappa = R/\mathfrak{m}$ be the residue field and $\Lambda = R/\mathfrak{m}^k$ for some $k > 0$. For a Λ -module M we have

the dual module $M^* = \text{Hom}_\Lambda(M, \Lambda)$ and the natural morphisms $M \rightarrow M^{**}$ and $(M^* \otimes M) \rightarrow (M \otimes M^*)^*$.

Lemma A.4.1. 1. *The morphism $M \rightarrow M^{**}$ is an isomorphism if and only if M is finitely generated.*

2. *Assuming M is finitely generated, the morphism $(M^* \otimes M) \rightarrow (M \otimes M^*)^*$ is an isomorphism if and only if M is free.*

Proof. For the “if” direction of the first point we note that the structure theorem for R -modules implies that a finitely generated Λ -module is a direct sum of finitely many cyclic Λ -modules, and each cyclic Λ -module is isomorphic to its own double dual.

Conversely assume that $M \rightarrow M^{**}$ is an isomorphism. We induct on k . For $k = 1$, Λ is a field and this is well-known. For general k we consider $N = M/\mathfrak{m}M$. Λ is an Artinian serial ring and hence injective as a module over itself. Thus the dualization functor is exact, and we get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{m}M & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\ & & \downarrow & & \cong \downarrow & & \downarrow & & \\ 0 & \longrightarrow & (\mathfrak{m}M)^{**} & \longrightarrow & M^{**} & \longrightarrow & N^{**} & \longrightarrow & 0, \end{array}$$

which shows that the right-most vertical map is surjective and the left-most vertical map is injective.

We have an isomorphism of Λ -modules $\mathfrak{m}^{m-1}\Lambda \rightarrow \kappa$, from which we obtain

$$N^* = \text{Hom}_\Lambda(N, \Lambda) = \text{Hom}_\Lambda(N, \mathfrak{m}^{m-1}\Lambda) \cong \text{Hom}_\kappa(N, \kappa).$$

Thus N^{**} is also the double dual of N in the category of κ -vector spaces, and it is easy to check that the right-most vertical map is the canonical map in that category. Thus, this map is an isomorphism, and N is finitely generated as a κ -vector space.

By the Snake Lemma, the left-most vertical arrow in the diagram is an isomorphism. We can apply the inductive hypothesis to the $(\Lambda/\mathfrak{m}^{m-1})$ -module $\mathfrak{m}M$ and conclude that it is finitely generated. Thus so is M .

For the proof of the second point, assume first that M is cyclic. Thus $M \cong \Lambda/\mathfrak{m}^i\Lambda$ for some $1 \leq i \leq k$. Let $\lambda \in \mathfrak{m}$ be a generator, and let $e \in M$ be a generator; then $M^* = \Lambda e^*$, where $\langle e, e^* \rangle = \lambda^{k-i}$. Similarly, since $M \otimes M^* \cong \Lambda/\lambda^i\Lambda$ with generator $e \otimes e^*$, the dual $(M \otimes M^*)^*$ is cyclic with generator f , where $\langle e \otimes e^*, f \rangle = \lambda^{k-i}$. The map $M^* \otimes M \rightarrow (M \otimes M^*)^*$

carries $e^* \otimes e$ onto the linear map $(e \otimes e^*) \mapsto \langle e^*, e \rangle \langle e, e^* \rangle = \lambda^{2(k-i)}$, which is to say that the image of $e^* \otimes e$ is $\lambda^{k-i}f$. This is an isomorphism precisely when $i = k$, i.e. when M is free.

A general finitely generated M is a direct sum of cyclic modules, $M = \bigoplus N_i$. The morphism $M^* \otimes M \rightarrow (M \otimes M^*)^*$ is an isomorphism if and only if the morphisms $N_i^* \otimes N_j \rightarrow (N_i \otimes N_j^*)^*$ are isomorphisms for all i, j . If M is free then $N_i \cong \Lambda$ and the above argument applies. Conversely, if the morphism $M^* \otimes M \rightarrow (M \otimes M^*)^*$ is an isomorphism, then so are the morphisms $N_i^* \otimes N_i \rightarrow (N_i \otimes N_i^*)^*$ for all i and thus $N_i \cong \Lambda$. \square

Lemma A.4.2. *Let M be a Λ -module. If M^* is at most countably generated, then M (and hence also M^*) is finitely generated.*

Proof. This is well known when Λ is a field. The general case reduces to this special case as follows. The nilpotency of \mathfrak{m} implies that any set of lifts of a generating set for the κ -vector space $M \otimes_{\Lambda} \kappa$ is a generating set for the Λ -module M . Thus, if M is not finitely generated, then $M \otimes_{\Lambda} \kappa$ is also not finitely generated. This implies that $\mathrm{Hom}_{\Lambda}(M, \mathfrak{m}^{k-1}) \cong \mathrm{Hom}_{\kappa}(M \otimes_{\Lambda} \kappa, \kappa)$ is not countably generated. But this is a submodule of M^* . Since Λ is noetherian, a submodule of an at most countably generated module N is also at most countably generated, as one sees by writing N as an increasing union of finitely generated modules. Thus M^* is not countably generated. \square

B A primer on reflexive sheaves, by David Hansen

In this appendix we discuss some basic examples and non-examples of reflexive sheaves, mostly in the context of classical rigid geometry. Although not strictly necessary in the main text of the paper, we hope these results might partially illuminate the hypotheses of reflexivity and strong reflexivity which recur throughout the paper. We also note that some closely related ideas were worked out almost simultaneously by Gaisin and Welliaveetil [GW17].

B.1 Results

Throughout what follows we fix an algebraically closed nonarchimedean base field C which we assume (for simplicity) is of mixed characteristic $(0, p)$. By an *adic space* we shall mean an adic space X over $S = \mathrm{Spa}(C, \mathcal{O}_C)$ which is locally of $+$ -weakly finite type, separated, taut and finite-dimensional. By a *rigid analytic space* we shall mean an adic space of the aforementioned type which is locally of topologically finite type and reduced; note that this last

condition is harmless, since replacing a rigid space by its nilreduction leaves the étale site unchanged.

Fix a prime power ℓ^n with $\ell \neq p$, and set $\Lambda = \mathbf{Z}/\ell^n\mathbf{Z}$. For any separated taut finite-dimensional morphism $f : X \rightarrow Y$ of adic spaces which is locally of +-weakly finite type, Huber [Hub96, §5.5 & §7.1] defined a functor $Rf_! : D(X_{\text{ét}}, \Lambda) \rightarrow D(Y_{\text{ét}}, \Lambda)$ admitting a right adjoint $Rf^!$. In particular, if X is an adic space with structure morphism $f : X \rightarrow S$, we may consider the dualizing complex $K_X \stackrel{\text{def}}{=} Rf^!\Lambda$ and the duality functor

$$\begin{aligned} D(X_{\text{ét}}, \Lambda) &\rightarrow D(X_{\text{ét}}, \Lambda) \\ \mathcal{F} &\mapsto \mathbf{D}\mathcal{F} \stackrel{\text{def}}{=} \underline{\text{RHom}}(\mathcal{F}, K_X). \end{aligned}$$

Definition B.1.1. An object $\mathcal{F} \in D(X_{\text{ét}}, \Lambda)$ is *reflexive* if the natural biduality map

$$\mathcal{F} \rightarrow \mathbf{D}\mathbf{D}\mathcal{F}$$

is an isomorphism.

As in the main text of the paper, this property is clearly étale-local on X , and is preserved under pullback along smooth maps and derived pushforward along proper maps. Moreover, reflexivity satisfies a 2-out-of-3 property: if $\mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H} \rightarrow$ is an exact triangle in $D(X_{\text{ét}}, \Lambda)$ such that two terms in the triangle are reflexive, then all three terms are reflexive. We also note that if $i : Z \rightarrow X$ is a closed embedding and $\mathcal{F} \in D(Z_{\text{ét}}, \Lambda)$ is such that $i_*\mathcal{F}$ is reflexive, then \mathcal{F} is reflexive. Finally, we observe that if \mathcal{F} is bounded with reflexive cohomology sheaves, then \mathcal{F} is reflexive itself.

One can make an analogous definition in the world of classical algebraic geometry, and it's a standard fact that *constructible* sheaves are reflexive in that setting. We remind the reader that if \mathcal{X} is a separated finite type C -scheme with associated rigid analytic space X , then pullback along the natural map of sites $\mu : X_{\text{ét}} \rightarrow \mathcal{X}_{\text{ét}}$ does not preserve constructibility, essentially because Zariski-open subsets of X are not quasicompact. Instead, the μ -pullback of a constructible sheaf on $\mathcal{X}_{\text{ét}}$ is an example of a *Zariski-constructible* sheaf on $X_{\text{ét}}$. There is also an intrinsic notion of constructible sheaf in the rigid analytic world, which is of a rather different flavor. Our first order of business is to check that these examples are all reflexive:

Proposition B.1.2. *If X is a rigid analytic space, then any object $\mathcal{F} \in D^b(X_{\text{ét}}, \Lambda)$ with constructible or Zariski-constructible cohomology sheaves is reflexive.*

In §B.3 below, we sketch a direct proof that constructible sheaves are reflexive. The idea is to first show that the constant sheaf Λ is reflexive on any rigid space X , which we then upgrade to the reflexivity of $j_!\Lambda$ where $j : U \rightarrow X$ is any separated étale map with affinoid source. For the reflexivity of Λ , we reduce to the smooth case using resolution of singularities.

However, it is more conceptual to deduce Proposition B.1.2 from a general criterion for reflexivity which was explained to us by Peter Scholze. To state this result, recall that for any (reduced) affinoid rigid space $U = \mathrm{Spa}(A, A^\circ)$ with its natural formal model $\mathfrak{U} = \mathrm{Spf}(A^\circ)$ over $\mathrm{Spf}(\mathcal{O}_C)$, there is a natural map of sites $\lambda_U : U_{\text{ét}} \rightarrow \mathfrak{U}_{\text{ét}}$ which induces a “nearby cycles” map $R\lambda_{U*} : D^b(U_{\text{ét}}, \Lambda) \rightarrow D^b(\mathfrak{U}_{\text{ét}}, \Lambda)$.

Proposition B.1.3 (Scholze). *Let X be a rigid analytic space. Suppose $\mathcal{F} \in D^b(X_{\text{ét}}, \Lambda)$ has the property that for every affinoid rigid space $U = \mathrm{Spa}(A, A^\circ)$ with an étale map $a : U \rightarrow X$, the nearby cycles $R\lambda_{U*}a^*\mathcal{F}$ are constructible. Then \mathcal{F} is reflexive.*

Combining this with a result of Huber, we deduce

Corollary B.1.4. *If X is a rigid analytic space and $\mathcal{F} \in D^b(X_{\text{ét}}, \Lambda)$ has quasi-constructible or oc-quasi-constructible cohomology sheaves in the sense of [Hub98a, Hub98b], then \mathcal{F} is reflexive.*

In the setting of sheaves on a finite type C -scheme, it may be true that reflexivity and constructibility coincide. One can thus ask whether reflexivity on rigid spaces is characterized by some variant of constructibility; however, this seems unlikely:

Proposition B.1.5. *There is an example of a reflexive sheaf on $\mathrm{Spa} C \langle T_1, \dots, T_6 \rangle$ with (some) infinite-dimensional stalks.*

Finally, we illustrate the failure of the Lefschetz fixed-point formula with an example of a reflexive sheaf which is not strongly reflexive.

Proposition B.1.6. *Let $X = \mathrm{Spa}(C \langle T \rangle, \mathcal{O}_C \langle T \rangle)$ be the one-dimensional rigid disk, and let $\bar{X} = \mathrm{Spa}(C \langle T \rangle, \mathcal{O}_C + T \cdot C \langle T \rangle^{\circ\circ})$ be its canonical adic compactification, with $j : X \rightarrow \bar{X}$ the natural inclusion. Then the sheaves $\Lambda_{\bar{X}}$ and $j_!\Lambda_X$ are reflexive but not strongly reflexive.*

This stands in contrast to the situation in classical algebraic geometry, where any constructible sheaf on a finite type C -scheme is strongly reflexive. In the course of building this example, we determine the dualizing complex of \bar{X} ; rather strangely, it turns out that $K_{\bar{X}} \simeq j_!\Lambda_X[2](1)$. In particular,

the dualizing complex of \overline{X} is not overconvergent, and some of its stalks vanish identically, in stark contrast with the case of rigid analytic spaces, cf. Proposition B.3.4. Morally, the failure of $K_{\overline{X}}$ to overconverge on the locus lying over the topological fixed points of $T \mapsto T + 1$ is “responsible” for the failure of the Lefschetz formula for this automorphism.

B.2 Nearby cycles and reflexivity

In this section we deduce Proposition B.1.3 from the following result, elaborating on a sketch explained to us by Peter Scholze.

Proposition B.2.1. *Let $X = \mathrm{Spa}(A, A^\circ)$ be an affinoid rigid analytic space over $\mathrm{Spa}(C, \mathcal{O}_C)$ as before; set $\mathfrak{X} = \mathrm{Spf}(A^\circ)$, so we get a nearby cycles map $R\lambda_{X^*} : D^b(X_{\acute{e}t}, \Lambda) \rightarrow D^b(\mathfrak{X}_{\acute{e}t}, \Lambda)$ as in the introduction. Then there is a natural equivalence $R\lambda_{X^*}\mathbf{D}_X \cong \mathbf{D}_{\mathfrak{X}}R\lambda_{X^*}$ compatible with étale localization and with the biduality maps, where \mathbf{D}_X and $\mathbf{D}_{\mathfrak{X}}$ denote the natural Verdier duality functors on $X_{\acute{e}t}$ and $\mathfrak{X}_{\acute{e}t}$.*

In most other settings where a nearby cycles functor is defined, commutation with Verdier duality is well-known (cf. [Ill94] and [Mas16], for example). However, the present situation is somewhat unique in that $R\lambda_{X^*}$ admits a useful left adjoint, which we’ll exploit heavily in the proof of Proposition B.2.1.

Proof of Proposition B.1.3. Fix $\mathcal{F} \in D^b(X_{\acute{e}t}, \Lambda)$ satisfying the conditions of the proposition. We need to show that the cone of the biduality map $\beta : \mathcal{F} \rightarrow \mathbf{D}_X\mathbf{D}_X\mathcal{F}$ is acyclic. Given any étale map $a : U = \mathrm{Spa}(B, B^\circ) \rightarrow X$, the constructibility hypothesis in the proposition guarantees that the biduality map

$$R\lambda_{U^*}a^*\mathcal{F} \rightarrow \mathbf{D}_{\mathfrak{U}}\mathbf{D}_{\mathfrak{U}}R\lambda_{U^*}a^*\mathcal{F}$$

is an isomorphism by Théorème 4.3 in [Del77, Th. de finitude]. We then see that the biduality map $a^*\mathcal{F} \rightarrow \mathbf{D}_U\mathbf{D}_Ua^*\mathcal{F} \cong a^*\mathbf{D}_X\mathbf{D}_X\mathcal{F}$ induces a map

$$R\lambda_{U^*}a^*\mathcal{F} \rightarrow R\lambda_{U^*}\mathbf{D}_U\mathbf{D}_Ua^*\mathcal{F} \cong \mathbf{D}_{\mathfrak{U}}R\lambda_{U^*}\mathbf{D}_Ua^*\mathcal{F} \cong \mathbf{D}_{\mathfrak{U}}\mathbf{D}_{\mathfrak{U}}R\lambda_{U^*}a^*\mathcal{F} \cong R\lambda_{U^*}a^*\mathcal{F}$$

whose composition is the identity; here the first two isomorphisms are obtained by applying Proposition B.2.1 twice, and the third isomorphism is given by the inverse of the biduality map for \mathfrak{U} . In particular, the map

$$R\lambda_{U^*}a^*\beta : R\lambda_{U^*}a^*\mathcal{F} \rightarrow R\lambda_{U^*}a^*\mathbf{D}_X\mathbf{D}_X\mathcal{F}$$

is an isomorphism. Passing to derived global sections on \mathfrak{U} gives an isomorphism

$$R\Gamma(U, \mathcal{F}|_U) \cong R\Gamma(\mathfrak{U}, R\lambda_{U*} a^* \mathcal{F}) \cong R\Gamma(\mathfrak{U}, R\lambda_{U*} a^* \mathbf{D}_X \mathbf{D}_X \mathcal{F}) \cong R\Gamma(U, \mathbf{D}_X \mathbf{D}_X \mathcal{F}|_U),$$

so in particular $R\Gamma(U, \text{Cone}(\beta)) \simeq 0$ for all affinoid étale maps $U \rightarrow X$. Since the stalks of the cohomology sheaves of $\text{Cone}(\beta)$ can be computed as colimits of $H^i(R\Gamma(U_j, \text{Cone}(\beta)))$ over suitable cofiltered inverse systems of affinoid étale maps $U_j \rightarrow X$, we deduce that $\text{Cone}(\beta)$ is acyclic, as desired. \square

This criterion is very useful in practice:

Proof of Proposition B.1.2 and Corollary B.1.4 . In [Hub98a, Hub98b], Huber defines classes of étale sheaves which he calls quasi-constructible and oc-quasi-constructible, which are preserved under arbitrary pullback and with the property that any constructible (resp. Zariski-constructible) sheaf is quasi-constructible (resp. oc-quasi-constructible). Moreover, he proves that the nearby cycles of such sheaves are always constructible, cf [Hub98a, Prop. 3.11] and [Hub98b, Prop. 2.12]. Combining these results with Proposition B.1.3, we get the result. \square

The proof of Proposition B.2.1 requires some work. Throughout the rest of this subsection, we fix X and \mathfrak{X} as in the statement of the result, and we let $f : X_{\text{ét}} \rightarrow \text{Spa}(C, \mathcal{O}_C)_{\text{ét}}$ and $\mathfrak{f} : \mathfrak{X}_{\text{ét}} \rightarrow \text{Spec}(\mathcal{O}_C/p)_{\text{ét}}$ denote the natural morphisms of étale sites. Note that we can identify the derived categories $D(\text{Spa}(C, \mathcal{O}_C)_{\text{ét}}, \Lambda)$ and $D(\text{Spec}(\mathcal{O}_C/p)_{\text{ét}}, \Lambda)$ with the derived category $D(\Lambda)$ of Λ -modules. The key non-formal ingredient in the argument is the existence of a natural equivalence $Rf_! \cong R\mathfrak{f}_! R\lambda_{X*}$, which can be proved as in [Hub98b, Lemma 2.13].

Proof. We first construct a natural transformation $\gamma_X : R\lambda_{X*} \mathbf{D}_X \rightarrow \mathbf{D}_{\mathfrak{X}} R\lambda_{X*}$. To do this, observe that for any $A \in D(\mathfrak{X}_{\text{ét}}, \Lambda)$ and $B \in D(X_{\text{ét}}, \Lambda)$, we have

a natural series of morphisms

$$\begin{aligned}
\mathrm{Hom}_{D(\mathfrak{x}_{\acute{e}t}, \Lambda)}(A, R\lambda_{X*} \mathbf{D}_X B) &\stackrel{(1)}{\cong} \mathrm{Hom}_{D(X_{\acute{e}t}, \Lambda)}(\lambda_X^* A, \mathbf{D}_X B) \\
&\stackrel{(2)}{\cong} \mathrm{Hom}_{D(X_{\acute{e}t}, \Lambda)}(\lambda_X^* A \otimes^{\mathbf{L}} B, Rf^! \Lambda) \\
&\stackrel{(3)}{\cong} \mathrm{Hom}_{D(\Lambda)}(Rf_!(\lambda_X^* A \otimes^{\mathbf{L}} B), \Lambda) \\
&\stackrel{(4)}{\cong} \mathrm{Hom}_{D(\Lambda)}(Rf_! R\lambda_{X*}(\lambda_X^* A \otimes^{\mathbf{L}} B), \Lambda) \\
&\stackrel{(5)}{\cong} \mathrm{Hom}_{D(\mathfrak{x}_{\acute{e}t}, \Lambda)}(R\lambda_{X*}(\lambda_X^* A \otimes^{\mathbf{L}} B), Rf^! \Lambda) \\
&\stackrel{(6)}{\xrightarrow{\sim}} \mathrm{Hom}_{D(\mathfrak{x}_{\acute{e}t}, \Lambda)}(A \otimes^{\mathbf{L}} R\lambda_{X*} B, Rf^! \Lambda) \\
&\stackrel{(7)}{\cong} \mathrm{Hom}_{D(\mathfrak{x}_{\acute{e}t}, \Lambda)}(A, \mathbf{D}_{\mathfrak{x}} R\lambda_{X*} B)
\end{aligned}$$

obtained as follows: (1) follows from adjointness of $R\lambda_{X*}$ and λ_X^* ; (2) and (7) are tensor-hom adjunction; (3) follows from adjointness of $Rf_!$ and $Rf^!$; (4) follows from the natural equivalence $Rf_! \cong Rf_! R\lambda_{X*}$ explained above; (5) follows from adjointness of $Rf_!$ and $Rf^!$; and (6) is dual to the natural “projection map” $A \otimes^{\mathbf{L}} R\lambda_{X*} B \rightarrow R\lambda_{X*}(\lambda_X^* A \otimes^{\mathbf{L}} B)$ obtained as the adjoint to the composition

$$\lambda_X^*(A \otimes^{\mathbf{L}} R\lambda_{X*} B) \cong \lambda_X^* A \otimes^{\mathbf{L}} \lambda_X^* R\lambda_{X*} B \rightarrow \lambda_X^* A \otimes^{\mathbf{L}} B,$$

cf. [Sta17, Tag 0943]. The composition of these morphisms induces a map

$$\mathrm{Hom}_{D(\mathfrak{x}_{\acute{e}t}, \Lambda)}(A, R\lambda_{X*} \mathbf{D}_X B) \rightarrow \mathrm{Hom}_{D(\mathfrak{x}_{\acute{e}t}, \Lambda)}(A, \mathbf{D}_{\mathfrak{x}} R\lambda_{X*} B)$$

which is functorial in A and B , so we obtain the desired natural transformation from the Yoneda lemma. Next we observe that when A is perfect, the map (6) is an isomorphism by [Sta17, Tag 0943]. In particular, taking $A = \Lambda[n]$ for varying n , we see that γ_X induces an isomorphism $R\Gamma(\mathfrak{x}, R\lambda_{X*} \mathbf{D}_X B) \xrightarrow{\sim} R\Gamma(\mathfrak{x}, \mathbf{D}_{\mathfrak{x}} R\lambda_{X*} B)$ for any B .

Now let $\mathfrak{Y} = \mathrm{Spf}(B^\circ)$ be any affine formal scheme with an étale map $j : \mathfrak{Y} \rightarrow \mathfrak{X}$, and let $j : Y = \mathrm{Spa}(B, B^\circ) \rightarrow X$ denote the induced étale map on rigid generic fibers. We then claim that formation of γ_X is compatible with étale localization, in the sense that the natural diagram

$$\begin{array}{ccc}
j^* R\lambda_{X*} \mathbf{D}_X B & \xrightarrow{\sim} & R\lambda_{Y*} \mathbf{D}_Y j^* B \\
\downarrow j^* \gamma_X & & \downarrow \gamma_Y \\
j^* \mathbf{D}_{\mathfrak{x}} R\lambda_{X*} B & \xrightarrow{\sim} & \mathbf{D}_{\mathfrak{Y}} R\lambda_{Y*} j^* B
\end{array}$$

commutes; here the horizontal isomorphisms are induced by the natural isomorphisms $j^*\mathbf{D}_{\mathfrak{X}} \cong \mathbf{D}_{\mathfrak{Y}}j^*$ and $j^*\mathbf{D}_X \cong \mathbf{D}_Yj^*$ together with the (easy) base change isomorphism $j^*R\lambda_{X*} \cong R\lambda_{Y*}j^*$. Granted the commutativity of this diagram, passing to derived global sections on \mathfrak{Y} induces a commutative diagram

$$\begin{array}{ccc}
R\Gamma(\mathfrak{Y}, (R\lambda_{X*}\mathbf{D}_X B)|_{\mathfrak{Y}}) & \xrightarrow{\sim} & R\Gamma(\mathfrak{Y}, R\lambda_{Y*}\mathbf{D}_Y j^* B) \\
\downarrow & & \downarrow \wr \\
R\Gamma(\mathfrak{Y}, (\mathbf{D}_{\mathfrak{X}}R\lambda_{X*}B)|_{\mathfrak{Y}}) & \xrightarrow{\sim} & R\Gamma(\mathfrak{Y}, \mathbf{D}_{\mathfrak{Y}}R\lambda_{Y*}j^* B)
\end{array}$$

where, crucially, the righthand vertical arrow is an isomorphism by arguing as in the previous paragraph with γ_Y in place of γ_X . Going around the diagram, we see that γ_X induces an isomorphism $R\Gamma(\mathfrak{Y}, R\lambda_{X*}\mathbf{D}_X B) \xrightarrow{\sim} R\Gamma(\mathfrak{Y}, \mathbf{D}_{\mathfrak{X}}R\lambda_{X*}B)$ for any affine étale map $\mathfrak{Y} \rightarrow \mathfrak{X}$, and therefore γ_X is an equivalence, as desired.

It remains to check the commutativity of the aforementioned square. This follows from a rather horrible diagram chase. More precisely, choose any $C \in D(\mathfrak{Y}_{\text{ét}}, \Lambda)$, and set $A = j_!C$; we then need to check that the diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{D(\mathfrak{X}_{\acute{e}t}, \Lambda)}(A, R\lambda_{X*} \mathbf{D}_X B) & \xrightarrow[\mathrm{i.}]{\sim} & \mathrm{Hom}_{D(\mathfrak{Y}_{\acute{e}t}, \Lambda)}(C, R\lambda_{Y*} \mathbf{D}_Y j^* B) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{D(X_{\acute{e}t}, \Lambda)}(\lambda_X^* A, \mathbf{D}_X B) & \xrightarrow[\mathrm{ii.}]{} & \mathrm{Hom}_{D(Y_{\acute{e}t}, \Lambda)}(\lambda_Y^* C, \mathbf{D}_Y j^* B) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{D(X_{\acute{e}t}, \Lambda)}(\lambda_X^* A \otimes^{\mathbf{L}} B, Rf^! \Lambda) & \xrightarrow[\mathrm{iii.}]{} & \mathrm{Hom}_{D(Y_{\acute{e}t}, \Lambda)}(\lambda_Y^* C \otimes^{\mathbf{L}} j^* B, j^! Rf^! \Lambda) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{D(\Lambda)}(Rf_! (\lambda_X^* A \otimes^{\mathbf{L}} B), \Lambda) & \xrightarrow[\mathrm{iv.}]{} & \mathrm{Hom}_{D(\Lambda)}(Rf_! j_! (\lambda_Y^* C \otimes^{\mathbf{L}} j^* B), \Lambda) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{D(\Lambda)}(Rf_! R\lambda_{X*} (\lambda_X^* A \otimes^{\mathbf{L}} B), \Lambda) & \xrightarrow[\mathrm{v.}]{} & \mathrm{Hom}_{D(\Lambda)}(Rf_! j_! R\lambda_{Y*} (\lambda_Y^* C \otimes^{\mathbf{L}} j^* B), \Lambda) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{D(\mathfrak{X}_{\acute{e}t}, \Lambda)}(R\lambda_{X*} (\lambda_X^* A \otimes^{\mathbf{L}} B), Rf^! \Lambda) & \xrightarrow[\mathrm{vi.}]{} & \mathrm{Hom}_{D(\mathfrak{Y}_{\acute{e}t}, \Lambda)}(R\lambda_{Y*} (\lambda_Y^* C \otimes^{\mathbf{L}} j^* B), j^! Rf^! \Lambda) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{D(\mathfrak{X}_{\acute{e}t}, \Lambda)}(A \otimes^{\mathbf{L}} R\lambda_{X*} B, Rf^! \Lambda) & \xrightarrow[\mathrm{vii.}]{} & \mathrm{Hom}_{D(\mathfrak{Y}_{\acute{e}t}, \Lambda)}(C \otimes^{\mathbf{L}} R\lambda_{Y*} j^* B, j^! Rf^! \Lambda) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{D(\mathfrak{X}_{\acute{e}t}, \Lambda)}(A, \mathbf{D}_{\mathfrak{X}} R\lambda_{X*} B) & \xrightarrow[\mathrm{viii.}]{\sim} & \mathrm{Hom}_{D(\mathfrak{Y}_{\acute{e}t}, \Lambda)}(C, \mathbf{D}_{\mathfrak{Y}} R\lambda_{Y*} j^* B)
\end{array}$$

commutes, functorially in C and B . Here the top and bottom horizontal arrows correspond to the horizontal isomorphisms in our original square, and the composition of all left and right vertical maps define $j^* \gamma_X$ and γ_Y , respectively, by Yoneda. Let us explain the intermediate horizontal arrows together with some of the commutativity checks, leaving a few details to the reader. The idea is to repeatedly use adjointness of the pairs $(j_!, j^* = j^!)$ and $(j_!, j^* = j^!)$, together with the base change isomorphism $j^* R\lambda_{X*} \cong R\lambda_{Y*} j^*$ and its adjoint incarnation $\lambda_X^* j_! \cong j_! \lambda_Y^*$. In particular, applying the latter to $A = j_! C$ gives a natural isomorphism $\lambda_X^* A \cong j_! \lambda_Y^* C$. Combining this isomorphism with the adjunction of $j_!$ and j^* induces the arrow labeled ii.; on the other hand, tensoring this isomorphism with B and applying the projection formula for $j_!$ gives

$$\lambda_X^* A \otimes^{\mathbf{L}} B \cong j_! (\lambda_Y^* C \otimes^{\mathbf{L}} j^* B),$$

which induces arrows iii. and iv. (here we've again used the adjunction of $j_!$ and j^*).

Next, by Lemma B.2.2 below, there is a natural equivalence $\tau : j_!R\lambda_{Y*} \rightarrow R\lambda_{X*}j_!$, which moreover is compatible with $Rf_!$ in the sense that the composite map

$$Rf_!j_!R\lambda_{Y*} \xrightarrow{Rf_!\tau} Rf_!R\lambda_{X*}j_! \cong Rf_!j_!$$

induces the natural equivalence

$$R(f \circ j)_!R\lambda_{Y*} \cong R(f \circ j)_!.$$

Applying this transformation to $\lambda_Y^*C \otimes^{\mathbf{L}} j^*B$ induces a map

$$j_!R\lambda_{Y*}(\lambda_Y^*C \otimes^{\mathbf{L}} j^*B) \rightarrow R\lambda_{X*}j_!(\lambda_Y^*C \otimes^{\mathbf{L}} j^*B) \cong R\lambda_{X*}(\lambda_X^*A \otimes^{\mathbf{L}} B),$$

which gives rise to arrows v. and vi. via suitable adjunctions; commutativity of the square spanned by arrows iv. and v. follows from the aforementioned compatibility of this transformation with $Rf_!$, and commutativity of the square spanned by arrows v. and vi. is straightforward. Next, we observe that the previous map together with the projection maps

$$A \otimes^{\mathbf{L}} R\lambda_{X*}B \xrightarrow{\pi_X} R\lambda_{X*}(\lambda_X^*A \otimes^{\mathbf{L}} B)$$

and

$$C \otimes^{\mathbf{L}} R\lambda_{Y*}j^*B \xrightarrow{\pi_Y} R\lambda_{Y*}(\lambda_Y^*C \otimes^{\mathbf{L}} j^*B)$$

fit together sits in a commutative square

$$\begin{array}{ccc} R\lambda_{X*}(\lambda_X^*A \otimes^{\mathbf{L}} B) & \longleftarrow & j_!R\lambda_{Y*}(\lambda_Y^*C \otimes^{\mathbf{L}} j^*B) \\ \pi_X \uparrow & & \uparrow j_!\pi_Y \\ A \otimes R\lambda_{X*}B & \longleftarrow & j_!(C \otimes R\lambda_{Y*}j^*B) \end{array}$$

where the lower horizontal arrow is given by the inverse of the composition

$$A \otimes R\lambda_{X*}B = j_!C \otimes R\lambda_{X*}B \cong j_!(C \otimes j^*R\lambda_{X*}B) \cong j_!(C \otimes R\lambda_{Y*}j^*B).$$

Applying $\mathrm{Hom}_{D(\mathfrak{X}_{\text{ét}}, \Lambda)}(-, Rf^!\Lambda)$ to this square and using the adjunction of $j_!$ and $j^!$ on the righthand column, we get arrows vi. and vii. together with the commutativity of the relevant square. \square

In the course of these arguments, we used the following lemma.

Lemma B.2.2. *Let X and \mathfrak{X} be as above, and let $j : \mathfrak{Y} = \mathrm{Spf}(B^\circ) \rightarrow \mathfrak{X}$ be an étale map of affine formal schemes, with $j : Y = \mathrm{Spa}(B, B^\circ) \rightarrow X$ the induced map on rigid generic fibers. Then the natural transformation $\tau : j_! R\lambda_{Y*} \rightarrow R\lambda_{X*} j_!$ defined as the adjoint to the composition*

$$R\lambda_{Y*} \rightarrow R\lambda_{Y*} j^* j_! \cong j^* R\lambda_{X*} j_!$$

is an equivalence, and is compatible with $Rf_!$ in the sense that the composite map

$$Rf_! j_! R\lambda_{Y*} \xrightarrow{Rf_! \tau} Rf_! R\lambda_{X*} j_! \cong Rf_! j_!$$

coincides with the natural equivalence

$$R(f \circ j)_! R\lambda_{Y*} \cong R(f \circ j)_!.$$

Proof. By a standard argument (cf. [Sta17, Tag 0AN8]), we can find an étale ring map $A^\circ \rightarrow B_0$ such that $B^\circ = \lim_{\leftarrow} B_0/p^n B_0$ as A° -algebras. By Zariski's main theorem, we can find a module-finite ring map $A^\circ \rightarrow D$ fitting into a diagram

$$\begin{array}{ccc} \mathrm{Spec}(B_0) & \xrightarrow{i} & \mathrm{Spec}(D) \\ & \searrow j & \downarrow h \\ & & \mathrm{Spec}(A^\circ) \end{array}$$

where i is an open immersion. Passing to p -adic completions induces a corresponding diagram

$$\begin{array}{ccc} \mathfrak{Y} = \mathrm{Spf}(B^\circ) & \xrightarrow{i} & \overline{\mathfrak{Y}} = \mathrm{Spf}(D) \\ & \searrow j & \downarrow \mathfrak{h} \\ & & \mathfrak{X} = \mathrm{Spf}(A^\circ) \end{array}$$

of p -adic formal schemes; here we used the fact that $D \cong \lim_{\leftarrow} D/p^n D$, which holds since $A^\circ \rightarrow D$ is module-finite and A° is p -adically separated and complete and Noetherian outside its ideal of definition, cf. [FGK11, Proposition 6.1.1(1)]. Passing to a similar diagram on the generic fibers and

going to étale sites, we get a commutative diagram

$$\begin{array}{ccc}
& Y_{\text{ét}} & \xrightarrow{\lambda_Y} & \mathfrak{Y}_{\text{ét}} \\
& \swarrow i & & \swarrow i \\
\bar{Y}_{\text{ét}} & \xrightarrow{j} & \bar{\mathfrak{Y}}_{\text{ét}} & \\
& \searrow h & & \searrow \mathfrak{h} \\
& X_{\text{ét}} & \xrightarrow{\lambda_X} & \mathfrak{X}_{\text{ét}}
\end{array}$$

where the λ_\bullet 's are the evident nearby cycles maps. Note that $h_* \cong Rh_*$ and $\mathfrak{h}_* \cong R\mathfrak{h}_*$ since both morphisms are finite. We then compute that

$$\begin{aligned}
j_! R\lambda_{Y*} &= \mathfrak{h}_* i_! R\lambda_{Y*} \\
&\cong \mathfrak{h}_* R\lambda_{\bar{Y}} i_! \\
&\cong R\lambda_{X*} h_* i_! \\
&= R\lambda_{X*} j_!
\end{aligned}$$

where the first isomorphism follows from [Hub96, Corollary 3.5.11.ii] and the second isomorphism is induced by the natural equivalence $\mathfrak{h}_* \lambda_{\bar{Y}*} = \lambda_{X*} h_*$. \square

Although not strictly necessary, let us develop a little more theory around the nearby cycles map $R\lambda_{X*}$.

Proposition B.2.3. *Let $X = \text{Spa}(A, A^\circ)$ and $\mathfrak{X} = \text{Spf}(A^\circ)$ be as above. Then $R\lambda_{X*}$ sends reflexive objects to reflexive objects, and the functor $R\lambda_X^! \stackrel{\text{def}}{=} \mathbf{D}_X \lambda_X^* \mathbf{D}_{\mathfrak{X}}$ defines a “weak” right adjoint of $R\lambda_{X*}$ on reflexive objects, in the sense that it induces functorial isomorphisms*

$$R\lambda_{X*} \underline{\mathbf{RHom}}_X(B, R\lambda_X^! C) \cong \underline{\mathbf{RHom}}_{\mathfrak{X}}(R\lambda_{X*} B, C)$$

and

$$\text{Hom}_{D(\mathfrak{X}_{\text{ét}}, \Lambda)}(R\lambda_{X*} B, C) \cong \text{Hom}_{D(X_{\text{ét}}, \Lambda)}(B, R\lambda_X^! C)$$

for any objects $B \in D(X_{\text{ét}}, \Lambda)$ and $C \in D(\mathfrak{X}_{\text{ét}}, \Lambda)$ with C reflexive.

Setting $C = K_{\mathfrak{X}}$ recovers Proposition B.2.1 as a special case of this result, although we use Proposition B.2.1 in the proof. We call $R\lambda_X^!$ a “weak” right adjoint because we’re not sure how it behaves on non-reflexive objects, or whether it preserves reflexivity; amusingly, the argument doesn’t require the latter fact.

Proof. The local result implies the global result upon applying $H^0(R\Gamma(\mathfrak{X}, -))$. For the local result, we calculate that

$$\begin{aligned}
\mathrm{Hom}_{D(\mathfrak{x}_{\acute{e}t}, \Lambda)}(A, R\lambda_{X*} \underline{\mathrm{RHom}}_X(B, R\lambda_X^! C)) &\stackrel{(1)}{\cong} \mathrm{Hom}_{D(X_{\acute{e}t}, \Lambda)}(\lambda_X^* A, \underline{\mathrm{RHom}}_X(B, R\lambda_X^! C)) \\
&\stackrel{(2)}{\cong} \mathrm{Hom}_{D(X_{\acute{e}t}, \Lambda)}(\lambda_X^* A, \underline{\mathrm{RHom}}_X(\lambda_X^* \mathbf{D}_{\mathfrak{X}} C, \mathbf{D}_X B)) \\
&\stackrel{(3)}{\cong} \mathrm{Hom}_{D(X_{\acute{e}t}, \Lambda)}(\lambda_X^* A \otimes^{\mathbf{L}} \lambda_X^* \mathbf{D}_{\mathfrak{X}} C, \mathbf{D}_X B) \\
&\stackrel{(4)}{\cong} \mathrm{Hom}_{D(X_{\acute{e}t}, \Lambda)}(\lambda_X^* (A \otimes^{\mathbf{L}} \mathbf{D}_{\mathfrak{X}} C), \mathbf{D}_X B) \\
&\stackrel{(5)}{\cong} \mathrm{Hom}_{D(\mathfrak{x}_{\acute{e}t}, \Lambda)}(A \otimes^{\mathbf{L}} \mathbf{D}_{\mathfrak{X}} C, R\lambda_{X*} \mathbf{D}_X B) \\
&\stackrel{(6)}{\cong} \mathrm{Hom}_{D(\mathfrak{x}_{\acute{e}t}, \Lambda)}(A \otimes^{\mathbf{L}} \mathbf{D}_{\mathfrak{X}} C, \mathbf{D}_{\mathfrak{X}} R\lambda_{X*} B) \\
&\stackrel{(7)}{\cong} \mathrm{Hom}_{D(\mathfrak{x}_{\acute{e}t}, \Lambda)}(A, \underline{\mathrm{RHom}}_{\mathfrak{X}}(\mathbf{D}_{\mathfrak{X}} C, \mathbf{D}_{\mathfrak{X}} R\lambda_{X*} B)) \\
&\stackrel{(8)}{\cong} \mathrm{Hom}_{D(\mathfrak{x}_{\acute{e}t}, \Lambda)}(A, \underline{\mathrm{RHom}}_{\mathfrak{X}}(R\lambda_{X*} B, \mathbf{D}_{\mathfrak{X}} \mathbf{D}_{\mathfrak{X}} C)) \\
&\stackrel{(9)}{\cong} \mathrm{Hom}_{D(\mathfrak{x}_{\acute{e}t}, \Lambda)}(A, \underline{\mathrm{RHom}}_{\mathfrak{X}}(R\lambda_{X*} B, C))
\end{aligned}$$

functorially in $A \in D(\mathfrak{x}_{\acute{e}t}, \Lambda)$. Here (1) and (5) follow from adjointness of λ_X^* and $R\lambda_{X*}$; (3) and (7) follow from tensor-hom adjunction; (4) is trivial; (6) follows from Proposition B.2.1; (9) follows from the reflexivity of C ; and, finally, in (2) and (8) we've used the fact that for $\bullet = X, \mathfrak{X}$ and any objects $C, D \in D(\bullet_{\acute{e}t}, \Lambda)$ we have

$$\underline{\mathrm{RHom}}_{\bullet}(C, \mathbf{D}_{\bullet} D) \cong \underline{\mathrm{RHom}}_{\bullet}(D, \mathbf{D}_{\bullet} C)$$

functorially in C and D , which is an easy exercise in tensor-hom adjunction. \square

B.3 Constructible sheaves on rigid spaces

In this section we sketch an alternative proof that constructible sheaves on rigid spaces are reflexive, which is in some ways more naive. The first non-formal input is the following claim.

Proposition B.3.1. *If X is a rigid analytic space, then the constant sheaf Λ is reflexive.*

Note that by definition, Λ is reflexive if and only if the natural map $\Lambda \rightarrow \underline{\mathrm{RHom}}(K_X, K_X)$ is an isomorphism.

Proof. The result clearly holds when X is smooth. For the general case, we argue by induction on the dimension of X . Thus, fix an integer $d \geq 1$. Assume the result holds for all rigid spaces of dimension $< d$, and let X be a d -dimensional (separated taut) rigid analytic space. We can assume that $X = \mathrm{Spa}(A, A^\circ)$ is affinoid and reduced. The ring A is an excellent Noetherian ring, so by Temkin [Tem12] we can find a projective birational morphism $f : \mathcal{X}' \rightarrow \mathrm{Spec}(A)$ where \mathcal{X}' is a regular C -scheme, such that f is an isomorphism over the regular locus of its target. This analytifies to a proper surjective map of rigid spaces

$$\pi : X' \rightarrow X$$

such that $X' \rightarrow S$ is smooth. In particular, $\Lambda_{X'} = \pi^* \Lambda_X$ is a reflexive sheaf on $X'_{\acute{e}t}$, so $R\pi_* \Lambda_{X'}$ is reflexive by stability under proper pushforward. Now, writing K for the cone of the adjunction map

$$\alpha : \Lambda_X \rightarrow R\pi_* \pi^* \Lambda_X \cong R\pi_* \Lambda_{X'},$$

the 2-out-of-3 property shows that Λ_X is reflexive if K is reflexive.

For reflexivity of K , consider the diagram

$$\begin{array}{ccccc} U & \xrightarrow{j'} & X' & \xleftarrow{i'} & Z' \\ \downarrow \iota & & \downarrow \pi & & \downarrow \tau \\ U & \xrightarrow{j} & X & \xleftarrow{i} & Z \end{array}$$

where U is the smooth locus in X with closed complement Z , and both squares are cartesian. Since $\pi|_{\pi^{-1}(U)}$ is an isomorphism, $j^* \alpha$ is an isomorphism in $D(U_{\acute{e}t}, \Lambda)$, so $j^* K \simeq 0$; applying the usual exact triangle $j_! j^* \rightarrow \mathrm{id} \rightarrow i_* i^*$, we get an isomorphism $K \simeq i_* i^* K$. Since i_* is a closed immersion and thus proper, we're now reduced to showing that $i^* K$ is a reflexive object of $D(Z_{\acute{e}t}, \Lambda)$. This pullback can be computed as

$$\begin{aligned} i^* K &= i^* \mathrm{Cone}(\Lambda_X \rightarrow R\pi_* \pi^* \Lambda_X) \\ &\cong \mathrm{Cone}(i^* \Lambda_X \rightarrow i^* R\pi_* \pi^* \Lambda_X) \\ &\cong \mathrm{Cone}(\Lambda_Z \rightarrow R\tau_* i'^* \pi^* \Lambda_X) \\ &\cong \mathrm{Cone}(\Lambda_Z \rightarrow R\tau_* \tau^* \Lambda_Z) \\ &\cong \mathrm{Cone}(\Lambda_Z \rightarrow R\tau_* \Lambda_{Z'}), \end{aligned}$$

where the third line follows by proper base change. Since Z and Z' are both of dimension $< d$, the sheaves Λ_Z and $\Lambda_{Z'}$ are reflexive by the induction hypothesis, and then $R\tau_* \Lambda_{Z'}$ is reflexive as well since τ is proper. Applying the 2-out-of-3 property again, we deduce that $i^* K$ is reflexive, as desired. \square

Corollary B.3.2. *If X is a rigid space and $j : U \rightarrow X$ is the inclusion of a Zariski-open subset with Zariski-closed complement $i : Z \rightarrow X$, then $j_!\Lambda$ and $i_*\Lambda$ are reflexive.*

Proof. Writing $i : Z \rightarrow X$ for the inclusion of the closed complement, the claim for $i_*\Lambda$ is immediate from preservation of reflexivity under proper pushforward. The exact triangle $j_!\Lambda \rightarrow \Lambda \rightarrow i_*\Lambda \rightarrow$ and the 2-out-of-3 property then imply reflexivity of $j_!\Lambda$. \square

Proposition B.3.3. *Let X be an adic space, and let $U \subset X$ be an open constructible subset with closure $\bar{U} \subset X$; let $j : U \rightarrow X$ and $\bar{j} : \bar{U} \rightarrow X$ denote the evident inclusions. Then for any overconvergent sheaf $\mathcal{F} \in Sh(X_{\acute{e}t}, \Lambda)$, we have natural identifications*

$$Rj_*j^*\mathcal{F} \cong j_*j^*\mathcal{F} \cong \bar{j}_*\bar{j}^*\mathcal{F}$$

in $Sh(X_{\acute{e}t}, \Lambda) \subset D(X_{\acute{e}t}, \Lambda)$.

Here we say a sheaf $\mathcal{F} \in Sh(X_{\acute{e}t}, \Lambda)$ is *overconvergent* if for any specialization of geometric points $\bar{x} \rightsquigarrow \bar{y}$, the associated map on stalks $\mathcal{F}_{\bar{y}} \rightarrow \mathcal{F}_{\bar{x}}$ is an isomorphism, cf. [Hub96, Definition 8.2.1]. We also say that $\mathcal{F} \in D(X_{\acute{e}t}, \Lambda)$ is overconvergent if it has overconvergent cohomology sheaves.

Proof. By [Hub96, Lemma 2.2.6], the functor j_* is exact, so the first isomorphism is clear. For the second, let $h : U \rightarrow \bar{U}$ be the evident open embedding, so

$$Rj_*j^*\mathcal{F} \cong \bar{j}_*Rh_*h^*\bar{j}^*\mathcal{F} \cong \bar{j}_*h_*h^*\bar{j}^*\mathcal{F}.$$

Here we used the fact that $h_* = Rh_*$ by another application of [Hub96, Lemma 2.2.6]. Now we need to see that $\bar{j}^*\mathcal{F} \cong h_*h^*\bar{j}^*\mathcal{F}$. Since $\bar{j}^*\mathcal{F}$ is overconvergent, this reduces us to checking that the natural map $\alpha : \mathcal{G} \rightarrow h_*h^*\mathcal{G}$ is an isomorphism for \mathcal{G} any overconvergent sheaf on \bar{U} . By [Hub96, Proposition 8.2.3], the sheaf $h_*h^*\mathcal{G}$ is overconvergent, so it suffices to check that α induces an isomorphism on stalks over any rank one (geometric) point. But every rank one point of \bar{U} is contained in U , so this is trivial. \square

Proposition B.3.4. *If X is a rigid analytic space, then the dualizing complex K_X is overconvergent.*

Proof. The proof is “dual” to the proof of Proposition B.3.1. More precisely, the result clearly holds for smooth X ; for a general X , we take a (global) resolution $\pi : X' \rightarrow X$ and consider the adjunction map

$$R\pi_!K_{X'} \cong R\pi_!\pi^!K_X \rightarrow K_X.$$

One now argues by induction as in the proof of Proposition B.3.1, making crucial use of the following facts:

- i. Overconvergence satisfies a 2-out-of-3 property.
- ii. Overconvergence is preserved by derived pushforward along proper maps [Hub96, Corollary 8.2.4]. \square

Granted these results, we now deduce the following key intermediate case.

Proposition B.3.5. *Let X be a rigid analytic space, and let $j : U \rightarrow X$ be the inclusion of an open constructible subset U . Then $j_! \Lambda$ is reflexive.*

The argument which follows is easily adapted to prove the more general statement that $j_! M$ is reflexive, where M is any finitely generated constant sheaf of Λ -modules on $X_{\text{ét}}$.

Proof. Set $\bar{V} = X \setminus U$, and let V be the interior of \bar{V} ; write $h : V \rightarrow X$ and $\bar{h} : \bar{V} \rightarrow X$ for the evident inclusions. Note that $\bar{U} = X \setminus V$. In particular, writing $\bar{j} : \bar{U} \rightarrow X$ for the evident inclusion, we get a canonical exact triangle

$$h_! h^* K_X \rightarrow K_X \rightarrow \bar{j}_* \bar{j}^* K_X \rightarrow .$$

By Propositions B.3.3 and B.3.4, the canonical map

$$\bar{j}_* \bar{j}^* K_X \rightarrow Rj_* j^* K_X = Rj_* K_U$$

is an isomorphism. Moreover, $Rj_* K_U \cong \mathbf{D}j_! \Lambda_U$, and $h^* K_X = K_V$. Thus we can rewrite the above triangle as

$$h_! K_V \rightarrow K_X \rightarrow \mathbf{D}j_! \Lambda_U \rightarrow ,$$

so dualizing this gives an exact triangle

$$\mathbf{D}^2 j_! \Lambda_U \rightarrow \mathbf{D}K_X \rightarrow \mathbf{D}h_! K_V \rightarrow .$$

Since $\mathbf{D}K_X \cong \Lambda_X$ and $\mathbf{D}h_! K_V \cong Rh_* \mathbf{D}K_V \cong Rh_* \Lambda_V$, we can rewrite the latter triangle as

$$\mathbf{D}^2 j_! \Lambda_U \rightarrow \Lambda_X \rightarrow Rh_* \Lambda_V \rightarrow .$$

This sits in a commutative diagram of exact triangles

$$\begin{array}{ccccc} j_! \Lambda_U & \longrightarrow & \Lambda_X & \longrightarrow & \bar{h}_* \Lambda_{\bar{V}} \longrightarrow \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{D}^2 j_! \Lambda_U & \longrightarrow & \Lambda_X & \longrightarrow & Rh_* \Lambda_V \longrightarrow \end{array}$$

where the lefthand vertical map is the biduality map, the central vertical map is the identity, and the righthand vertical map is the canonical map $a : \bar{h}_* \Lambda_{\bar{V}} \rightarrow Rh_* \Lambda_V$. By [Hub98a, Theorem 3.7], the map a is an isomorphism. Therefore the biduality map $j_! \Lambda_U \rightarrow \mathbf{D}^2 j_! \Lambda_U$ is an isomorphism, as desired. \square

Corollary B.3.6. *Let $f : U \rightarrow X$ be any étale map of affinoid rigid spaces, and let M be a constant sheaf of finitely generated Λ -modules on $U_{\text{ét}}$. Then $f_! M$ is reflexive.*

Proof. The claim is local on X . However, locally on X , we can factor f as the composite of an open embedding $j : U \rightarrow W$ and a finite étale map $g : W \rightarrow X$, cf. [Hub96, Lemma 2.2.8]. By the previous proposition, $j_! M$ is a reflexive sheaf on $W_{\text{ét}}$, and g is finite, hence proper, so $g_* = g_!$ preserves reflexivity. Therefore $f_! M = g_! j_! M$ is reflexive. \square

Now, fix an affinoid rigid space X . Let us say a constructible sheaf \mathcal{G} on X is *elementary* if there exists an affinoid rigid space U and an étale map $j : U \rightarrow X$ such that $\mathcal{G} \simeq j_! \Lambda$. Note that any finite direct sum of elementary sheaves is elementary. By the previous corollary, any elementary sheaf is reflexive. Moreover, any bounded complex of elementary sheaves is reflexive; this follows by an easy induction on the length of the complex, using the exact triangle associated with the “stupid” truncation functors together with the 2-out-of-3 property. Arguing as in the schemes case, one easily checks that any constructible sheaf \mathcal{F} admits a surjection $s_0 : \mathcal{G}^0 \rightarrow \mathcal{F}$ from an elementary sheaf \mathcal{G}^0 . The kernel of s_0 is again constructible, so we may choose a surjection $s_{-1} : \mathcal{G}^{-1} \rightarrow \ker s_0$ with \mathcal{G}^{-1} elementary; iterating this procedure, we can find an isomorphism $\mathcal{F} \simeq \mathcal{G}^\bullet = [\dots \xrightarrow{s_{-2}} \mathcal{G}^{-1} \xrightarrow{s_{-1}} \mathcal{G}^0]$ in the derived category, where all the \mathcal{G}^i 's are elementary. Set $\mathcal{H}_n = \ker s_{-n}$; playing with truncations, we get an exact triangle

$$\tau_{\leq -1} \sigma_{\geq -n}(\mathcal{G}^\bullet) \simeq \mathcal{H}_n[n] \rightarrow \sigma_{\geq -n}(\mathcal{G}^\bullet) \rightarrow \tau_{\geq 0} \sigma_{\geq -n}(\mathcal{G}^\bullet) \simeq \mathcal{F} \rightarrow$$

for any $n \geq 1$. Note that $\sigma_{\geq -n}(\mathcal{G}^\bullet)$ is a bounded complex of elementary sheaves, and hence is reflexive. Since any exact triangle induces a corresponding exact triangle whose terms are the cones of the evident biduality maps, we get an isomorphism

$$\text{Cone}(\mathcal{F} \rightarrow \mathbf{D}^2 \mathcal{F}) \simeq \text{Cone}(\mathcal{H}_n \rightarrow \mathbf{D}^2 \mathcal{H}_n)[n+1]$$

for any n . Using the cohomological dimension bounds proved in [Hub96], one easily checks that there is an integer N depending only on X such that

the i th cohomology sheaf of $\text{Cone}(\mathcal{H}_n \rightarrow \mathbf{D}^2\mathcal{H}_n)$ is zero for any $i \notin [-N, N]$ (in fact, $N = 1 + 2 \dim X$ is sufficient). Taking n arbitrarily large, we then see that the cohomology sheaves of $\text{Cone}(\mathcal{F} \rightarrow \mathbf{D}^2\mathcal{F})$ are zero in any given degree. Therefore $\text{Cone}(\mathcal{F} \rightarrow \mathbf{D}^2\mathcal{F})$ is acyclic, as desired.

B.4 Some counterexamples

In this section we prove Propositions B.1.5 and B.1.6.

Proof of Proposition B.1.5. We construct an explicit example as follows. Let $X = \text{Gr}(2, 5)$ be the usual rigid analytic Grassmannian over C , which we regard as parametrizing modifications of the bundle $\mathcal{O}(2/5)$ on the Fargues-Fontaine curve at the distinguished point. Let X^{adm} be the admissible locus inside X . According to an unpublished computation of the author and Jared Weinstein, the closed subdiamond $Z \subset X^\diamond$ corresponding to the closed subset $|X| \setminus |X^{\text{adm}}|$ can be explicitly described by an isomorphism

$$Z \simeq \left(\text{Spa} \left(\mathcal{O}_C[[T^{1/p^\infty}]] \right) \setminus V(pT) \right)^\diamond / \underline{D}_{1/3}^\times,$$

for a certain free action of $\underline{D}_{1/3}^\times$ on the diamond $(\text{Spa}(\mathcal{O}_C[[T^{1/p^\infty}]]) \setminus V(pT))^\diamond$; here $\underline{D}_{1/3}$ denotes the division algebra over \mathbf{Q}_p of invariant $1/3$. In particular, there is a natural smooth map $Z \rightarrow [\text{pt}/\underline{D}_{1/3}^\times]$. Let π be any infinite-dimensional admissible representation of $\underline{D}_{1/3}^\times$, and let \mathcal{L}_π be the corresponding pro-étale local system on Z . The sheaf \mathcal{L}_π is then reflexive, since it's the pullback of a reflexive sheaf on $[\text{pt}/\underline{D}_{1/3}^\times]$ along a smooth map, and therefore its pushforward along the closed embedding $i : Z \rightarrow X^\diamond$ is a reflexive sheaf on $X_{\text{ét}} \cong X_{\text{ét}}^\diamond$. Now, choose any open affinoid subset $j : U \rightarrow X$ meeting Z together with a finite map $f : U \rightarrow \text{Spa } C \langle T_1, \dots, T_6 \rangle$. The sheaf $\mathcal{F} \stackrel{\text{def}}{=} f_* j^* i_* \mathcal{L}_\pi$ is then an example of the type we seek. \square

This example is closely related to Example 3.6.7.

We now turn to Proposition B.1.6. For brevity, we prove that $\mathcal{F} = \Lambda_{\overline{X}}$ is reflexive but not strongly reflexive; the case of $j_! \Lambda_X$ is dual. The only non-formal ingredient we need is

Proposition B.4.1. *The sheaves $\Lambda_{\overline{X}}$ and $j_! \Lambda_X$ are reflexive, and the dualizing complex of \overline{X} coincides with $j_! \Lambda_X[2](1)$ where $j : X \rightarrow \overline{X}$ is the natural inclusion.*

Proof. We first show that $j_!\Lambda_X$ is reflexive. To see this, let $i : \bar{X} \rightarrow \mathbf{P}_C^1$ be the natural closed embedding; then $i_*j_!\Lambda_X \cong (i \circ j)_!\Lambda_X$ is a constructible sheaf on \mathbf{P}_C^1 , and therefore is reflexive. Thus $j_!\Lambda$ is reflexive by our previous remarks. We now calculate

$$\mathbf{D}_{\bar{X}}j_!\Lambda_X \simeq j_*\mathbf{D}_X\Lambda_X \simeq j_*K_X \simeq j_*\Lambda_X[2](1) \simeq \Lambda_{\bar{X}}[2](1),$$

where we've used the smoothness of X to identify K_X . Since this calculation exhibits $\Lambda_{\bar{X}}$ as the dual of a reflexive sheaf, it is reflexive itself. Applying $\mathbf{D}_{\bar{X}}$ again and using reflexivity, we get

$$j_!\Lambda_X \simeq \mathbf{D}_{\bar{X}}\mathbf{D}_{\bar{X}}j_!\Lambda_X \simeq \mathbf{D}_{\bar{X}}(\Lambda_{\bar{X}}[2](1)) \simeq K_{\bar{X}}[-2](-1),$$

as desired. \square

Consider the cartesian diagram

$$\begin{array}{ccc} & X \times X & \\ f_1 \swarrow & & \searrow f_2 \\ X \times \bar{X} & & \bar{X} \times X \\ h_1 \searrow & & \swarrow h_2 \\ & \bar{X} \times \bar{X} & \end{array}$$

of adic spaces, where all fiber products are taken over $\mathrm{Spa} C$. Using the previous proposition, it is easy to see that

$$\mathbf{D}\mathcal{F} \boxtimes \mathcal{F} \simeq h_{1!}\Lambda_{X \times \bar{X}}[2](1).$$

By the symmetry of the situation, we have

$$\mathcal{F} \boxtimes \mathbf{D}\mathcal{F} \simeq h_{2!}\Lambda_{\bar{X} \times X}[2](1),$$

so then

$$\mathbf{D}(\mathcal{F} \boxtimes \mathbf{D}\mathcal{F}) \simeq \mathbf{D}(h_{2!}\Lambda_{\bar{X} \times X}[2](1)) \simeq h_{2*}K_{\bar{X} \times X}[-2](-1).$$

To calculate $K_{\bar{X} \times X}$, we use that the projection $\mathrm{pr} : \bar{X} \times X \rightarrow \bar{X}$ is smooth of relative dimension one, so

$$K_{\bar{X} \times X} = \mathrm{pr}^!K_{\bar{X}} = \mathrm{pr}^*K_{\bar{X}}[2](1) \simeq f_{2!}\Lambda_{X \times X}[4](2)$$

by the previous proposition. Thus

$$\mathbf{D}(\mathcal{F} \boxtimes \mathbf{D}\mathcal{F}) \simeq h_{2*} f_{2!} \Lambda_{X \times X}[2](1).$$

It is now clear that $\mathbf{D}\mathcal{F} \boxtimes \mathcal{F}$ and $\mathbf{D}(\mathcal{F} \boxtimes \mathbf{D}\mathcal{F})$ cannot be isomorphic. For example, let U be the open subspace of $\overline{X} \times \overline{X}$ defined by the conditions $|T_1| \leq |T_2| \neq 0$. Then

$$H^{-2}(R\Gamma(U, \mathbf{D}\mathcal{F} \boxtimes \mathcal{F})) \simeq h_{1!} \Lambda_{X \times \overline{X}}(U) = 0,$$

since $U \not\subseteq X \times \overline{X}$, while on the other hand

$$H^{-2}(R\Gamma(U, \mathbf{D}(\mathcal{F} \boxtimes \mathbf{D}\mathcal{F}))) \simeq (f_{2!} \Lambda_{X \times X})(U \cap (\overline{X} \times X)) \simeq \Lambda,$$

which one easily checks using the fact that $U \cap (\overline{X} \times X)$ is a nonempty connected open subset of $X \times X$.

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